An effective mass theorem for a bidimensional electron gas under a strong magnetic field

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Motivation

We present an asymptotic model for the transport of 3D quantum gas *strongly* confined in one direction ($z \in \mathbb{R}$) and subject to a *strong* magnetic field whose direction is in the transport plane (the horizontal (x, y) plane).

Hence, we want to derive, from asymptotic analysis, a quantum equivalent to the guiding center approximation for classical systems.

The three directions will play different roles :

- in z, the dynamics is frozen by the confinement,
- in y, direction of B, the dynamics is not perturbed by B,
- in x, the dynamics is averaged over the cyclotron motion.

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The nonlinear model

The Schrödinger-Poisson system

$$i\hbar\partial_t \Psi = \frac{\hbar^2}{2m}(i\nabla)^2\Psi + \mathbb{V}\Psi,$$

$$-\epsilon\Delta\mathbb{V} = e^2 N|\Psi^\varepsilon|^2.$$

The nonlinear model

The Schrödinger-Poisson system perturbed by a confinement potential

$$i\hbar\partial_t\Psi = \frac{\hbar^2}{2m}(i\nabla)^2\Psi + V_c\Psi + \mathbb{V}\Psi,$$

$$\mathbb{V} = \frac{e^2 N}{4\pi\epsilon\sqrt{x^2+y^2+z^2}} * \left(|\Psi|^2\right).$$

The nonlinear model

The Schrödinger-Poisson system perturbed by a confinement potential and a strong uniform magnetic potential :

$$i\hbar\partial_t\Psi = \frac{1}{2m}\left(i\hbar\nabla - \frac{eB}{c}z\right)^2\Psi + V_c\Psi + \mathbb{V}\Psi,$$

$$= \frac{e^2N}{4\pi\epsilon\sqrt{x^2+y^2+z^2}} * \left(|\Psi|^2\right).$$

with $V_c = V_c(z) \to +\infty$ as $z \to +\infty$,

and B has a constant value.

Choice of the scales

Let us put the system in dimensionless form. We assume that there are two energy scales in the problem, $E_{conf} \gg E_{transp}$, related with two lengthes $\ell \ll L$:

- z is at the scale ℓ and (x, y) are at the scale L,
- $E_{conf} \sim V_c \sim \frac{1}{2} m \left(\frac{eB}{mc}\right)^2 \ell^2 \sim \frac{\hbar^2}{2m\ell^2}$,
- $E_{transp} \sim \mathbb{V} \sim \frac{e^2 N}{\epsilon L^2} \sim \frac{\hbar^2}{2mL^2} \sim \frac{\hbar}{t}$.

Introduce a small parameter $\frac{\ell}{L} = \varepsilon \ll 1$. Then we have

$$\frac{E_{conf}}{E_{transp}} = \frac{1}{\varepsilon^2}.$$

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Rescaling : the initial singularly perturbed system

This leads to the following dimensionless system :

$$i\partial_t \psi^\varepsilon = \frac{1}{\varepsilon^2} H_z \psi^\varepsilon - \frac{2iB}{\varepsilon} z\partial_x \psi^\varepsilon - \partial_x^2 \psi^\varepsilon - \partial_y^2 \psi^\varepsilon + V^\varepsilon \psi^\varepsilon,$$

where

$$H_z = -\partial_z^2 + B^2 z^2 + V_c(z)$$

and

$$V^\varepsilon = \frac{1}{4\pi\sqrt{|x|^2 + \varepsilon^2 z^2}} * |\psi^\varepsilon|^2.$$

A simplified case

The following case is well-known by physicists : when V_c is an harmonic confinement potential and V^ε (the Poisson potential) is replaced by a given potential.

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$$V^\varepsilon = V(t, x, y, \varepsilon z) \text{ given, smooth enough.}$$

In this case, there is a usual trick in order to transform the equation

$$i\partial_t \psi^\varepsilon = \frac{1}{\varepsilon^2}\left(-\partial_z^2 + (\alpha^2 + B^2) z^2 - 2iB\varepsilon z\partial_x - \varepsilon^2 \partial_x^2\right) - \partial_y^2 \psi^\varepsilon + V^\varepsilon \psi^\varepsilon.$$

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Indeed,

$$-\partial_z^2 + (\alpha^2 + B^2)z^2 - 2iB\varepsilon z\partial_x - \varepsilon^2\partial_x^2$$
$$= -\partial_z^2 + (\alpha^2 + B^2)\left(z - \frac{B}{\alpha^2 + B^2} i\varepsilon\partial_x\right)^2 - \frac{\alpha^2}{\alpha^2 + B^2}\varepsilon^2\partial_x^2$$

Introduce the following operator : for a function u, we set

$$(\Theta u)(x, y, z) = u\left(x, y, z - \frac{B}{\alpha^2 + B^2} i\varepsilon\partial_x\right).$$

We obtain a simplified system on the unknown $u^\varepsilon = \Theta\Psi^\varepsilon$:

$$i\partial_t u^\varepsilon = \frac{1}{\varepsilon^2}\widetilde{H}_z u^\varepsilon - \frac{\alpha^2}{\alpha^2 + B^2}\partial_x^2 u^\varepsilon - \partial_y^2 u^\varepsilon + \Theta V^\varepsilon \Theta^{-1} u^\varepsilon,$$

where

$$\widetilde{H}_z = -\partial_z^2 + (\alpha^2 + B^2)z^2$$

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The cyclotron effective mass

As $\varepsilon \to 0$, there is combination of two effects :

- Θ becomes the identity operator : $\Theta V^\varepsilon \Theta^{-1} \to 0$.
- $V^\varepsilon(t,x,y,z) \to V(t,x,y,0)$, so the dynamics along z and (x,y) are decoupled.

Indeed, one can filter out the oscillations by setting

$$\phi^\varepsilon = \exp(it\widetilde{H}_z/\varepsilon^2)\psi^\varepsilon.$$

Then, asymptotically, we obtain a bidimensional Schödinger equation with the cyclotron effective mass :

$$i\partial_t \phi^\varepsilon = -\frac{\alpha^2}{\alpha^2+B^2}\partial_x^2\phi^\varepsilon - \partial_y^2\phi^\varepsilon + V(t,x,y,0)\phi^\varepsilon.$$

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To go further in the more general situation, two difficulties :

1. With a general confinement potential $V_c(z)$ one cannot use the trick.
2. The Poisson nonlinearity might induce non trivial effects such as resonances between fast oscillating terms.

Our strategy :

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As ε tends to 0, the solution V^ε of the Poisson equation behaves as follows :

$$V(t,x,z) \sim W(t,x) = \frac{1}{4\pi\sqrt{x^2+y^2}} * \int_{\mathbb{R}} |\psi^\varepsilon(t,\cdot,z')|^2 dz'$$

The Schrödinger-Poisson system confined on the plane :

$$i\partial_t \psi^\varepsilon = -\Delta_{x,y}\psi^\varepsilon + \frac{1}{\varepsilon^2} H_z \psi^\varepsilon + V^\varepsilon \psi^\varepsilon$$

We filter out the fast oscillations :

$$\phi^\varepsilon(t,x,y,z) = e^{itH_z/\varepsilon^2}\psi^\varepsilon(t,x,y,z)$$

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It converges towards the limit model :

$$i\partial_t \phi = -\Delta_{x,y}\phi + W\phi$$

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A few previous works on nonlinear quantum confinement problems...

1. Ben Abdallah, Méhats, Pinaud, '05 : the case $B = 0$ for Schrödinger-Poisson (with polarized initial data)
2. Ben Abdallah, Castella, Méhats, '08 : confinement on the plane or on a line for the Gross-Pitaevski equation,
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Now, both operators $-\Delta_x$ and W commute with H_z. Filtering by e^{itH_z/ε^2} in the intermediate system : the function

$$\phi^\varepsilon(t,x,z) = e^{itH_z/\varepsilon^2} \psi^\varepsilon(t,x,z)$$

satisfies the filtered system

$$i\partial_t \phi^\varepsilon = \frac{2B}{\varepsilon} e^{itH_z/\varepsilon^2} z e^{-itH_z/\varepsilon^2} (-i\partial_x \phi^\varepsilon) - \Delta_{x,y} \phi^\varepsilon + W \phi^\varepsilon.$$

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Assumptions

Confinement assumption

V_c is assumed to be even, positive and smooth such that :

$$V_c(z) \underset{|z|\to\infty}{\longrightarrow} +\infty, \text{ at most polynomially.}$$

+ a spectral assumption : If $(E_p)_{p\geq 0}$ denote the eigenvalues of operator $-\partial_z^2 + B^2 z^2 + V_c(z)$, we assume that

$$\exists n_0 > 0,\ \exists C > 0 : \quad \forall p \geq 0,\ |E_{p+1} - E_p| > Cp^{-n_0}.$$

Examples :

- $V_c(z) = \alpha^2 z^2 + V_1(z)$, with $\|V_1\|_{L^\infty}$ small,
- $V_c(z) \sim \alpha^2 |z|^k$ with $k > 2$.

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Assumptions

The initial datum

At the initial time, we have

$$\psi^{\varepsilon}(t=0)=\psi_0,$$

with ψ_0 in the energy space

$$B_1=\left\{u\in H^1(\mathbb{R}^3),\ \sqrt{V_c}\,u\in L^2(\mathbb{R}^3),\ zu\in L^2(\mathbb{R}^3)\right\}.$$

Main result

Theorem

There exists $T > 0$ such that the following convergence holds :

$$\sup_{t \in [0,T]} \left\| \psi^\varepsilon(t,x,y,z) - \sum_{p \geq 0} e^{-itE_p/\varepsilon^2} \varphi_p(t,x,y)\, \chi_p(z) \right\|_{B_1} \underset{\varepsilon \to 0}{\longrightarrow} 0.$$

where, for all $p \in \mathbb{N}$:

- E_p and χ_p are the pth eigenvalue and eigenfunction of the confinement operator $H_z = -\partial_z^2 + B^2 z^2 + V_c(z)$
- the functions φ_p satisfies the following infinite system of coupled bidimensional Schrödinger equations.

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For all $p \geq 0$,

$$i\partial_t \varphi_p = -\alpha_p \partial_x^2 \varphi_p - \partial_y^2 \varphi_p + W\varphi_p \,,$$

$$\varphi_p(t=0,x,y) = \int \psi_0(x,y,z)\chi_q(z)dz,$$

and the selfconsistent potential is given by

$$W(t,x) = \frac{1}{4\pi\sqrt{x^2+y^2}} * \left(\sum_{p\geq 0} |\psi_p|^2\right).$$

Moreover, the coefficients α_p are given by

$$\forall p \geq 0, \;\; \alpha_p = 1 - 4B^2 \sum_{q\neq p} \frac{\int z\chi_p(z), \chi_q(z)dz)^2}{E_q - E_p}.$$

Comments

- In the case $V(z) = \alpha^2 z^2$, one computes explicitely E_p and χ_p, and
$$\alpha_p = \frac{\alpha^2}{\alpha^2 + B^2}.$$
- In the general case, the 2D dynamics is diagonal on the eigenmodes of H_z but the cyclotron effective mass depends on the label p of the considered eigenmode.

Fast oscillating ODEs

Let us introduce

$$\tau \mapsto f(\tau)u = 2Be^{i\tau H_z} z e^{-i\tau H_z}(-i\partial_x u)$$

and

$$g(u) = -\Delta_{x,y} u + W(u)u.$$

Then our initial system reads as the following fast oscillating ODE :

$$iy'(t) = \frac{1}{\varepsilon} f\left(\frac{t}{\varepsilon^2}\right) y(t) + g(y(t))$$

Assuming that y is estimated in sufficiently regular spaces, let us analyze this equation.

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An integration by parts in time

Duhamel representation of the solution :

$$y(t) = y_0 - \frac{i}{\varepsilon}\int_0^t f\left(\frac{s}{\varepsilon^2}\right) y(s)ds - i\int_0^t g(y(s))ds.$$

Key Idea : replace $\frac{i}{\varepsilon}\int_0^t f(\frac{s}{\varepsilon^2}y(s)ds$ by a sum of terms that are not of order $\mathcal{O}(\frac{1}{\varepsilon})$.

Consider

$$F(t)u = \int_0^t f(s)uds,$$

then

$$\begin{aligned}\tfrac{1}{\varepsilon}f(\tfrac{s}{\varepsilon^2})y(s) &= \varepsilon\frac{\partial}{\partial s}\left[F(\frac{s}{\varepsilon^2})y(s)\right] - \varepsilon F(\frac{s}{\varepsilon^2})\frac{\partial y}{\partial s}(s) \\ &= \varepsilon\frac{\partial}{\partial s}\left[F(\frac{s}{\varepsilon^2})y(s)\right] + iF(\frac{s}{\varepsilon^2})\left(f\left(\frac{t}{\varepsilon^2}\right)y(t) + \varepsilon g(y(t))\right)\end{aligned}$$

An integration by parts in time

Duhamel representation of the solution :

$$y(t) = y_0 - \frac{i}{\varepsilon}\int_0^t f\left(\frac{s}{\varepsilon^2}\right) y(s)ds - i\int_0^t g(y(s))ds.$$

Key Idea : replace $\frac{i}{\varepsilon}\int_0^t f(\frac{s}{\varepsilon^2}y(s)ds$ by a sum of terms that are not of order $\mathcal{O}(\frac{1}{\varepsilon})$.
Consider

$$F(t)u = \int_0^t f(s)uds,$$

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$$\begin{aligned} \frac{1}{\varepsilon}f(\tfrac{s}{\varepsilon^2})y(s) &= \varepsilon\frac{\partial}{\partial s}\left[F(\frac{s}{\varepsilon^2})y(s)\right] - \varepsilon F(\frac{s}{\varepsilon^2})\frac{\partial y}{\partial s}(s) \\ &= \varepsilon\frac{\partial}{\partial s}\left[F(\frac{s}{\varepsilon^2})y(s)\right] + iF(\frac{s}{\varepsilon^2})\left(f\left(\frac{t}{\varepsilon^2}\right)y(t) + \varepsilon g(y(t))\right) \end{aligned}$$

Finally we obtain the twice iterated Duhamel representation :

$$y(t) = y_0 + \int_0^t F(\frac{s}{\varepsilon^2})f(\frac{s}{\varepsilon^2})y(s)ds - i\int_0^t g(y(s))ds$$
$$- i\varepsilon F(\frac{t}{\varepsilon^2})y(t) + \varepsilon\int_0^t F(\frac{s}{\varepsilon^2})g(y(s))ds.$$

One can prove that the terms in ε are small. It remains to consider the term

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Convergence towards the asymptotic model

Direct computations give, if $a_{pq} = \int z\chi_p(z), \chi_q(z) dz$:

$$f(\tau)y = 2B \sum_{n\geq 0} \sum_{p\geq 0} a_{pn} e^{-it(E_p - E_n)} \partial_x y_p \chi_n.$$

Key fact : we have that $\forall p \geq 0,\ a_{pp} = \int z|\chi_p(z)|^2 dz = 0$ because V_c is even. This is crucial to avoid dangerous terms of order τ (which will be $1/\varepsilon^2$). Hence

$$f(\tau)y = 2B \sum_{n\geq 0} \sum_{p\neq n} a_{pn} e^{-it(E_p - E_n)} \partial_x y_p \chi_n$$

and

$$F(\tau)y = 2B \sum_{n\geq 0} \sum_{m\geq 0} a_{mn} \frac{e^{-it(E_m - E_n)} - 1}{E_m - E_n} \partial_x y_m \chi_n.$$

$$F(t)f(t)y = -4iB^2 \sum_{m\geq 0} \sum_{n\neq m} \sum_{r\neq m} a_{mn} a_{mr} \frac{e^{-it(E_m - E_n)} - 1}{E_m - E_n} e^{-it(E_r - E_m)} \partial_x^2 y_r \chi_n$$

Convergence towards the asymptotic model

We need to compute the limit :

$$\lim_{\varepsilon \to 0} \int_0^t F(\frac{s}{\varepsilon^2}) f(\frac{s}{\varepsilon^2}) y(s) ds.$$

Each term of the series is of the form :

$$\int_0^t e^{-i\frac{s}{\varepsilon^2}(E_m - E_n)} y_r(s) ds \quad \text{or} \int_0^t e^{-i\frac{s}{\varepsilon^2}(E_r - E_n)} y_r(s) ds.$$

The arguments for the convergence :

1. The terms with $m \neq n$ and $n \neq r$ are small if we have some regularity in time for $y(s)$. It remains only the terms $n = r \neq m$.
2. Small denominators $E_m - E_n$ seem to appear. But, the assumption $|E_{p+1} - E_p| > Cp^{-n_0}$ and the regularity in space of the function $y(s)$ garantee the summability.

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Scale of adapted functional spaces

The natural scale of functional spaces adapted to the positive self-adjoint operators $-\Delta_{x,y}$ and $H_z = -\partial_z^2 + z^2 + V_c(z)$ is the Sobolev B_m scale defined for $m \geq 0$ by

$$B^m := \left\{ u \in L^2(\mathbb{R}^3),\ \Delta_{x,y}^{m/2} u \in L^2(\mathbb{R}^3),\ H_z^{m/2} u \in L^2(\mathbb{R}^3) \right\}.$$

They form a scale of Hilbert spaces for the following norm :

$$\|u\|_{B^m}^2 := \|u\|_{L^2(\mathbb{R}^3)}^2 + \|(-\Delta_{x,y})^{m/2} u\|_{L^2(\mathbb{R}^3)}^2 + \|H_z^{m/2} u\|_{L^2(\mathbb{R}^3)}^2$$

Using Weyl-Hörmander pseudodifferential calculus, on can prove that it is equivalent to the norm

$$\|u\|_{B^m}^2 \sim \|u\|_{H^m(\mathbb{R}^3)}^2 + \|(V_c(z) + z^2)^{m/2} u\|_{L^2(\mathbb{R}^3)}.$$

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1. Introduction
   - The singularly perturbed problem
2. Heuristics and main result
   - Harmonic confinement
   - The strategy in the general case
   - The case $B = 0$
   - On the way to the asymptotic model
   - Main theorem
3. Second order averaging
4. The nonlinear analysis : main tools
   - Adapted functional framework
   - Asymptotics for the Poisson kernel
   - A priori local-in-time estimates
5. Towards a global in time result

In order to justify the approximation of the initial system by the system where the Poisson kernel V^ε is replaced by W, we need to precise the asymptotic behavior of the Poisson kernel.

We prove the following result. Consider $\psi \in B^2$ and define the potentials

$$V^\varepsilon(x,z) =,$$

$$W(x,z) = \frac{1}{4\pi\sqrt{x^2+y^2}} * \left(\int_{\mathbb{R}} |\psi(x,z')|^2 dz'\right).$$

Then there exists $\eta < 1$ such that :

$$\|V^\varepsilon \psi - W\psi\|_{B^1} \leq C\varepsilon^{1-\eta}\|\psi\|_{B^2}^3$$

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B^m estimates

To justify the proof of second order averaging, we use a regularization procedure. In that view, we need a local-in-time B^m estimate for m arbitrary large and for regularized initial data ψ_0^ε in B^m.

To this aim, we need in a crucial way the following tame estimate for the Poisson nonlinearity :.

tame estimate

For all $m \in \mathbb{N}^*$, there exists $C_m > 0$ such that, for all $\psi \in B^m$,

$$\left\| \left(\frac{1}{4\pi\sqrt{x^2+y^2+\varepsilon^2 z^2}} * |\psi|^2 \right) \psi \right\|_{B^m} \leq C_m \|\psi\|_{B^1}^2 \|\psi\|_{B^m}$$

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Consider the limit system. In order to get global in time estimates, we deduce from the energy estimates :

$$\sum_{n\geq 0} \alpha_n \|\partial_x \varphi_n\|^2_{L^2(\mathbb{R}^2)} + \|\partial_y \varphi_n\|^2_{L^2(\mathbb{R}^2)} \leq C$$

where C does not depend on ε.

In the case of harmonic confinement, i.e $V_c(z) = \alpha^2 z^2$, then

$$\alpha_n = \frac{\alpha^2}{\alpha^2 + B^2}.$$

What is remarkable here is :

1. The fact that α_n does not depend on n : the effective mass is the same for any energy level.
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