NESS as adiabatic limit on the potential bias

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Based on work in collaboration with

Horia Cornean, Pierre Duclos and Gheorghe Nenciu

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In a a previous paper:
H. Cornean, P. Duclos, Gh. Nenciu, R. Purice:
Adiabatically switched-on electrical bias and the LandauerBttiker formula, **Journal of Mathematical Physics 49** (2008), 20 pp.

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we have studied the linear response approximation for the electric curent appearing in a system composed of two conductors communicating through a 'small' sample, when a potential difference is applied adiabatically on the two conductors.

Introduction

Now, our problem is to prove the existence of a stationary limit state for the same problem.

Plan of the talk

- The System
- The Adiabatic Limit
- Proof of the Main Result

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where:

1. $\mathcal{I}_- := (-\infty, -a)$ and $\mathcal{I}_+ := (a, \infty)$ for some $a > 0$.
2. $\mathfrak{D} \subset \mathbb{R}^d$ is a bounded open set awith regular boundary $\partial\mathfrak{D}$,
3. $\mathcal{C} \subset \mathbb{R}^{d+1}$ is bounded and satisfies: $[-a, a] \times \mathfrak{D} \subset \mathcal{C}$,

$$[\{-a\} \times \mathfrak{D} \cup \{a\} \times \mathfrak{D}] \subset \partial\mathcal{C},$$

$$\boldsymbol{\Sigma} := \left[\mathcal{I}_- \times (\partial\mathfrak{D})\right] \cup \left[\partial\mathcal{C} \setminus (\{-a\} \times \mathfrak{D} \cup \{a\} \times \mathfrak{D})\right] \cup \left[\mathcal{I}_+ \times (\partial\mathfrak{D})\right]$$

is a regular surface in $\mathbb{R}^{d+1}$.

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We shall suppose that $\mathfrak{H}_C \geq 0$ (by just adding a constant term)

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- We denote by $R(z)$ the rezolvent of H.

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We shall suppose that all the iterated commutators of the form

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We denoted by Q_1 the operator of multiplication with the variable $x \in \mathbb{R}$ on $\mathcal{H}$ and by $P_1 := -i\partial_x$

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The time-dependent Hamiltonian

$$K_\eta(t) := H + V_\eta(t)$$

with domain

$$\mathbb{H}_D(\mathcal{L}) := H_0^1(\mathcal{L}) \cap H^2(\mathcal{L})$$

.

The non-homogenous evolution

For $-\infty < s \leq t \leq 0$, the unitary propagator $W_\eta(t,s)$ solution of the Cauchy problem:

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For any $\eta > 0$ the family $\{K_\eta(t)\}_{t\in\mathbb{R}}$ are self-adjoint operators in $\mathcal{H}$, having a common domain equal to $\mathbb{H}_D(\boldsymbol{\mathcal{L}}) \otimes \mathcal{K}$ and depending differentiable on $t \in \mathbb{R}$ with a bounded self-adjoint norm derivative

$$\partial_t K_\eta(t) = \eta\,\chi(\eta t) V.$$

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Thus it is described by a quasi-free state having as two-point function the usual Fermi-Dirac density at temperature T and chemical potential μ:

$$\rho(E) := \frac{1}{1 + e^{(E-\mu)/kT}}$$

applied to the total Hamiltonian $H = \big(-\Delta_D \big) \otimes 1 + \Pi_0 \mathfrak{H}_C \Pi_0$.

Initial state at $t = -\infty$: $\rho(H)$.

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Proposition

The following limit exists

$$\Omega_\eta(t) := s - \lim_{s \searrow -\infty} \Omega_\eta(t,s).$$

but, not uniformly with respect to η.

Proof of the Proposition:
Let us write the equation in integral form:

$$\Omega_\eta(t,s) \;=\; 1 + i\int_s^t \chi(\eta r)\Omega_\eta(t,r)e^{i(r-t)H}V(Q)e^{-i(r-t)H}\,dr$$

so that

$$\|\Omega_\eta(t,s_1) \;-\; \Omega_\eta(t,s_2)\| \;\le\; \int_{s_2}^{s_1} \chi(\eta r)\|V(Q)\|\,dr$$

verifying thus the Cauchy criterion for convergence with respect to the uniform topology on $\mathbb{B}[\mathcal{H}]$ due to the integrability of χ.

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This trick will allow us to compare in a more precise way
the asymptotic evolution $W_\eta(t, s)$
with the one associated to the Hamiltonian H.

The Adiabatic Limit

The Decoupled System

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- $\overset{\circ}{\Delta}_D : \overset{\circ}{\mathbb{H}}_D(\boldsymbol{\mathcal{L}}) \to L^2(\boldsymbol{\mathcal{L}})$ the self-adjoint Laplace operator with Dirichlet conditions on $\partial\boldsymbol{\mathcal{L}} \cup \boldsymbol{\mathcal{D}}_- \cup \boldsymbol{\mathcal{D}}_+$;

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We can write $\overset{\circ}{\Delta}_{D,\pm} = \mathfrak{l}_\pm \otimes 1 + 1 \otimes \mathfrak{L}_{\mathcal{D}}$ with:

- $\mathfrak{L}_{\mathcal{D}}$ the Laplacean on the bounded domain $\mathcal{D} \subset \mathbb{R}^d$ with Dirichlet conditions on the boundary $\partial\mathcal{D}$
- $\mathfrak{l}_\pm$ the operator of second derivative on $\mathcal{I}_\pm$ with Dirichlet condition at $\pm a$.

The Decoupled System

The decoupled Hamiltonian

$$\overset{\circ}{H} := \big(-\overset{\circ}{\Delta}_D\big) \otimes 1 + \Pi_0 \mathfrak{H}_C \Pi_0 : \quad \overset{\circ}{\mathbb{H}}_D(\boldsymbol{\mathcal{L}}) \otimes \mathcal{K} \quad \longrightarrow \quad \mathcal{H}$$

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The decoupled non-homogeneous evolution

$\mathring{W}_\eta(t,s)$ defined as the solution of the following Cauchy problem:

$$\begin{cases} -i\partial_t \mathring{W}_\eta(t,s) = -\mathring{K}_\eta(t)\mathring{W}_\eta(t,s) \\ \mathring{W}_\eta(s,s) = 1 \end{cases} .$$

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- All the above operators commute with $\Pi_\pm$ and thus with V.
- We have the formula $\overset{\circ}{W}_\eta(t,s) =$

$$= e^{-i(t-s)\overset{\circ}{H}} \left[1 + \Pi_- \left(e^{iv_- \int_s^t \chi(\eta u) du} \right) + \Pi_+ \left(e^{iv_+ \int_s^t \chi(\eta u) du} \right) \right]$$

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Hypothesis

$$\sigma_{pp}(K_1) = \emptyset.$$

The Main Result

Theorem

1. The limit $\rho_\eta(t) := \lim_{s \searrow -\infty} \rho_\eta(t,s)$ exists for any $t \leq 0$, in the strong operator topology on $\mathbb{B}(\mathcal{H})$, uniformly with respect to $\eta > 0$.
2. The wave operator Ξ_- associated to the pair $\{\mathring{K}_1, K_1\}$ exists and is complete.
3. The limit $\lim_{\eta \searrow 0} \rho_\eta(t)$ exists in the strong operator topology on $\mathbb{B}(\mathcal{H})$ and we have the equality

$$s - \lim_{\eta \searrow 0} \rho_\eta(t) = (\Xi_-)\rho(\mathring{H})(\Xi_-)^*,$$

so that the 'asymptotic state' is stationary.

Proof of the main result

Proof

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- Using the decoupled evolution we may write

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$$= W_\eta(t,s)\mathring{W}_\eta(t,s)^* \times$$
$$\times \left[1 + \Pi_-\left(e^{iv_- \int_s^t \chi(\eta u)du}\right) + \Pi_+\left(e^{iv_+ \int_s^t \chi(\eta u)du}\right)\right] \times$$
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$$\times e^{-i(t-s)\overset{\circ}{H}} e^{i(t-s)H}$$

- We recall that we know that the above limit exists (even for the uniform topology on $\mathbb{B}(\mathcal{H})$)
but not uniformly with respect to $\eta > 0$.

Proposition A - *The wave operator ω_-*

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Corollary

With the above notations we have ($\Pi = \Pi_\pm$ or Π_0)

$$s - \lim_{s \searrow -\infty} \left[W_\eta(t,s)\overset{\circ}{W}_\eta(t,s)^* \Pi e^{-i(t-s)\overset{\circ}{H}} e^{i(t-s)H} - W_\eta(t,s)\overset{\circ}{W}_\eta(t,s)^* \Pi E_{ac}(\overset{\circ}{H})\omega_- \right] = 0.$$

Proposition B - *The wave operator $\Xi_\eta(t)$*

The following limit exists with respect to the strong operator topology on $\mathbb{B}(\mathcal{H})$
and uniformly with respect to $\eta > 0$:

$$s - \lim_{s \searrow -\infty} W_\eta(t,s)\overset{\circ}{W}_\eta(t,s)^* E_{ac}(\overset{\circ}{H}) =: \Xi_\eta(t).$$

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$$s - \lim_{s \searrow -\infty} \left[W_\eta(t,s) \mathring{W}_\eta(t,s)^* \Pi e^{-i(t-s)\mathring{H}} e^{i(t-s)H} - \Xi_\eta(t) \Pi E_{ac}(\mathring{H}) \omega_- \right] = 0.$$

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Corollary

we may conclude that ($\Pi = \Pi_\pm$ or Π_0)

$$s - \lim_{s \searrow -\infty} \left[e^{-i(t-s)H} e^{i(t-s)\overset{\circ}{H}} \Pi W_\eta(t,s) W_\eta(t,s)^* - \omega_-^* E_{ac}(\overset{\circ}{H}) \Pi \Xi_\eta(t)^* \right] = 0.$$

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Conclusion 1

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- We have the equality

$$\rho_\eta(t) = \Xi_\eta(t)\rho(\overset{\circ}{H})\Xi_\eta(t)^*.$$

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Proposition E - *The adiabatic limit*

The following limits exist with respect to the strong operator topology on $\mathbb{B}(\mathcal{H})$ and we have the equalities:

$$s - \lim_{\eta \searrow 0} \Xi_\eta(t) E_{ac}(\mathring{H}) = \Xi_-,$$

$$s - \lim_{\eta \searrow 0} \Xi_\eta(t)^* = \Xi_-^*.$$

Proof - Some remarks on $\mathring{H}$

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- $$\sigma_{pp}(\overset{\circ}{H}) = \sigma_{pp}(\Pi_0 \overset{\circ}{H} \Pi_0) = \sigma(\Pi_0 \overset{\circ}{H} \Pi_0) \subset \mathbb{R}_+,$$
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  with $\pi_\pm : L^2(\mathbb{R}) \to L^2(\mathcal{I}_\pm)$ the usual orthogonal projections.
- the integral kernel of the rezolvent $\left(\mathfrak{l}_\pm - z)\right)^{-1}$ has exponential decay like $e^{\mp\alpha(z)x}$ for some $\alpha(z) > 0$.

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- exponential decay
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- application of Kuroda-Birman results.

Proof of Proposition A - Exponential decay

For any $\alpha < \alpha(z)$ let $\Psi_\alpha \in C^\infty(\mathbb{R})$ be such that

$$\Psi_\alpha(x) \geq 1, \ \forall x \in \mathbb{R}; \quad \Psi_\alpha(x) = e^{\pm\alpha x}, \ \forall x \in \mathcal{I}_\pm,$$

$$|(\partial\Psi_\alpha)(x)| \leq \alpha, \ |(\partial^s\Psi_\alpha)(x)| \leq C, \forall s \geq 2, \ \forall x \in \mathbb{R}.$$

Then $\Psi_\alpha(x)$ is invertible and $\Psi_\alpha^{-1} \in L^k(\mathbb{R})$ for any $k \geq 1$.

Lemma

For any $\alpha < \alpha(z)$

$$\Psi_\alpha(Q_1)\mathring{H}\Psi_\alpha(Q_1)^{-1} = \mathring{H} + \mathring{T}_\alpha; \ \Psi_\alpha(Q_1)H\Psi_\alpha(Q_1)^{-1} = H + T_\alpha$$

where for any $k \geq 1$, $\mathring{T}_\alpha$ is a bounded operator
$H^k(\mathcal{L}_-) \oplus H^k(\mathcal{C}) \oplus H^k(\mathcal{L}_+) \longrightarrow H^{k-1}(\mathcal{L}_-) \oplus H^{k-1}(\mathcal{C}) \oplus H^{k-1}(\mathcal{L}_+)$
and T_α is a bounded operator $H^k(\mathcal{L}) \longrightarrow H^{k-1}(\mathcal{L})$.

Proof of Proposition A - A Hilbert-Schmidt property

Thus the range of $\overset{\circ}{R}(z)^k$ and of $\Psi_\alpha(Q_1)\overset{\circ}{R}(z)^k\Psi_\alpha(Q_1)^{-1}$ are contained in $H^{2k}(\mathcal{L}_-) \oplus H^{2k}(\mathcal{C}) \oplus H^{2k}(\mathcal{L}_+)$
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Using the usual Sobolev embeding theorems we get:

Lemma

There exists $k_d \in \mathbb{N}$ depending on the dimension d such that for any $z \in \mathbb{C} \setminus [0,\infty)$, any $k \geq k_d$, any $\alpha < \alpha(z)$ and for any measurable function $w \in L^2(\mathbb{R})$, we have that

- $w(Q_1)\overset{\circ}{R}(z)^k$ and $w(Q_1)\Psi_\alpha(Q_1)\overset{\circ}{R}(z)^k\Psi_\alpha(Q_1)^{-1}$ are Hilbert-Schmidt operators on $\mathcal{H}$;
- $w(Q_1)R(z)^k$ and $w(Q_1)\Psi_\alpha(Q_1)R(z)^k\Psi_\alpha(Q_1)^{-1}$ are Hilbert-Schmidt operators on $\mathcal{H}$.

Proof of Proposition A - Difference of the rezolvents

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$$K_{0,\kappa} := \big(-\Delta_D\big) \otimes 1 + \Pi_0 \mathfrak{H}_C \Pi_0 : C_0^\infty(\overset{\circ}{\mathcal{L}}_- \cup \overset{\circ}{\mathcal{C}} \cup \overset{\circ}{\mathcal{L}}_+) \to \mathcal{H}.$$

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Let K_0^* be its adjoint. It extends both operators H and $\overset{\circ}{H}$ so that

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Let K_0^* be its adjoint. It extends both operators H and $\overset{\circ}{H}$ so that

$$\left[R(z) - \overset{\circ}{R}(z) \right] \mathcal{H} \subset \mathbb{K}er \left(K_0^* - z \right).$$

For elements $u \in \mathbb{K}er \left(K_0^* - z \right)$ the distribution $K_0^* u - zu$ has support in the border $\mathcal{D}_- \cup \mathcal{D}_+$ and thus on $\mathcal{L}_- \cup \mathcal{L}_+$ it satisfies:

$$\overset{\circ}{\Delta}_{D,\pm} u = -(z - v_\pm) u$$

with boundary condition $u_\pm|_{\mathcal{I}_\pm \times \partial \mathcal{D}} = 0.$

We deduce

Proposition

For any fixed $z \in \mathbb{C} \setminus [0, \infty)$ there exists $\gamma_0(z) > 0$ such that for $0 < \gamma_\pm \leq \gamma_0(z)$ we have:

$$\left\| e^{\pm\gamma_\pm Q_1} \Pi_\pm \big(R(z) - \mathring{R}(z) \big) \right\| \leq c,$$

Proof of Proposition A - Difference of the rezolvents

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Corollary

For any fixed $z \in \mathbb{C} \setminus [0, \infty)$, for $\alpha < \gamma_0(z)$, let μ_α be a strictly positive smooth function such that $\mu_\alpha(x) \geq 1$ for any $x \in \mathbb{R}$ and $\mu_\alpha(x) = e^{\pm \alpha x}$ for $\pm x \geq 2a$. Then

$$\left\| \mu_\alpha(Q_1) \left(R(z) - \overset{\circ}{R}(z) \right) \right\| \leq c,$$

Proof of Proposition A - End

Proposition

There exists $n_d \in \mathbb{N}$ depending on the dimension d such that for $z \in \mathbb{C} \setminus [0, \infty)$ and $n \geq n_d$ we have $\left[R(z)^n - \overset{\circ}{R}(z)^n\right] \in \mathbb{B}_1(\mathcal{H})$.

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Proof: For any $p \in \mathbb{N}$:

$$R(z)^p - \mathring{R}(z)^p = \sum_{0 \leq j \leq p-1} R(z)^j \left(R(z) - \mathring{R}(z)\right) \mathring{R}(z)^{p-1-j}.$$

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If $p \geq 2k_d + 1$, then either $j \geq k_d$ or $p - j - 1 \geq k_d$ so that each term is Hilbert-Schmidt by writing:

$$R(z)^j \left(R(z) - \overset{\circ}{R}(z)\right) = R(z)^j \mu_\alpha(Q_1)^{-1} \mu_\alpha(Q_1) \left(R(z) - \overset{\circ}{R}(z)\right)$$

or

$$\left(R(z) - \overset{\circ}{R}(z)\right) \overset{\circ}{R}(z)^{p-j-1} = \left(R(z) - \overset{\circ}{R}(z)\right) \mu_\alpha(Q_1) \mu_\alpha(Q_1)^{-1} \overset{\circ}{R}(z)^{p-j-1}.$$

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If $\beta > 0$ such that $\alpha + \beta < \min\{\alpha(z), \gamma_0(z)\}$ we conclude in a similar way that $\mu_\beta(Q_1)\left(R(z)^p - \mathring{R}(z)^p\right)$ is also Hilbert-Schmidt.

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Thus for $p \geq 2k_d + 1$ there exist a Hilbert-Schmidt operator $S_p(z)$ such that $R(z)^p = \overset{\circ}{R}(z)^p + S_p(z)$,
$\mu_\beta(Q_1)S_p(z)$ is also Hilbert-Schmidt for $\beta < \min\{\alpha(z), \gamma_0(z)\}$ and

$$R(z)^{2p} = \overset{\circ}{R}(z)^{2p} + S_p(z)\overset{\circ}{R}(z)^p + \overset{\circ}{R}(z)^p S_p(z) + S_p(z)^2.$$

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We can apply the usual Kato-Birman procedure.

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Proposition B - *The wave operator $\Xi_\eta(t)$*

The following limit exists with respect to the strong operator topology on $\mathbb{B}(\mathcal{H})$
and uniformly with respect to $\eta > 0$:

$$s - \lim_{s \searrow -\infty} W_\eta(t,s) \overset{\circ}{W}_\eta(t,s)^* E_{ac}(\overset{\circ}{H}) =: \Xi_\eta(t).$$

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We have evidently:

$$\sigma_{sc}(\mathring{K}_\kappa) = \emptyset, \quad \mathcal{H}_{ac}(\mathring{K}_\kappa) = \mathcal{H}_- \oplus \mathcal{H}_+, \quad \mathcal{H}_{pp}(\mathring{K}_\kappa) = \mathcal{H}_0, \quad \forall \kappa \in [0,1],$$

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$\sigma_{ac}(\overset{\circ}{H}) = [0,\infty)$ has the set of thresholds $\mathcal{T} = \sigma_{pp}(\mathfrak{L}_{\mathcal{D}})$.

For any $\delta > 0$ let $\mathcal{V}_\delta$ be the set of vectors $f \in \mathcal{H}_{ac}(\overset{\circ}{H})$ with compact spectral support with respect to $\overset{\circ}{H}$ at distance δ from all the thresholds.

Proof of Proposition B

If we denote by

$$\Psi_\eta(t,s) := W_\eta(t,s)\left[\left(K_\eta(s)+1\right)^{-1} - \left(\mathring{K}_\eta(s)+1\right)^{-1}\right]\mathring{W}_\eta(t,s)^*$$

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we have that

$$\Xi_\eta(t,s) =$$

$$= \left(K_\eta(t)+1\right)^{-1}\left(\overset{\circ}{K}_\eta(t)+1\right)^{-1} +$$

$$+ \int_t^s \Psi_\eta(t,u)\,du\left(\overset{\circ}{K}_\eta(s)+1\right)^2 - \Psi_\eta(t,s)\left(\overset{\circ}{K}_\eta(s)+1\right).$$

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For $f \in \mathcal{H}_- \cap \mathcal{V}_\delta$ of the form $f = v_1 \otimes w$ we have

$$\|\Psi_\eta(t,s)v\| = \left\| \left[\left(K_\eta(s)+1\right)^{-1} - \left(\mathring{K}_\eta(s)+1\right)^{-1} \right] \mathring{W}_\eta(t,s)^* v \right\| \leq$$

$$\leq \left\| e^{\gamma_- |Q_1|} \Pi_- \left(R_{\chi_\eta(s)} - \mathring{R}_{\chi_\eta(s)} \right) \right\| \left\| e^{-\gamma_- |Q_1|} \mathring{W}_\eta(t,s)^* v \right\| \leq$$

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But $e^{i(t-s)\mathring{H}} v = e^{i(t-s)\mathfrak{l}_-} v_1 \otimes e^{i(t-s)\mathfrak{L}_{\mathcal{D}}} w$

where $\left\| e^{i(t-s)\mathfrak{L}_{\mathcal{D}}} w \right\| = \|w\|$

Proof of Proposition B

For $t \geq t_0 > 0$ with $t_0\sqrt{\delta} \geq 2x$, for any $N \in \mathbb{N}$, by integration by parts:

$$\left|\left(e^{it\mathfrak{l}_-}v_1\right)(x)\right| = \left|\int_{|k|\geq\sqrt{\delta}} dk\, e^{i(tk^2+kx)}\widehat{v_1}(k)\right| \leq$$

$$\leq C_N(t\sqrt{\delta} - |x|)^{-N}\|\mathfrak{l}_-^{1+(N/2)}v_1\|_2.$$

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Thus, for $t \geq t_0 > 0$, and $N \in \mathbb{N}$ there exists $C_N > 0$ such that

$$\left\|e^{-\gamma_-|Q_1|} e^{it\mathring{H}} v\right\|_2^2 \leq$$

$$\leq C_N\|w\|_2^2 \left\{\int_{-\infty}^{-(t\sqrt{\delta})/2} dx\, e^{-2\gamma_-|x|}\left|\left(e^{it\mathfrak{l}_-} v_1\right)(x)\right|^2 + \int_{-(t\sqrt{\delta})/2}^{0} dx\, e^{-2\gamma_-|x|}\left|\left(e^{it\mathfrak{l}_-} v_1\right)(x)\right|^2\right\} \leq$$

$$\leq 2C_N t^{-2N}\|w\|_2^2\left\{\left(\sqrt{\delta}\right)^{-2N}\left\|< Q_1 >^{2+N} e^{-\gamma_-|Q_1|} e^{it\mathfrak{l}_-} v_1\right\|_2^2 + \left(\sqrt{\delta}/2\right)^{-2N}\|\mathfrak{l}_-^{1+(N/2)} v_1\|_2^2\right\} \leq$$

$$\leq C'_{N,\gamma_-} t^{-2N}\left(\sqrt{\delta}\right)^{-2N}\|(\mathfrak{l}_-^{1+(N/2)} \otimes 1)v\|_2^2.$$

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Conclusion

$$\exists\ \Xi_\eta(t) := s - \lim_{s \searrow -\infty} \Xi_\eta(t,s) =$$

$$= \left(K_\eta(t)+1\right)^{-1}\left(\overset{\circ}{K}_\eta(t)+1\right)^{-1} + \Phi_\eta(t,-\infty)\left(\overset{\circ}{H}+1\right)^2,$$

uniformly with respect to $\eta > 0$.

Proof of Proposition C

Proposition C - *The wave operator $\Xi_\eta(t)^*$*

- For any $\eta > 0$ the limit $s - \lim_{s \searrow -\infty} \overset{\circ}{W}_\eta(t,s) W_\eta(t,s)^*$, exists with respect to the strong operator topology on $\mathbb{B}(\mathcal{H})$, and its image is contained in $E_{ac}(\overset{\circ}{H})\mathcal{H}$.
- For any $\eta > 0$, $s - \lim_{s \searrow -\infty} \overset{\circ}{W}_\eta(t,s) W_\eta(t,s)^* = E_{ac}(\overset{\circ}{H})\Xi_\eta(t)^*$ that will be an isometry.

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We may write: $\overset{\circ}{W}_\eta(t,s) W_\eta(t,s)^* =$

$$= \overset{\circ}{W}_\eta(t,s) e^{i(t-s)\overset{\circ}{H}} e^{-i(t-s)\overset{\circ}{H}} e^{i(t-s)H} e^{-i(t-s)H} W_\eta(t,s)^* =$$

$$= \left[1 + \Pi_- \left(e^{iv_- \int_s^t \chi(\eta u) du} \right) + \Pi_+ \left(e^{iv_+ \int_s^t \chi(\eta u) du} \right) \right] \times$$

$$\times \left(e^{-i(t-s)\overset{\circ}{H}} e^{i(t-s)H} \right) \left(e^{-i(t-s)H} W_\eta(t,s)^* \right)$$

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Let us denote by $\Xi_\eta(t,s) := W_\eta(t,s)\mathring{W}_\eta(t,s)^* E_{ac}(\mathring{H})$.

Proof of Proposition C

Let us denote by $\Xi_\eta(t,s) := W_\eta(t,s)\overset{\circ}{W}_\eta(t,s)^* E_{ac}(\overset{\circ}{H})$.
Then from Proposition B we know that there exists

$$s-\lim_{s\searrow-\infty}\Xi_\eta(t,s) = \Xi_\eta(t)$$

and hence also

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Thus $\Theta_\eta(t) = \Xi_\eta(t)^*$,
which being the strong limit of unitary operators is an isometry.

Proposition D - *The wave operator Ξ_-*

1. For any $\kappa \in [0, 1]$ the following spectral projections coincide

$$E_{ac}(\mathring{K}_\kappa) = E_{ac}(\mathring{H}).$$

2. The following wave operators exist and are complete:

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Let us consider the second point.

Proof of Proposition D

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For $s > 1/2$ let us consider $< Q_1 >^{-s} R_1(z) < Q_1 >^{-s}$ that we shall compare with a 'quasi-decoupled' rezolvent.

Proof of Proposition D

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For $s > 1/2$ let us consider $< Q_1 >^{-s} R_1(z) < Q_1 >^{-s}$ that we shall compare with a 'quasi-decoupled' rezolvent.
We introduce a quadratic partition of the unity:

$$\chi_-^2 + \chi_0^2 + \chi_+^2 = 1, \quad \chi_\pm, \in C^\infty(\mathbb{R}),$$

$$\chi_\pm(x) = 1 \text{ for } \pm x > 2a, \quad \chi_\pm(x) = 0 \text{ for } \pm x < a$$

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Lemma 1

$$\sigma_{sc}(K_1) = \emptyset$$

For $s > 1/2$ let us consider $< Q_1 >^{-s} R_1(z) < Q_1 >^{-s}$ that we shall compare with a 'quasi-decoupled' rezolvent.
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and the operator: $\widetilde{R}_1(z) :=$

$$= \chi_-(Q_1)\overset{\circ}{R}_1(z)\chi_-(Q_1) + \chi_0(Q_1)S(z)\chi(Q_1) + \chi_+(Q_1)\overset{\circ}{R}_1(z)\chi_+(Q_1),$$

Proof of Proposition D

- $S(z)$ is the rezolvent of the operator $K_{1,L}$, that is just K_1 with Dirichlet boundary conditions on some $L > 2a$; it clearly has an analytic extension to the plane $\mathbb{C} \setminus \mathfrak{N}$ with $\mathfrak{N} \subset \mathbb{R}_+$ the discret set of eigenvalues of $K_{1,L}$.

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- On the range of $\chi_\pm(Q_1)$ the operators K_1 and $\overset{\circ}{K}_1$ coincide. On the range of $\chi_0(Q_1)$ the operators K_1 and $K_{1,L}$ coincide.
- Thus we can write $(K_1 - z)\widetilde{R}_1(z) = 1 + X(z)$ with $X(z)$ containing on the left side only commutators that have compact support in $x \in \mathbb{R}$.

In conclusion

$$< Q_1 >^{-s} R_1(z) < Q_1 >^{-s} = \\ = < Q_1 >^{-s} \widetilde{R}_1(z) < Q_1 >^{-s} [1 - < Q_1 >^{s} X(z)]^{-1}$$

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and we can use the analytic Frdholm alternative on any open set
$\{x + iy | x \in I, 0 < y < \delta\}$
for intervals $I \subset \mathbb{R}_+ \setminus (\{0\} \cup \mathfrak{N} \cup \sigma_{pp}(\boldsymbol{\mathcal{L}}_D)))$
and the continuity to the border of $< Q_1 >^{-s} \overset{\circ}{R}_1(z) < Q_1 >^{-s}$
(the limiting absorption principle for the Laplace operator)
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LAP for K_1

The Hamiltonian K_1 has no singular spectrum,
and its rezolvent verifies the estimation (for any $s > 1/2$)

$$\sup_{z \in \{x+iy | x \in I, 0<y<\delta\}} \left\| < Q_1 >^{-s} R_1(z) < Q_1 >^{-s} \right\| \leq C(I, \delta, s) < \infty.$$

Proof of Proposition D

- We remark that the perturbation V is still relatively bounded with respect to H with 0 relative bound but it is no longer relatively compact with respect to H and its commutator with H defined as a sesquilinear form on the domain $\mathbb{H}_D(\boldsymbol{\mathcal{L}}) \otimes \mathcal{K}$ of H is singular.

Proof of Proposition D

- We remark that the perturbation V is still relatively bounded with respect to H with 0 relative bound but it is no longer relatively compact with respect to H and its commutator with H defined as a sesquilinear form on the domain $\mathbb{H}_D(\boldsymbol{\mathcal{L}}) \otimes \mathcal{K}$ of H is singular.
- Nevertheless (V commuting with $\overset{\circ}{H}$),

  the exponential decay for $\overset{\circ}{R}_1(z)$,

  the Hilbert-Schmidt property similar to that of $\overset{\circ}{R}(z)$,
  and the exponential decay of the difference of the rezolvents, can stil be obtained by the same argument as in the proof of Proposition A.

Proof of Proposition D

- We shall consider a modified potential $\widetilde{V}(Q_1)$ with $\widetilde{V} \in C^\infty(\mathbb{R})$ and $V(x) = v_\pm$ for $x \pm x > (a/2)$

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- and the rezolvents $\widetilde{R}_1(z)$ and $\overset{\circ}{\widetilde{R}}_1(z)$ associated to the Hamiltonians $H + \widetilde{V}$ and $\overset{\circ}{H}$.

To this pair we can apply exactly the arguments used for the pair H and $\overset{\circ}{H}$ in order to get

Proposition

There exists $k'_d \in \mathbb{N}$ depending on the dimension d such that for any $z \in \mathbb{C} \setminus [0, \infty)$ and any $k \geq k'_d$ we have that $w(Q_1)\widetilde{R}_1(z)^k$, $w(Q_1)\overset{\circ}{\widetilde{R}}_1(z)^k$ and $w(Q_1)\Psi_\alpha(Q_1)\widetilde{R}_1(z)^k\Psi_\alpha(Q_1)^{-1}$, $w(Q_1)\Psi_\alpha(Q_1)\overset{\circ}{\widetilde{R}}_1(z)^k\Psi_\alpha(Q_1)^{-1}$ with $\alpha < \alpha(z)$, are Hilbert-Schmidt operators on $\mathcal{H}$ for any measurable function $w \in L^2(\mathbb{R})$.

Proof of Proposition D

The arguments used in the proof of Proposition A may be repeated identically to get

proposition

Let $z \in \mathbb{C} \setminus [0, \infty)$ and $\gamma_\pm \in \mathbb{R}_+ \setminus \{0\}$ be such that $0 < \gamma_\pm \leq \gamma_0(z)$, then we have:

$$\left\| e^{\pm\gamma_\pm Q_1} \Pi_\pm \left(\widetilde{R}_1(z) - \overset{\circ}{\widetilde{R}}_1(z) \right) \right\| \leq c,$$

$$\left\| \mu_\alpha(Q_1) \left(\widetilde{R}_1(z) - \overset{\circ}{\widetilde{R}}_1(z) \right) \right\| \leq c,$$

where for $\alpha < \gamma_0(z)$, μ_α is a strictly positive smooth function such that $\mu_\alpha(x) \geq 1$ for any $x \in \mathbb{R}$ and $\mu_\alpha(x) = e^{\pm\alpha x}$ for $\pm x \geq 2a$.

Proof of Proposition D

Thus we can use the Kuroda-Birman theory to get the existence and asymptotic completness of the wave operators for the pait of Hamiltonians $\{H + \widetilde{V}, \mathring{H} + \widetilde{V}\}$.

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Now we observe that $E_{ac}(\overset{\circ}{H} + V) = E_{ac}(\overset{\circ}{H} + \widetilde{V}) = E_{ac}(\overset{\circ}{H})$ and moreover

$$e^{it(\overset{\circ}{H} + \widetilde{V})} E_{ac}(\overset{\circ}{H}) = e^{it(\overset{\circ}{H} + V)} E_{ac}(\overset{\circ}{H})$$

so that we deduce the existence and asymptotic completness of the wave operators for the pair $\{H + \widetilde{V}, \overset{\circ}{H} + V\}$.

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Now using Proposition LAP-K1 and the fact that $K_1 - (H + \widetilde{V}(Q_1) = w(Q_1)$ a bounded function with compact support, we conclude that the wave operators for the pair $\{K_1, H + \widetilde{V}\}$ also exist and are complete.

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Puting these results together we get the existence and asymptotic completness of the wave operators for the pair $\{K_1, \mathring{K}_1\}$ and thus we end the proof of Proposition D.

Proposition E - *The adiabatic limit*

The following limits exist with respect to the strong operator topology on $\mathbb{B}(\mathcal{H})$ and we have the equalities:

$$s-\lim_{\eta\searrow 0}\Xi_\eta(t)E_{ac}(\mathring{H})=\Xi_-,$$
$$s-\lim_{\eta\searrow 0}\Xi_\eta(t)^*=\Xi_-^*.$$

We shall first consider the limit $\eta\searrow 0$ for the operators $\Xi_\eta(t)$.

Proof of Proposition E

All the estimations in the proof of Proposition B have been independent of the value of $\eta > 0$ and we deduce that the limit $s - \lim_{s \searrow -\infty} \Phi_\eta(t, s)$ is uniform in $\eta > 0$.

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Repeating the arguments in the proof of Proposition B we denote by

$$\Xi_1(t) = e^{-itK_1} e^{it\overset{\circ}{K}_1},$$

and can write

$$\Xi_1(t) = \left(K_1 + 1\right)^{-1} \left(\overset{\circ}{K}_1 + 1\right) u + \int_0^t \Psi(s)\, ds \left(\overset{\circ}{K}_1 + 1\right)^2 u - \Psi(t) \left(\overset{\circ}{K}_1 + 1\right).$$

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With similar notations:

$$\Psi(t) := e^{-itK_1}\left[(K_1+1)^{-1} - (\overset{\circ}{K}_1+1)^{-1}\right] e^{it\overset{\circ}{K}_1}$$

$$\Phi(t) := e^{-itK_1}(K_1+1)^{-1}(\overset{\circ}{K}_1+1)^{-1} e^{it\overset{\circ}{K}_1}.$$

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We repeat the same arguments of the proof of Proposition B to prove that the right-hand side has a strong limit for $t \searrow -\infty$, while due to Proposition D the left-hand side has strong limit Ξ_-. Starting from the usual approximation of the non-homogeneous propagator by products of unitary groups associated to Hamiltonians at sets of fixed points during the evolution it is easy to see that for any fixed $s \leq t \leq 0$ we have

$$s - \lim_{\eta \searrow 0} \Psi_\eta(t, s) = \Psi(t - s)$$

and this finishes the first part of the Proposition E, concerning the strong limit of $\Xi_\eta(t)$ for $\eta \searrow 0$.

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- We have just seen that $s - \lim_{\eta \searrow 0} \Xi_\eta(t) = \Xi_-$.
- Thus $w - \lim_{\eta \searrow 0} \Xi_\eta(t)^* = \Xi_-^*$.

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- The completness of the wave operator Ξ_- (Proposition B) implies $\Xi_-^* = s - \lim_{\eta \searrow 0} e^{-is\mathring{K}_1} e^{isK_1}$

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Now, if the weak limit of a family of operators with norms bounded by 1 is an isometry, then for any $f \in \mathcal{H}$

$$\left\| \left[\Xi_-^* - \Xi_\eta(t)^* \right] f \right\|_{\mathcal{H}} \leq 2\|f\|_{\mathcal{H}} - 2\Re\left(\left\langle \Xi_-^* f, \Xi_\eta(t)^* f \right\rangle \right) \to 0$$

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Thus we have strong convergence of $\Xi_\eta(t)^*$ to Ξ_-^* when $\eta \searrow 0$.