Transport in Open Systems of Fermions

DIAS conference

joint work with Claude-Alain Pillet

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Introduction

- Transport Problems
- Statistical Mechanics and NESS
- Open Systems

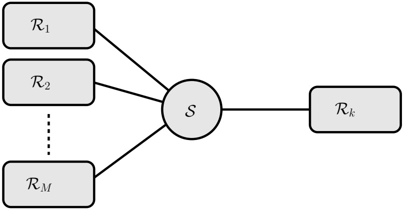

Figure: Open System $\mathcal{S}$ in contact with reservoirs $\mathcal{R}_1, \ldots, \mathcal{R}_M$.

- NESS et currents in open systems
- Systems of fermions and C^*-algebra
- Construction of NESS in such systems
- Formalism of Landauer-Büttiker

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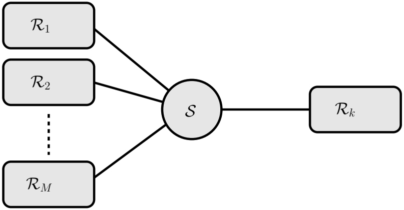

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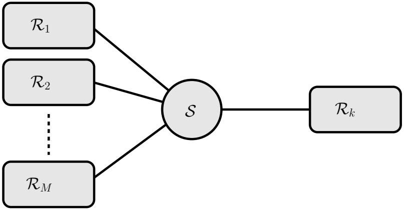

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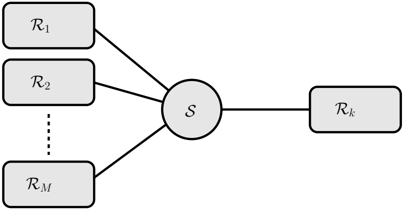

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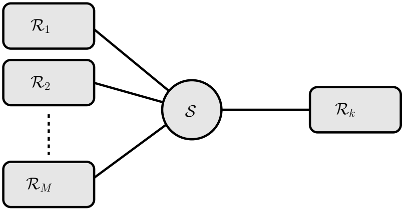

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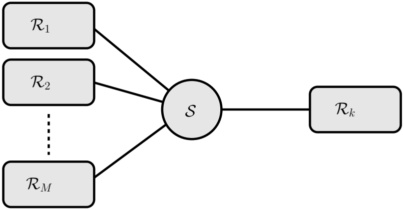

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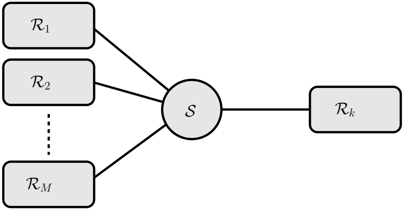

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CAR Algebra and free dynamics

- We consider a one particle Hilbert space $\mathcal{H}$.
- Creation and annihlation operators $a^*(f)$, $a(f)$ on the Fermionic Fock space $\Gamma_-(\mathcal{H})$.
- C^*-algebra CAR $(\mathcal{H})$=C^*-algebra generated by $\{a^*(f), a(f)\}$.
- Dynamics is given by group of *-automorphismes:

$$\tau^t(a(f)) = \mathrm{e}^{\mathrm{i}td\Gamma(H)}a(f)\mathrm{e}^{-\mathrm{i}td\Gamma(H)} = a(\mathrm{e}^{\mathrm{i}tHf})$$

  H is the one particule Hamiltonian.
- The group of jauge play an important role

$$\vartheta^t(a(f)) = \mathrm{e}^{\mathrm{i}sd\Gamma(Q)}a(f)\mathrm{e}^{-\mathrm{i}sd\Gamma(Q)} = a(\mathrm{e}^{\mathrm{i}sQf})$$

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Gauge invariant states

- A state ω on a C^*-algebra is a normalized positive linear functional.
- ω on $\mathrm{CAR}(\mathcal{H})$ is completly characterized by

$$F_{m,n}(g_1,\ldots,g_m,f_1,\ldots,f_n)=\omega(a^*(g_m)\ldots a^*(g_1)a(f_1)\ldots a(f_n)).$$

  Such a state ϑ-invariant is called gauge invariant quasi free state if there exists a self adjoint operator T on $\mathcal{H}$ such that $0\leq T\leq I$ and

$$\omega(a^*(g_m)...a^*(g_1)a(f_1)...a(f_n))=\delta_{n,m}\det\{(f_j,Tg_k)\}_{j,k=1,\ldots,n},$$

- If c is a trace class operator on $\mathcal{H}$ then $d\Gamma(c)\in CAR(\mathcal{H})$ and

$$\omega(d\Gamma(c))=Tr(Tc).$$

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KMS states

- ω is a (τ, β)-KMS if $\omega(A\tau^{t+i\beta}(B)) = \omega(\tau^t(B)A)$, such a state characterizes a thermal equilibrium of the system at inverse temperature β.
- $$\omega \text{ is } (\tau^t \circ \vartheta^{-\mu t}, \beta) - KMS \text{ state on } \mathrm{CAR}(\mathcal{H}) \text{ for } \mu \in \mathbb{R}$$

  $$\Updownarrow$$

  $$\omega \text{ is a quasi free state with density } T = \frac{1}{1 + e^{\beta(h-\mu)}}$$

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Open Systems of fermions

- Let a free Fermi gas (ideal) confined in a connexe geometric structure $\mathfrak{M}$ for example a domain $\mathfrak{M} \subset \mathbb{R}^d$ such that $\mathfrak{M}$ is the disjoint union of a compact part $\mathfrak{S}$ and M infinite ends $\mathfrak{R}_1, \ldots \mathfrak{R}_M$ with a tubular or cylindric structure.

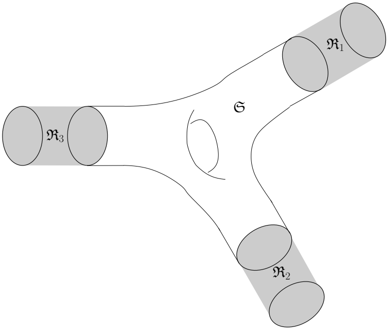

Figure: Geometrical Structure of Fermi gas.

Quasi free NESS

- Let T the generator of a jauge invariant quasi free state $\omega_T \in \mathrm{CAR}(\mathcal{H})$.
- For all $f_1, \ldots, g_1, \ldots \in \mathcal{H}$ we have

$$\omega_T \circ \tau^t (a^*(g_m) \cdots a^*(g_1) a(f_1) \cdots a(f_n)) \\ = \omega_{T_t}(a^*(g_m) \cdots a^*(g_1) a(f_1) \cdots a(f_n)),$$

where $T_t \equiv \mathrm{e}^{-\mathrm{i}tH} T \mathrm{e}^{\mathrm{i}tH}$.
- We conclude that $\omega \circ \tau^t$ is the jauge invariant quasi free state on $\mathrm{CAR}(\mathfrak{h})$ generated by T_t.

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Mourre estimate

Definition

Let H and A self adjoint operators on the Hilbert space $\mathcal{H}$.

(i) *H satisfies a Mourre estimate in $E \in \mathbb{R}$ with the conjuguate operator A if it exists $\theta > 0$, $g \in C_0^\infty(\mathbb{R})$ and a compact operator K such that $0 \leq g \leq 1$, $g(E) = 1$ and*

$$g(H)\mathrm{i}[H, A]g(H) \geq \theta g(H)^2 + K. \tag{1}$$

(ii) *Let an open $O \subset \mathbb{R}$. We say H satisfies a Mourre estimate on O with the conjugate operator A if, for all $E \in O$, H satisfies a Mourre estimate with the conjugate operator A in E.*

(iii) *If we can take $K = 0$ in (1), we say H satisfies a strict Mourre estimate in E (resp on O) with A.*

Definition

If H is selfadjoint on the Hilbert space $\mathcal{H}$, $\Lambda = (I + |H|)$, and $s \in \mathbb{R}$ we denote $\mathcal{H}_H^s$ the Banach space obtained by completion of $\mathrm{Dom}\,(\Lambda^s)$ with the norm

$$\|u\|_{H,s} = \|\Lambda^s u\|.$$

Definition

Let A a selfadjoint operator on $\mathcal{H}$. For all integers $n \geq 0$ we define

$$\mathcal{B}_A^n(\mathcal{H}_H^s, \mathcal{H}) \equiv \{B \in \mathcal{B}(\mathcal{H}_H^s, \mathcal{H}) \,|\, \mathrm{ad}_A^k(B) \in \mathcal{B}(\mathcal{H}_H^s, \mathcal{H}), k = 0, 1, \ldots, n\}.$$

for $s > 0$ and

$$\mathrm{ad}_A^0(B) \equiv B, \qquad \mathrm{ad}_A^k(B) = \mathrm{i}[A, \mathrm{ad}_A^{k-1}(B)], (k = 1, 2, \ldots).$$

The first consequences of the Mourre estimate concern the singular spectrum of H(Commun .Math. Phys. **78**, 391 (1981)).

Definition

A selfadjoint operator H is $C^n_{\mathrm{loc}}(A)$ class if, for all $u \in \mathcal{H}$ and for all $f \in C_0^\infty(\mathbb{R})$ the function
$\mathbb{R} \ni \theta \longrightarrow \tau_A^\theta(f(H))u \equiv \mathrm{e}^{\mathrm{i}\theta A} f(H) \mathrm{e}^{-\mathrm{i}\theta A} u = f(\tau_A^\theta(H))u$ *is C^n class.*

Theorem

We assume that the selfadjoint operator H on $\mathcal{H}$ satisfies a Mourre estimate on the open $O \subset \mathbb{R}$ with the conjugate operator A.

(i) *If $H \in C^1_{\mathrm{loc}}(A)$ and $I \subset O$ is compact then $\mathrm{spec}_{\mathrm{pp}}(H) \cap I$ is finite. This ensemble is empty if the Mourre estimate is strict in I.*

(ii) *If $H \in C^2_{\mathrm{loc}}(A)$, $\mathrm{spec}_{\mathrm{sc}}(H) \cap O$ est empty.*

Our main tool ta control dynamics is the propagation estimate (Sigal-Soffer,Preprint, Princeton University, (1988),Hunziker-Sial-Soffer Commun. Partial Differential Equations **24**, 2279 (1999).)

Proposition

Let A and H selfadjoint operators on $\mathcal{H}$ such that:

(i) *$H \in C^n_{\mathrm{loc}}(A)$ for an integer $n \geq 2$.*

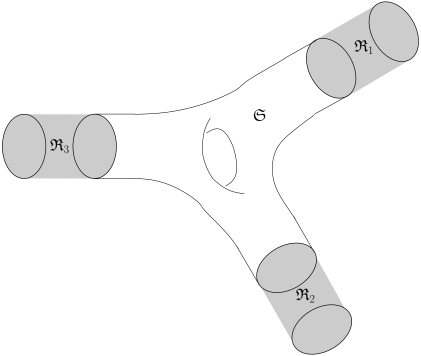
(ii) *H satisfies a Mourre estimate strict with the conjugate operator A on the open $O \subset \mathbb{R}$.*

Then, for $s < n-1$ and $g \in C_0^\infty(O)$, there exist constants $\vartheta > 0$ and c such that

$$\|F(\pm A \leq a - b + \vartheta t)\mathrm{e}^{\mp \mathrm{i}tH} g(H) F(\pm A \geq a)\| \leq c \langle b + \vartheta t \rangle^{-s}, \quad (2)$$

for all $a \in \mathbb{R}$, $b \geq 0$ et $t \geq 0$.

We consider a system of quasi free fermions propagating in the structure $\mathfrak{M}$ of 3. The fermions visiting the compact part $\mathfrak{S} \subset \mathfrak{M}$ consist in the small system $\mathcal{S}$. The reservoirs $\mathcal{R}_1, \ldots, \mathcal{R}_M$ are composed of the fermions contenained in the infinite ends



$\mathfrak{R}_1, \ldots, \mathfrak{R}_M$.

Figure: Geometrical structure of the Fermi gas.

Construction of NESS

The system of fermions is described by the C^*-algebra $\mathcal{O} \equiv \mathrm{CAR}(\mathcal{H})$ with the Bogoliubov $*$-automorphisms group $\tau^t(a(f)) = a(\mathrm{e}^{\mathrm{i}tH} f)$.

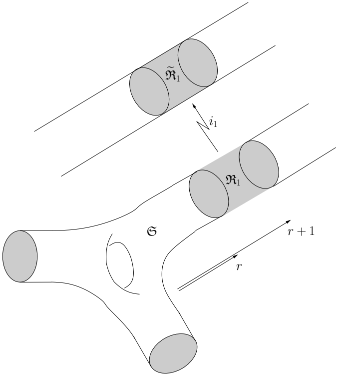


Figure: Immersion de la branche étendue $\mathfrak{R}_1$ dans le super-réservoir $\widetilde{\mathfrak{R}}_1$. The parameter r describes the deep of immersion. $M_k f = (\widetilde{\chi}_k f) \circ i_k$.

(H1) Imbedding.

For $k \in \{1, \ldots, M\}$ there exists an Hilbert space $\widetilde{\mathcal{H}}_k$ and an identification operateur $M_k \subset \mathcal{B}(\widetilde{\mathcal{H}}_k, \mathcal{H})$ we dénote by $M_k = J_k \widetilde{\chi}_k$ the polar decomposition, $\widetilde{\chi}_k \equiv |M_k| = (M_k^* M_k)^{1/2}$, $\chi_k \equiv (M_k M_k^*)^{1/2}$, $\tilde{1}_k \equiv J_k^* J_k$ et $1_k \equiv J_k J_k^*$. We assume:

(i) $\|M_k\| = 1$.

(ii) $J_k^* J_l = 0$ for $k \neq l$.

(H2) Coupling.

For $k \in \{1, \ldots, M\}$ it exists a selfadjoint hamiltonian $\widetilde{H}_k$ on $\widetilde{\mathcal{H}}_k$ such that

(i) $M_k \mathrm{Dom}\,(\widetilde{H}_k) \subset \mathrm{Dom}\,(H)$ and $M_k^* \mathrm{Dom}\,(H) \subset \mathrm{Dom}\,(\widetilde{H}_k)$.

(ii) H coincides with $\widetilde{H}_k$ on $M_k \mathrm{Dom}\,(\widetilde{H}_k)$:

$$HM_k u = J_k \widetilde{H}_k \widetilde{\chi}_k u,$$

for all $u \in \mathrm{Dom}\,(\widetilde{H}_k)$.

(iii) $(I - \tilde{1}_k)\widetilde{H}_k \widetilde{\chi}_k u = 0$ for all $u \in \mathrm{Dom}\,(\widetilde{H}_k)$.

(iv) The operator $B_k \equiv [\widetilde{H}_k, \widetilde{\chi}_k]$ is $\widetilde{H}_k$-compact.

(v) The operator $B_k^* J_k^*$ is H-compact.

(vi) 1_0 is H-compact.

(H3)Mourre.

For $k \in \{1, ..., M\}$, there exists a selfadjoint operator $\widetilde{A}_k$ on $\widetilde{\mathcal{H}}_k$ and a denombrable closed subset $\Sigma_k \subset \mathbb{R}$ such that , for all $E \in \mathbb{R} \setminus \Sigma_k$, $\widetilde{H}_k$ satisfies a stict Mourre estimate with the conjugate operator $\widetilde{A}_k$. Moreover:

(i) $\widetilde{\chi}_k \mathrm{Dom}\,(\widetilde{A}_k) \subset \mathrm{Dom}\,(\widetilde{A}_k)$.

(ii) $\mathrm{e}^{\mathrm{i}\theta \widetilde{A}_k} \mathrm{Dom}\,(\widetilde{H}_k) \subset \mathrm{Dom}\,(\widetilde{H}_k)$ for $\theta \in \mathbb{R}$.

(iii) $\widetilde{H}_k \in \mathcal{B}^n_{\widetilde{A}_k}(\widetilde{\mathcal{H}}^1_k, \widetilde{\mathcal{H}}_k)$ for an integer $n \geq 2$.

$\widetilde{\mathcal{H}}^1_k$ denotes the Sobolev scale associated to $\widetilde{H}_k$.

Theorem

Under assumptions **(H1)**, **(H2)** *and* **(H3)** *, the operator A defined by* $A \equiv \sum_{k=1}^{M} J_k \widetilde{A}_k J_k^*$*, is selfadjoint and satisfies*

(i) $\mathrm{e}^{\mathrm{i}\theta A}\mathrm{Dom}\,(H) \subset \mathrm{Dom}\,(H)$ *for all* $\theta \in \mathbb{R}$.

(ii) $H \in \mathcal{B}^n_{A_k}(\mathcal{H}^1_H, \mathcal{H})$ *for* $k \in \{1, \ldots, M\}$.

(iii) $H \in C^n_{\mathrm{loc}}(A)$.

(iv) *There exists a closed and denombrable set* $\Sigma_H \subset \mathbb{R}$ *such that* H *satisfies a strict Mourre estimate on* $\mathbb{R} \setminus \Sigma_H$ *with the conjugate operator* A.

Remarque The system $\mathcal{S}$ is localised in $\mathrm{Ran}\,1_0$, it is also localised in $\mathrm{Ran}\,F(A=0)$. For $a>0$, $\mathrm{Ran}\,F(\pm A_k > a) \subset \mathrm{Ran}\,1_k$, it results that $F(\pm A_k > a)$ localise in the interior of reservoir $\mathcal{R}_k$.

Theory of multi-canal scattering

Definition

We call Møller partiels operators the strong limits

$$\Omega_k^{\pm} \equiv \mathrm{s}-\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH} J_k \mathrm{e}^{-\mathrm{i}t\widetilde{H_k}} P_{\mathrm{ac}}(\widetilde{H_k}),$$

if it exists. $\Omega_k^{\pm}$ *is a partial isometry from initial space* $\mathcal{H}_{\widetilde{\mathcal{R}_k},\mathrm{ac}}(\widetilde{H_k})$ *and we have* $f(H)\Omega_k^{\pm} = \Omega_k^{\pm} f(\widetilde{H_k})$.
$\Omega_k^{\pm}$ *are complete if* $\oplus_k \mathrm{Ran}\,\Omega_k^{\pm} = \mathcal{H}_{\mathrm{ac}}(H)$. *We call asymptotic projections on* $\mathcal{H}_k$ *(respectivement* $\widetilde{\mathcal{H}_k}$*) the operators*

$$\widetilde{P}_k^{\pm} \equiv \mathrm{s}-\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}t\widetilde{H}_k} 1_k P_{\mathrm{ac}}(\widetilde{H}_k) \mathrm{e}^{-\mathrm{i}t\widetilde{H}_k},$$

$$P_k^{\pm} \equiv \mathrm{s}-\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH_k} 1_k P_{\mathrm{ac}}(H_k) \mathrm{e}^{-\mathrm{i}tH_k}.$$

Proposition

Under (H1), (H2), et (H3) the partiels Møller operators $\Omega_k^{\pm}$ *exist, and satisfy*

$$\Omega_k^{\pm *}\Omega_l^{\pm} = \delta_{kl}\widetilde{P}_k^{\pm}, \qquad \sum_{k=1}^{M}\Omega_k^{\pm}\Omega_k^{\pm *} = P_{\mathrm{ac}}(H).$$

In particular, the Møller operators $\Omega^{\pm} = \oplus_{k=1}^{M}\Omega_k^{\pm} : \oplus_{k=1}^{M}\widetilde{\mathcal{H}}_k \to \mathcal{H}$ *are complete:* $\mathrm{Ran}\,\Omega^- = \mathrm{Ran}\,\Omega^+ = \mathcal{H}_{\mathrm{ac}}(H)$.

The scattering matrix $S : \oplus_k \mathcal{H}_{\widetilde{\mathcal{R}}_k} \to \oplus_k \mathcal{H}_{\widetilde{\mathcal{R}}_k}$ given by

$$S_{jk} = \Omega_j^{+*}\Omega_k^{-},$$

is unitary and

$$\Omega_k^{\pm *} = \mathrm{s} - \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}t\widetilde{H}_k} J_k^* \mathrm{e}^{-\mathrm{i}tH}.$$

Definition

ω^+ is the NESS associated to the initial state ω if

$$\lim_{T\to+\infty}\frac{1}{T}\int_0^T \omega\circ\tau^t(A)dt=\omega^+(A)$$

$\forall A\in\mathcal{O}$.

Theorem

Under (H1) and (H2). For all $k\in\{1,\ldots,M\}$ let $T_k=f_k(H)$ the jauge invariant quasi free generator and τ-invariant on $\mathcal{O}$.

$$T=\sum_{k=1}^{M}1_k T_k 1_k,$$

generates jauge invariant quasi free state on $\mathcal{O}$. If (H3) is true, the NESS ω_T^+ associated to ω_T exists.

Theorem ((suite))

(i) *The restriction* $\omega_T^+|_{\mathrm{CAR}(\mathcal{H}_{\mathrm{ac}})}$ *is the invariant gauge quasi free state generated by* $T^+ = \sum_{k=1}^M P_k^- T_k P_k^-$.

(ii) *For all gauge invariant state and* ω_T*-normal* η *on* $\mathcal{O}$ *and for* $A \in \mathrm{CAR}(\mathcal{H}_{\mathrm{ac}}(H))$,

$$\lim_{t\to\infty} \eta \circ \tau^t(A) = \omega_{T^+}(A).$$

(iii) *For all trace class operator* c *on* $\mathcal{H}$,

$$\omega_T^+(\mathrm{d}\Gamma(c)) = \mathrm{Tr}(T^+ c) + \sum_{\varepsilon \in \mathrm{spec}_{\mathrm{pp}}(H)} \mathrm{Tr}(P_\varepsilon T P_\varepsilon c), \qquad (3)$$

where P_ε *denote the orthogonal projection on the eigenvectors space of* H *for the eigenvalue* ε.

Landauer-Büttiker formula

(H4)

There exists an integer $m \geq 2$ such that, for $k \in \{1, \ldots, M\}$,

(i) $(\widetilde{A}_k + \mathrm{i})^{-m} g(\widetilde{H}_k)$ is trace class for all $g \in C_0^\infty(\mathbb{R} \setminus \Sigma_k)$.

(ii) $\widetilde{H}_k \in \mathcal{B}^{m+2}_{\widetilde{A}_k}(\widetilde{\mathcal{H}}^1_k, \widetilde{\mathcal{H}}_k)$.

(iii) $1_0 g(H)$ is trace class for all $g \in C_0^\infty(\mathbb{R})$.

(H5)

There exists an integer ν such that, for $k \in \{1, \ldots, M\}$,

(i) $\mathrm{ad}^j_{\widetilde{H}_k}(\widetilde{\chi}_k) \in \mathcal{B}(\widetilde{\mathcal{H}}^{j/2}_k, \widetilde{\mathcal{H}}_k)$ for $j = 1, \ldots, 4\nu$.

(ii) $(I - \tilde{1}_k)\,\mathrm{ad}^j_{\widetilde{H}_k}(\widetilde{\chi}_k) = 0$ for $j = 1, \ldots, 2\nu$.

Charges

Definition

A charge of one particule system is an observable, described by a selfadjoint operator Q on $\mathcal{H}$, such that

(i) $\mathrm{e}^{\mathrm{i}tH} Q \mathrm{e}^{-\mathrm{i}tH} = Q$ *for all* $t \in \mathbb{R}$.

(ii) *For each* $k \in \{1, \ldots, M\}$ *there exists a selfadjoint operator* $\widetilde{Q}_k$ *on* $\widetilde{\mathcal{H}}_k$ *such that* $\mathrm{e}^{\mathrm{i}t\widetilde{H}_k} \widetilde{Q}_k \mathrm{e}^{-\mathrm{i}t\widetilde{H}_k} = \widetilde{Q}_k$ *for all* $t \in \mathbb{R}$.

(iii) *For all diffusion state* $u \in \mathcal{H}_{\mathrm{ac}}(H)$ *we have*

$$\lim_{t \to \pm\infty} \sum_{k=1}^{M} (M_k^* \mathrm{e}^{-\mathrm{i}tH} u, \widetilde{Q}_k M_k^* \mathrm{e}^{-\mathrm{i}tH} u) = (u, Qu).$$

Courants

Soit Q une charge du système à une particule. La charge totale située à l'intérieur du réservoir $\mathcal{R}_k$ est décrite par l'observable $\mathrm{d}\Gamma(1_k Q 1_k)$. L'observable de courant correspondante est

$$\frac{\mathrm{d}}{\mathrm{d}t}\tau^t(\mathrm{d}\Gamma(1_k Q 1_k))\Big|_{t=0} = \mathrm{d}\Gamma(\mathrm{i}[H, 1_k Q 1_k]).$$

Notre but dans ce paragraphe est de donner un sens au courant stationnaire

$$\omega_T^+(\mathrm{d}\Gamma(\mathrm{i}[H, 1_k Q 1_k])), \tag{4}$$

valeur moyenne de l'observable de courant dans le NESS que nous avons construit dans le paragraphe précédent.

Courants régularisés

Definition

Une charge Q est tempérée si $\mathrm{Dom}\,(|H|^{\alpha}) \subset \mathrm{Dom}\,(Q)$ *pour un* $\alpha > 0$.

Si Q est une charge tempérée alors $Q_\epsilon \equiv Q(1+\epsilon H^2)^{-\alpha/2}$ est une charge bornée. Régularisons le commutateur en le localisant à l'aide d'une fonction $g \in C_0^\infty(\mathbb{R} \setminus \Sigma_H)$. On remarque que si $f \in C_0^\infty(\mathbb{R})$ est telle que $g(x)f(x) = xg(x)$ pour tout $x \in \mathbb{R}$, alors l'expression

$$\Phi_{Q_\epsilon,g,k} \equiv g(H)[H, 1_k Q_\epsilon 1_k]g(H) = g(H)[f(H), 1_k Q_\epsilon 1_k]g(H),$$

est à trace.

Représentations spectrales des hamiltoniens $\widetilde{H}_k$

Pour $k \in \{1, \dots, M\}$ il existe une famille $(\mathfrak{h}_k(\varepsilon))_{\varepsilon\in\mathbb{R}}$ d'espaces de Hilbert et un opérateur unitaire

$$U_k : \widetilde{\mathcal{H}}_{k,\mathrm{ac}} \to \int^{\oplus} \mathfrak{h}_k(\varepsilon)\,\mathrm{d}\varepsilon, \tag{5}$$

tel que $(U_k\widetilde{H}_k u)(\varepsilon) = \varepsilon(U_k u)(\varepsilon)$ pour tout $u \in \widetilde{\mathcal{H}}_{k,\mathrm{ac}}$. La représentation (5) fait correspondre à tout opérateur $B \in \mathcal{B}(\widetilde{\mathcal{H}}_k, \widetilde{\mathcal{H}}_j)$, tel que $f(\widetilde{H}_j)B = Bf(\widetilde{H}_k)$ pour toute fonction bornée f, une famille mesurable $b(\varepsilon) \in \mathcal{B}(\mathfrak{h}_k(\varepsilon), \mathfrak{h}_j(\varepsilon))$ telle que $(U_jBu)(\varepsilon) = b(\varepsilon)(U_k u)(\varepsilon)$ pour tout $u \in \widetilde{\mathcal{H}}_{k,\mathrm{ac}}$. On a en particulier les correspondances

$$\begin{aligned} \widetilde{Q}_k &\longrightarrow q_k(\varepsilon), \\ \widetilde{P}_k^{\pm} &\longrightarrow p_k^{\pm}(\varepsilon), \\ S_{kj} &\longrightarrow s_{kj}(\varepsilon). \end{aligned}$$

Formule de Landauer-Büttiker

Theorem

Supposons que les hypothèses **(H1)-(H5)** *soient satisfaites. Soit* ω_T^+ *le NESS et* Q *une charge tempérée tels que*

$$\mathrm{Ran}\, \widetilde{T}_j \subset \mathrm{Dom}\, \widetilde{H}_j^{\nu+\alpha+1}, \qquad \mathrm{Dom}\, H^\alpha \subset \mathrm{Dom}\, Q.$$

Alors, pour toute suite $g_n \in C_0^\infty(\mathbb{R} \setminus \Sigma_H)$ *telle que* $0 \leq g_n \leq 1$ *et* $\lim_n g_n(x) = 1$ *presque partout la limite*

$$\omega_T^+(\mathrm{d}\Gamma(\Phi_{Q,k})) \equiv \lim_n \lim_{\epsilon \to 0} \omega_T^+(\mathrm{d}\Gamma(\Phi_{Q_\epsilon, g_n, k})),$$

$$\omega_T^+(\mathrm{d}\Gamma(\Phi_{Q,k})) = \sum_{j=1}^{M} \int \mathrm{Tr}_{\mathfrak{h}_j(\varepsilon)} \left\{ f_j(\varepsilon) \left(s_{jk}^*(\varepsilon) q_k(\varepsilon) s_{kj}(\varepsilon) - \delta_{kj} p_j^-(\varepsilon) q_j(\varepsilon) p_j^-(\varepsilon) \right) \right\} \frac{\mathrm{d}\varepsilon}{2\pi}$$

Equivalences des courants

Soit $h \in C^\infty(\mathbb{R})$ une fonction réelle telle que $0 \leq h \leq 1$ et

$$h(x) = \begin{cases} 0 & \text{si} \quad x < -1; \\ 1 & \text{si} \quad x > 1. \end{cases}$$

Pour $a \geq 1$ on, pose $h_\pm^{(a)}(x) \equiv h(\pm x - a)$ et $h^{(a)} \equiv h_-^{(a)} + h_+^{(a)}$ et $g^{(a)} = 1 - h^{(a)} \in C_0^\infty(\mathbb{R})$ avec $\mathrm{supp}\, g^{(a)} \subset [-a-1, a+1]$

Theorem

Si les hypothèses **(H4)** *et* **(H5)** *sont satisfaites et $a \geq 1$ l'opérateur*

$$\Psi_{Q,g,k}^{(a)} \equiv g(H) i [f(H), h^{(a)}(A_k) Q h^{(a)}(A_k)] g(H). \tag{6}$$

est à trace. De plus, si T est le générateur d'un état quasi-libre invariant de jauge et τ-invariant sur $\mathcal{O}$

$$\mathrm{Tr}(T\Psi_{Q,g,k}^{(a)}) = \mathrm{Tr}(T\Phi_{Q,g,k}^{(1)}).$$

Calcul du courant

L'idée est de développer l'opérateur de courant

$$\Psi^{(a)}_{Q,g,k} = \sum_{\sigma,\sigma'\in\{\pm\}} \Psi^{(a,\sigma,\sigma')}_{Q,g,k}, \tag{7}$$

où

$$\Psi^{(a,\sigma,\sigma')}_{Q,g,k} \equiv g(H)i[f(H), h^{(a)}_{\sigma}(A_k)Qh^{(a)}_{\sigma'}(A_k)]g(H), \tag{8}$$

et d'exploiter une propriété du commutateur $[H, h^{(a)}_{\sigma}(A_k)]$. En développant ce commutateur

$$[H, h^{(a)}_{\sigma}(A_k)] \sim h^{(a)'}_{\sigma}(A_k)[H, A_k],$$

on remarque qu'il est bien localisé, dans un voisinage spectral de $A_k = \sigma a$. L'estimation de Mourre nous renseigne sur la façon dont se propagent les états localisés au voisinage de $A_k = \sigma a$,

$$\mathrm{e}^{\mathrm{i}\sigma tH} h^{(a)'}_{\sigma}(A_k) \sim h^{(a)'}_{\sigma}(A_k + \theta t)\mathrm{e}^{\mathrm{i}\sigma tH}.$$

Le système $\mathcal{S}$ étant confiné au sous-espace $A_k = 0$ par l'hypothèse (H5), on en déduit que ces états ne subissent pas de diffusion lorsque $t \to \sigma\infty$. L'opérateur de Møller $\Omega^{\sigma *}$ donc agit trivialement sur de tels états. alors que $\Omega^{-\sigma *} = \Omega^{-\sigma *}\Omega^{\sigma}\Omega^{\sigma *}$ agit comme la matrice de diffusion $\Omega^{-\sigma *}\Omega^{\sigma}$ ou son adjointe.

Le résultat de cette première réduction du problème est le suivant.

Theorem

On suppose les hypothèses (H3), (H4), (H5) et (H4) satisfaites. Soit $\widetilde{T}_j = f_j(\widetilde{H}_j)$ *le générateur d'un état quasi-libre invariant de jauge et* τ_j*-invariant sur* $\mathcal{O}_j$ *tel que* $\mathrm{Ran}\,\widetilde{T}_j^{1/2} \subset \mathrm{Dom}\,\widetilde{H}_j$*. Pour tout* $g \in C_0^\infty(\mathbb{R} \setminus \Sigma_H)$ *on a*

$$\lim_{a\to\infty} \Big[\mathrm{Tr}(P_k^- T_j P_k^- \Psi_{Q,g,k}^{(a)}) - \mathrm{Tr}\Big(f_j(\widetilde{H}_j) \Big\{ S_{jk}^* \widetilde{\Psi}_{Q,g,k}^{(a,+,+)} S_{kj} + \delta_{jk} \Big(\widetilde{P}_k^- \widetilde{\Psi}_{Q,g,k}^{(a,-,-)} \widetilde{P}_k^- + S_{kk}^* \widetilde{\Psi}_{Q,g,k}^{(a,+,-)} \widetilde{P}_k^- + \widetilde{P}_k^- \widetilde{\Psi}_{Q,g,k}^{(a,-,+)} S_{kk} \Big) \Big\} \Big) \Big] = 0. \tag{9}$$

où $\widetilde{\Psi}_{Q,g,k}^{(a,\sigma,\sigma')} \equiv g(\widetilde{H}_k) \mathrm{i} [f(\widetilde{H}_k), h_\sigma^{(a)}(\widetilde{A}_k) M_k^* Q M_k h_{\sigma'}^{(a)}(\widetilde{A}_k)] g(\widetilde{H}_k)$.

Le principal outil technique nécessaire à la démonstration de ce résultat est le lemme de localisation suivant du à Avron-Graf et al(Commun. Pure and Applied Math. **57**, 528 (2004).)

Lemma

Si les hypothèses (H3), (H4), (H5) et (H4) sont satisfaites alors, pour tout $f \in C_0^\infty(\mathbb{R})$ *et* $g \in C_0^\infty(\mathbb{R} \setminus \Sigma_H)$*:*

(i) $\sup_{a \geq 1} \|[f(H), h_\pm^{(a)}(A_k)]g(H)\|_1 < \infty$.

(ii) *Il existe des constantes* $s > 1$ *et* C *telles que, pour tous* $a, \alpha \geq 1$,

$$\|F(\pm A < a - \alpha)[f(H), h_\pm^{(a)}(A_k)]g(H)\|_1 < C\langle\alpha\rangle^{-s}.$$

Remarque. Il découle de (i) que les composantes $\Psi_{Q,g,k}^{(a,\sigma,\sigma')}$ du courant sont des opérateurs à trace.

Opérateurs à trace

Theorem

Si C est un opérateur à trace sur $\mathfrak{H} \equiv \int^{\oplus} \mathfrak{h}_\varepsilon \, \mathrm{d}\varepsilon$ *alors:*

(i) *Il existe un ensemble mesurable* $\Delta_0 \subset \mathbb{R}$ *tel que* $\mathbb{R} \setminus \Delta_0$ *soit négligeable au sens de Lebesgue et, pour tout* $\langle \varepsilon', \varepsilon \rangle \in \Delta_0 \times \Delta_0$, *un opérateur à trace* $c(\varepsilon', \varepsilon) : \mathfrak{h}_\varepsilon \rightarrow \mathfrak{h}_{\varepsilon'}$ *tel que* $\langle \varepsilon', \varepsilon \rangle \mapsto (u(\varepsilon'), c(\varepsilon', \varepsilon) v(\varepsilon))_{\mathfrak{h}_{\varepsilon'}}$ *soit mesurable pour tout* $u, v \in \mathfrak{H}$.

(ii) *Pour tout* $u, v \in \mathfrak{H}$, $(u, Cv) = \int (u(\varepsilon'), c(\varepsilon', \varepsilon) v(\varepsilon))_{\mathfrak{h}_{\varepsilon'}} \, \mathrm{d}\varepsilon \mathrm{d}\varepsilon'$.

(iii) $\int \| c(\varepsilon, \varepsilon) \|_1 \, \mathrm{d}\varepsilon \leq \| C \|_1$.

(iv) $\int \mathrm{Tr}_{\mathfrak{h}_\varepsilon}(c(\varepsilon, \varepsilon)) \, \mathrm{d}\varepsilon = \mathrm{Tr}(C)$.

La diagonale

La diagonale $c(\varepsilon,\varepsilon)$ du noyau intégral d'un opérateur à trace C sur une intégrale directe de Lebesgue $\int^{\oplus} \mathfrak{h}_\varepsilon \, \mathrm{d}\varepsilon$ est définie presque partout. Pour calculer cette diagonale, le résultat suivant est souvent utile.

Lemma

Soit E l'opérateur autoadjoint sur $\mathfrak{H} \equiv \int^{\oplus} \mathfrak{h}_\varepsilon \, \mathrm{d}\varepsilon$ défini par $(Eu)(\varepsilon) = \varepsilon u(\varepsilon)$. Si $C \in \mathcal{L}^1(\mathfrak{H})$ et $c(\varepsilon',\varepsilon)$ dénote son noyau intégral alors il existe un sous-espace dense $\mathfrak{M} \subset \mathcal{H}$ tel que

$$\int (u(\varepsilon), c(\varepsilon,\varepsilon)v(\varepsilon)) \, \mathrm{d}\varepsilon = \lim_{\eta\downarrow 0} \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{-\eta|t|} (\mathrm{e}^{\mathrm{i}tE} u, C\mathrm{e}^{\mathrm{i}tE} v) \, \mathrm{d}t,$$

pour tout $u, v \in \mathfrak{M}$.

Représentation spectrale du courant

Theorem

Supposons que les hypothèses (H3), (H4) et (H5) soient satisfaites. Si $g \in C_0^\infty(\mathbb{R} \setminus \Sigma_H)$ *et si l'opérateur*

$$\widetilde{\Psi}_{Q,g,k}^{(a,\sigma,\sigma')} = g(\widetilde{H}_k)\mathrm{i}[f(\widetilde{H}_k), h_\sigma^{(a)}(\widetilde{A}_k)M_k^* Q M_k h_{\sigma'}^{(a)}(\widetilde{A}_k)]g(\widetilde{H}_k),$$

est à trace alors:

(i) $\widetilde{\Psi}_{Q,g,k}^{(a,\sigma,\sigma')}$ *se réduit à sa partie dans* $\widetilde{\mathcal{H}}_{k,\mathrm{ac}}$, *c'est à dire que*

$$\widetilde{\Psi}_{Q,g,k}^{(a,\sigma,\sigma')} = P_{\mathrm{ac}}(\widetilde{H}_k)\widetilde{\Psi}_{Q,g,k}^{(a,\sigma,\sigma')}P_{\mathrm{ac}}(\widetilde{H}_k).$$

(ii) *Pour tout* $\varepsilon \in \Delta$,

$$\psi_{Q,g,k}^{(a,\sigma,\sigma')}(\varepsilon,\varepsilon) = \frac{\sigma}{2\pi}\, g(\varepsilon)^2\, p_k^\sigma(\varepsilon) q_k(\varepsilon) p_k^\sigma(\varepsilon)\delta_{\sigma\sigma'}. \tag{10}$$