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Mean-Field Interacting Boson Random Point Processes in Weak (Harmonic) Traps

Valentin A. ZAGREBNOV

Centre de Physique Théorique - Luminy - UMR 6207

- **Random Fermion and Boson Processes =**
Random (Quantum) Point Fields or Determinantal and Permanental Processes $\subseteq$ *Cox Processes* (1955).
- Condensation in **"Weak Harmonic Traps"**.

1. Random Point Processes (RPP)

(a) N.B. Keep in mind just: $\Lambda \subseteq \mathbb{R}^d$ is an open subset, ν is the Lebesgue measure, $K(x,y)$ is a kernel of non-negative self-adjoint locally Tr-class operator on $L^2(\Lambda)$, and $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space with $\omega \in \Omega$.

(b) Definition: A *random point processes* in a locally compact (*Polish*) space Λ is a random integer-valued positive Radon measure μ^ω on Λ. For a *simple* point process the measure μ^ω assigns a.-s. $\mu^\omega(x) \leq 1$ for any $x \in \Lambda$ and $\mu^\omega(D) := N_D^\omega$, the number of points that fall in D for *locally-finite* point configurations $Q(\Lambda)$.

(c) Example:(*The Poisson random point field*, intensity $\lambda \geq 0$)
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let the random counting measure $\{\mu_\lambda^\omega(dx)\}_{\omega\in\Omega}$ be such that for $n \in N \cup \{0\}$ and any $k \geq 1$:

$$\mathbb{P}\{\omega \in \Omega : \mu_\lambda^\omega(D) = n\} = \frac{(\lambda\nu(D))^n}{n!}\, e^{-\lambda\nu(D)}$$

$$\mathbb{E}(\mu_\lambda^\omega(D_1)\ldots\mu_\lambda^\omega(D_k) = \lambda^k \nu(D_1)\ldots\nu(D_k)(= \mathbb{E}\mu_\lambda^\omega(D_1)\ldots\mathbb{E}\mu_\lambda^\omega(D_k)).$$

(d) Definition: For any family of mutually disjoint subsets $\{D_n \subset \Lambda\}_{n\geq 1}$ the correlation functions (*joint intensities*) of the RPP μ^ω are defined by the densities $\{\rho_n : \Lambda^n \mapsto \mathbb{R}^1_+\}_{n\geq 1}$ with respect to the measure ν:

$$\mathbb{E}(\mu^\omega(D_1)\ldots\mu^\omega(D_n)) = \int_{D_1\times\ldots\times D_n} \nu(dx_1)\ldots\nu(dx_n)\ \rho_n(x_1,\ldots,x_n)$$

(e) Definition: A RPP is called determinantal with (a *locally* Tr-class) kernel K is it is *simple* and its correlation functions:

$$\rho_k(x_1,\ldots,x_k) = \det\|K(x_i,x_j)\|_{1\leq i,j\leq n}$$

for any $k \geq 1$ and $x_1,\ldots,x_n \in \Lambda$.

(f) Definition: A RPP is called permanental with (a *locally* Tr-class) kernel K is it is *simple* and its correlation functions:

$$\rho_n(x_1,\ldots,x_n) = \mathrm{per}\|K(x_i,x_j)\|_{1\leq i,j\leq n}$$

for any $k \geq 1$ and $x_1,\ldots,x_n \in \Lambda$.

N.B. $\det_\alpha A := \sum_{\sigma\in\mathfrak{S}_n} \alpha^{n-c(\sigma)} \prod_{1\leq i\leq n} a_{i\sigma(i)}$, $\alpha = \pm 1 \Leftrightarrow \mathrm{per}/\det$

2. Fermion/Boson Random Point Processes

2.1 Quantum (Statistical) Mechanics: Fermions

- Let $\mathfrak{H}_L := L^2(\Lambda_L), \Lambda_L = [-L/2, L/2]^d$ and $\Delta_{L,p}$ be Laplacian with *periodic* boundary conditions on $\partial\Lambda_L$, i.e.

$$\mathrm{spec}(-\Delta_{L,p}) = \{\varepsilon(k) = (2\pi/L)^2\|k\|^2 : k \in \mathbb{Z}^d\}.$$

The Gibbs semigroup kernel has the form:

$$G_L(x,y) := (e^{\beta\Delta_L})(x,y) = \sum_{k\in\mathbb{Z}^d} e^{-\beta\varepsilon(k)}\phi_{k,L}(x)\overline{\phi_{k,L}(y)} =$$

$$\sum_{k\in\mathbb{Z}^d} G(x, y + kL),$$

where

$$G(x,y) = \lim_{L\to\infty} G_L(x,y) = (4\pi\beta)^{-d/2}\exp(-\|x-y\|^2/4\beta).$$

- **Remark:** Any n-particle free-fermion wave function is the *Slater* determinant:

$$\Psi_{k_1,\dots,k_n}(x_1,\dots,x_n) = \frac{1}{\sqrt{n!}} \det \|\phi_{k_i,L}(x_j)\|_{1\le i,j\le n}$$

- The corresponding n-point free-fermion joint *probability distribution* density: $p_{n,L}(x_1,\dots,x_n) := |\Psi_{k_1,\dots,k_n}(x_1,\dots,x_n)|^2$, or

$$p_{n,L}(x_1,\dots,x_n) = \frac{1}{n!} \det \|\phi_{k_i,L}(x_j)\|_{1\le i,j\le n} \, \overline{\det \|\phi_{k_i,L}(x_j)\|_{1\le i,j\le n}}$$

- Since $\det A \det B = \det A\,B$ one gets:

$$p_{n,L}(x_1,\dots,x_n) = \frac{1}{n!} \det \|K_{n,L}(x_i,x_j)\|_{1\le i,j\le n}\,,$$

where $K_{n,L}(x,y) = \sum_{1\le i\le n} \phi_{k_i,L}(x)\overline{\phi_{k_i,L}(y)}$ is the kernel of orthogonal projection on the Env$\{\phi_{k_1,L},\dots,\phi_{k_n,L}\}$.

- Since the k-point *marginal* correlation functions are

$$p_{n,L}^{(k)}(x_1,\ldots,x_n) := \frac{n!}{(n-k)!}\int p_{n,L}(x_1,\ldots,x_n)dx_{k+1},\ldots,dx_n = \det\|K_{n,L}(x_i,x_j)\|_{1\le i,j\le k}\ ,$$

the determinantal RPP $\mu_{n,L}^{\omega,F}$ generated by the joint probability distribution density $p_{n,L}$ is correctly defined for n free fermions in the cube Λ_L.

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cubic box $\Lambda = L \times L \times L$, $|\Lambda| = V$ with *periodic boundary conditions* for single-particle Hamiltonian $t_L := (-\Delta/2)_{\Lambda_L,P}$.

• **Generalized BEC** - take a *prism* $\Lambda = L_1 \times L_2 \times L_3$ of the *same* volume with sides of length $L_j = V^{\alpha_j}$, $j = 1, 2, 3$, such that $\alpha_1 \geq \alpha_2 \geq \alpha_3 > 0$ and $\alpha_1 + \alpha_2 + \alpha_3 = 1$, with the *Periodic boundary conditions* for single-particle Hamiltonian $t_L := (-\Delta/2)_{\Lambda_L,P}$ on the boundary of this prism (**Casimir boxes** (1968)).

• **Proposition 1:** Generalized BEC $\neq$ Conventional BEC.

Rewrite the *finite-volume* equation for a **fixed** *total* particle density ρ (*grand-canonical* ensemble $(\beta \geq 0, \mu < 0)$) in the form:

$$\rho = \frac{1}{V}\frac{1}{e^{-\beta\mu}-1} + \frac{1}{V}\sum_{k\in\{\Lambda^*:n_1\neq 0,n_2=n_3=0\}}\sum \frac{1}{e^{\beta(\varepsilon_k-\mu)}-1}$$
$$+\frac{1}{V}\sum_{k\in\{\Lambda^*:n_j\neq 0,j=2\text{ or }3\}}\sum \frac{1}{e^{\beta(\varepsilon_k-\mu)}-1}.$$

• Here the *dual space* Λ^* of momenta w.r.s. to the periodic boundary conditions is:

$$\Lambda^* := \left\{k_j := \frac{2\pi}{V^{\alpha_j}} n_j : n_j \in \mathbb{Z}\right\}_{j=1}^{d=3} \quad \text{and} \quad \varepsilon_k := \sum_{j=1}^{d} \frac{k_j^2}{2}$$

• **Cube**: $\alpha_1 = \alpha_2 = \alpha_3 = 1/3,\ V = L^3$. If $\mu < 0$ and $\Lambda \nearrow \mathbb{R}^3$:

$$\begin{aligned}\rho &= \lim_{\Lambda} \rho_\Lambda(\beta,\mu) := \lim_{\Lambda} \frac{1}{V}\{\frac{1}{e^{-\beta\mu}-1} + \sum_{k\in\{\Lambda^*/0\}} \frac{1}{e^{\beta(\varepsilon_k-\mu)}-1}\} \\ &= \lim_{L\to\infty} \frac{1}{L^3} \sum_{n_j\in\mathbb{Z}\backslash 0} \left\{e^{\beta(\sum_{j=1}^{d}(2\pi n_j V^{-\alpha_j})^2/2-\mu)} - 1\right\}^{-1} \\ &= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} d^3k \left\{e^{\beta(k^2/2-\mu)} - 1\right\}^{-1} =: \mathfrak{I}(\beta,\mu).\end{aligned}$$

• For $d > 2$ the *free* Bose-gas *critical density* $\rho_c(\beta) := \lim_{\mu\nearrow 0} \mathfrak{I}(\beta,\mu)$ is finite: then **if** $\rho > \rho_c(\beta) \Rightarrow$ **BEC** at $k = 0$, $\rho_0(\beta) := \rho - \rho_c(\beta)$.

- **Saturation Mechanism** (*conventional condensation*):

Let $\mu_\Lambda(\beta,\rho)$ be solution of the equation

$$\rho = \rho_\Lambda(\beta,\mu) \quad \Leftrightarrow \quad \rho \equiv \rho_\Lambda(\beta, \mu_\Lambda(\beta,\rho)).$$

Then either:

- $\lim_\Lambda \ \mu_\Lambda(\beta, \rho < \rho_c(\beta)) = \mu_\Lambda(\beta,\rho) < 0$ or
- $\lim_\Lambda \ \mu_\Lambda(\beta, \rho \geq \rho_c(\beta)) = 0$, and

$$\rho_0(\beta) = \rho - \rho_c(\beta) = \lim_\Lambda \frac{1}{V} \ \left\{e^{-\beta\, \mu_\Lambda(\beta,\rho\geq\rho_c(\beta))} - 1\right\}^{-1} \Rightarrow$$

$$\mu_\Lambda(\beta,\rho \geq \rho_c(\beta)) = -\frac{1}{V}\frac{1}{\beta(\rho-\rho_c(\beta))} + o(1/V)$$

- Since $\varepsilon_k = \sum_{j=1}^{d}(2\pi n_j/V^{1/3})^2/2$ the BEC is *unique* (**type I**):

$$\lim_\Lambda \frac{1}{V}\left\{e^{\beta(\varepsilon_{k\neq 0}-\mu_\Lambda(\beta,\rho))} - 1\right\}^{-1} = 0.$$

- **Saturation Mechanism** (*generalised condensation*):

- **The Casimir Box:** Let $\alpha_1 = 1/2$, i.e. $\alpha_2 + \alpha_3 = 1/2$. Then

$$\lim_{\Lambda} \frac{1}{V} \left\{ e^{\beta(\varepsilon_{k \neq 0} - \mu_\Lambda(\beta,\rho))} - 1 \right\}^{-1} \neq 0, \varepsilon_{k_1,0,0} = (2\pi n_1 / V^{1/2})^2/2 \sim \mu_\Lambda(\beta,\rho).$$

$$\lim_{\Lambda} \frac{1}{V} \left\{ e^{\beta(\varepsilon_{k \neq 0} - \mu_\Lambda(\beta,\rho))} - 1 \right\}^{-1} = 0, \varepsilon_{0,k_{2,3} \neq 0} \sim (2\pi n_j / V^{\alpha_j})^2/2 > \mu_\Lambda(\beta,\rho)$$

- Hence again the solution $\mu_\Lambda(\beta,\rho)$ of the equation

$$\rho = \rho_\Lambda(\beta,\mu) \quad \Leftrightarrow \quad \rho \equiv \rho_\Lambda(\beta, \mu_\Lambda(\beta,\rho)).$$

has the asymptotics $\mu_\Lambda(\beta, \rho \geq \rho_c(\beta)) = -A/V + o(1/V),\ A \geq 0$.

• Generalised BEC condensation **type II** [van den Berg-Lewis-Pulé (1978)]:

$$\rho - \rho_c(\beta) = \lim_{L\to\infty} \frac{1}{V} \sum_{n_1\in\mathbb{Z}} \left\{ e^{\beta((2\pi n_1/V^{1/2})^2/2-\mu_\Lambda(\beta,\rho))} - 1 \right\}^{-1}$$

$$= \sum_{n_1\in\mathbb{Z}} \frac{1}{(2\pi n_1)^2/2 + A} \,.$$

Here $A \geq 0$ is a *unique root* of the above equation.

• **N.B.** For $\alpha_1 = 1/2$ the BEC is still **microscopical** , but **infinitely** *fragmented.* Experiment with rotating condensate (2000).

• **The van den Berg-Lewis-Pulé Box:** $\alpha_1 > 1/2$.

• **N.B.** No macroscopic occupation of any level:

$$\lim_{\Lambda} \frac{1}{V} \left\{ e^{\beta(\varepsilon_k-\mu_\Lambda(\beta,\rho))} - 1 \right\}^{-1} = 0.$$

- Generalised BEC of the **type III:**

$$\lim_{\delta\to 0+}\lim_{\Lambda}\frac{1}{V}\sum_{\{k\in\Lambda^*,0\leq\|k\|\leq\delta\}}\left\{e^{\beta(\varepsilon_k-\mu_\Lambda(\beta,\rho))}-1\right\}^{-1}=\rho-\rho_c(\beta)$$

- Chemical potential ($\alpha_1 > 1/2$):

$$\mu_\Lambda(\beta,\rho>\rho_c(\beta))=-\frac{B}{V^\delta}+o\left(\frac{1}{V^\delta}\right),\ B>0,\ \delta=2(1-\alpha_1)<1, \tag{1}$$

- Equation for B

$$\rho-\rho_c(\beta)=(2\pi\beta)^{-1/2}\int_0^{+\infty}d\xi e^{-\beta B\xi}\xi^{-1/2},$$

- ρ_m-**PROBLEM**: (van den Berg-Lewis-Pulé)

$\rho_c \leq \rho_m \leq \infty$ such that **type II or III** $\rightarrow$ **type I**, for $\rho \geq \rho_m$?

YES!

II Free Bose-Gas

2.1 One-Particle Integrated Density of States

- Let $\Lambda_L \subset \mathbb{R}^d$, with a smooth boundary $\partial\Lambda_L$ and $|\Lambda_L| = V_L$.
- $\mathcal{H}_L := L^2(\Lambda_L)$, and (free) one-particle Hamiltonian $t_{\Lambda_L} := (-\Delta/2)_{\Lambda_L, D} = t^*_{\Lambda_L}$, with (for example) *D=Dirichlet boundary* conditions.
- t_{Λ_L} has a discrete spectrum $\sigma(t_{\Lambda_L}) = \left\{E_{k,L}\right\}_{k\geq 1}$:

$$t_{\Lambda_L}\psi_{k,L} = E_{k,L}\psi_{k,L}, \;\; 0 < E_{1,L} < E_{2,L} \leq E_{3,L} \leq \ldots$$

of finite multiplicity, and $\exp(-\beta t_{\Lambda_L}) \in$ Tr-class($\mathcal{H}_L$) for $\beta > 0$.

Definition 2.1 The finite-volume *integrated density of states* (**IDS**) of t_{Λ_L} is the specific (by a *unit* volume) eigenvalue counting function $\mathcal{N}_{\Lambda_L}(E) := \max\left\{k : E_{k,L} < E\right\}/|\Lambda_L|$.

Proposition 2.2 There exists a *limiting* integrated density of states: $\mathcal{N}^{(0)}(E) = w{-}\lim_{L\to\infty}\mathcal{N}_{\Lambda_L}(E)$, where $\mathcal{N}^{(0)}(E) = C_d E^{d/2}$.

2.2 BEC of the Free Bose-Gas

• **Definition 2.3** The grand-canonical **non**-interacting bosons **without** external potential are called the (β, μ)-**free** Bose-gas.
• **Proposition 2.4** By the *Bose-statistics* and by **Definition 1** of the *finite-volume* **IDS**, the *mean value* of the *total* particle-density $\rho_{\Lambda_L}(\beta, \mu)$ in the volume Λ_L is:

$$\rho_{\Lambda_L}(\beta, \mu) = -\int_0^\infty dE\ \mathcal{N}_{\Lambda_L}(E)\ \partial_E \left\{ \frac{1}{e^{\beta(E-\mu)} - 1} \right\}\ ,\ \mu < 0\ .$$

• By **Proposition 2.5**, the limiting density $\rho(\beta, \mu)$ exists for *negative* chemical potentials $\mu \in (-\infty, 0)$:

$$\rho(\beta, \mu) = -\int_0^\infty dE\ \mathcal{N}^{(0)}(E)\ \partial_E \left\{ \frac{1}{e^{\beta(E-\mu)} - 1} \right\}\ .$$

• The **critical** density $\rho_c(\beta) := \rho(\beta, -0) < \infty$ is **finite** for $d > d_c = 2$, since $\mathcal{N}^{(0)}(dE) \sim E^{d/2-1} dE$.

We resume the above observations as the main statement about the BEC for the case of the free boson gas:

• **Proposition 2.6** Let $\rho_c(\beta) < \infty$ and $\mu_{\Lambda_L}(\beta, \rho)$ be unique root of equation $\rho = \rho_L(\beta, \mu)$. For $\rho \geq \rho_c(\beta)$, $\lim_{L\to\infty} \mu_{\Lambda_L}(\beta, \rho) = 0$ and the BEC density $\rho_0(\beta, \rho) := \rho - \rho_c(\beta) > 0$ is

$$\rho_0(\beta, \rho) = - \lim_{\epsilon \downarrow 0} \lim_{L\to\infty} \int_0^{\epsilon} dE \; \mathcal{N}_{\Lambda_L}(E) \; \partial_E \left\{ \frac{1}{e^{\beta(E - \mu_{\Lambda_L}(\beta,\rho))} - 1} \right\}$$

• **N.B.** If $\rho_c(\beta) = \infty$, this statement has no sense, **but** the value of critical density $\rho_c(\beta)$ may be **changed**, if the non-interacting gas is placed in an **external potential**: since the value of $\rho_c(\beta)$ is a function of the critical dimensionality d_c and the latter is a functional of the **One-Particle Density of States**: $\mathcal{N}^{(0)}(dE)$.

2.3 Why BEC of the Free Bose-Gas is a Subtle Matter ?

- Let $\Lambda_{L,\mathbf{D}} = \times_{j=1}^{3}[-L/2, L/2]$ be a **cube**. Then

$$\rho_0(\beta, \rho > \rho_c(\beta)) = \lim_{L\to\infty} \frac{1}{L^3} \left\{ e^{\beta(E_{k,L}-\mu_L(\beta,\rho))} - 1 \right\}^{-1}$$
$$= (\rho - \rho_c(\beta))\delta_{\mathbf{1},k} \ , \ E_{\mathbf{1}=(1,1,1),L} = \{3(\pi/L)^2\}/2$$

is the **ground-state BEC** (*type* I), $E_{\mathbf{1},L} - \mu_L(\beta,\rho)) \sim L^{-3}$.

$E_{gr} = 0 \bullet - - - - - - - E_* \bullet - - - - - - - - - - - - - - - \longrightarrow$ **E**

- Let $\rho_c(\beta) = \int_0^\infty \mathcal{N}^{(0)}(dE)\{e^{\beta E} - 1\}^{-1} = \infty$ $\Leftrightarrow$ **high** *density of states* $\mathcal{N}^{(0)}(dE)$ at $E = 0$ (e.g. $E^{d/2-1}dE$ for $d \leq 2$) $\Leftrightarrow$ "*leaking*" of the BEC into **excited** states $\Rightarrow$
- **Conclusion:** To *preserve* the BEC one has to *suppress* density of states in the *vicinity* of the **ground-state** ($E_{gr} = 0$), e.g., a **spectral gap**: $\mathcal{N}^{(0)}(E) = \theta(E - E_{gr})$ for $E < E_*$, where $E_{gr} < E_*$ [Buffet,Pulé,Lauwers,Verbeure,Z].

III Perfect Bose-Gas in Magnetic Field

3.1 Hamiltonian

- Let open $\Lambda_{L=1} \subset \mathbb{R}^{d=3}$ with $|\Lambda_{L=1}| = 1$ and piecewise continuously differentiable boundary $\partial\Lambda_{L=1}$ contain the origin $\{x = 0\}$. Put $\Lambda_L := \{x \in \mathbb{R}^3 : L^{-1}x \in \Lambda_{L=1}\}$, $L > 0$.
- Take a magnetic *vector-potential* in the form: $a(x) = \omega\ a_0(x)$, $\omega \geq 0$. For two types of gauges: symmetric (*transverse*): $a_0(x) = 1/2(-x_2, x_1, 0)$, or *Landau*: $a_1(x) = (0, x_1, 0)$, this generates a unit magnetic field $\mathbf{B}$ *parallel* to the third direction.
- The one-particle Hamiltonian with *Dirichlet* boundary conditions (D) on $\partial\Lambda_L$ is defined in $L^2(\Lambda_L)$ by

$$h_{\Lambda_L}(\omega) := (-i\nabla - a)^2 + V_{\Lambda_L} \equiv t_{\Lambda_L}(\omega) + V_{\Lambda_L} \ ,$$

where V_{Λ_L} is an external "*electric*" potential. Then $h_{\Lambda_L}(\omega)$ has purely discrete spectrum.

3.2 No-Go Theorem for BEC in a Constant Magnetic Field

• Let a continuous external potential $V(x) = v(x_1)$ (v is $\mathbb{Z}$-periodic) and use the Landau gauge $a_1(x) = (0, x_1, 0) \in \mathbb{R}^3$ (a particular gauge is irrelevant since the density of states is gauge invariant). Then the *bulk* Hamiltonian is:

$$h_\infty(\omega) = (-i\nabla - \omega a_1)^2 + v = -\partial_{x_1}^2 + v(x_1) + (-i\partial_{x_2} - \omega x_1)^2 - \partial_{x_3}^2 \,,$$

acting in $L^2(\mathbb{R}^3)$, where $\omega \geq 0$.

• **Proposition 3.1** Let $E_0(\omega) := \inf \sigma(h_\infty(\omega))$. Then:

$$\mathcal{N}_{\infty,\omega}(E) = B_{\omega,d} \cdot (E - E_0(\omega))^{d/2-1} + o((E - E_0(\omega))^{d/2-1})$$

for $E \searrow E_0(\omega)$. Hence, for $d = 3$ and any $\omega > 0$ the critical density

$$\rho_c(\beta) = -\lim_{\mu \nearrow E_0(\omega)} \int_{E_0(\omega)}^{\infty} dE\ \mathcal{N}_{\infty,\omega}(E)\partial_E \left\{\frac{1}{e^{\beta(E-\mu)} - 1}\right\}$$

is infinite, i.e. the BEC is destroyed.

• **Remark 3.2** Operator $h_\infty(\omega)$ is unitary equivalent to the sum of ω-harmonic oscillator (Landau levels) and one-dimensional v-Schrödinger operator in the *third direction*. If $v = 0$, then

$$\mathcal{N}_{\infty,\omega}(E) = \omega(E-\omega)^{1/2}/2\pi^2$$

between the first two Landau levels: $E \in (\omega, 3\omega)$, i.e.

$$d = 3 \text{ and } \omega > 0 \Leftrightarrow d = 1 \text{ and } \omega = 0$$

• **Proposition 3.3 [BCZ (2004)]** Assume that $\omega = 2\pi$. Then there exists an external "*electric*" potential of the form:

$$V_\epsilon(x) = \epsilon \cdot [v_1(x_1) + v_2(x_2)] + v_3(x_3),$$

where $\epsilon > 0$ and small, each of the functions $\{v_j\}_{j=1}^3$ is a smooth $\mathbb{Z}$-periodic potential, and *neither* one of v_1 and v_2 is constant, that critical density is bounded.

3.3 Another example gives a Generalized BEC (Casimir (1968), van den Berg - Lewis - Pulé (1978))

- **Generalized** (*fragmented*) BEC in the *Casimir boxes*:

$$\Lambda_L = \times_{j=1}^{3}[-V^{\alpha_j}/2, V^{\alpha_j}/2] \ , \ \ \alpha_1 + \ldots + \alpha_3 = 1.$$

- $\alpha_1 < 1/2 \Rightarrow$ **BEC type I**

- $\alpha_1 = 1/2 \Rightarrow$ **BEC type II**

- $1 > \alpha_1 > 1/2 \Rightarrow$ **BEC type III**

IV Bose-Condensation in Random Potentials

4.1 Random Schrödinger Operator (RSO)

• **Random Repulsive Impurities:** $u(x) \geq 0$, $x \in \mathbb{R}^d$, continuous function with a *compact* support is a local single-impurity potential. The Random *Poisson* Potential (*RPP*):

$$v^\omega(x) := \int_{\mathbb{R}^d} \mu_\tau^\omega(dy) u(x-y) = \sum_j u(x - y_j^\omega) \geq 0,$$

where impurity positions $\left\{y_j^\omega\right\} \subset \mathbb{R}^d$ are the atoms of the random Poisson measure:

$$\mathbb{P}\left(\{\omega \in \Omega : \mu_\tau^\omega(\Lambda) = n\}\right) = \frac{(\tau\,|\Lambda|)^n}{n!} e^{-\tau|\Lambda|}$$

$n \in \mathbb{N} \cup \{0\}$, $\Lambda \subset \mathbb{R}^d$, $\mathbb{E}\left(\mu_\tau^\omega(\Lambda)\right) = \tau\,|\Lambda|$, the parameter τ is concentration of impurities.

PROPERTIES:

- This potential is *homogeneous* and *ergodic*.

- *RSO* is a family of random (*a.s*) self-adjoint operators

$$\{h^{\omega} := t + v^{\omega}\}_{\omega \in \Omega} .$$

Proposition 4.1 For *RSO* with *RPP* the spectrum $\sigma(h^{\omega})$ of h^{ω} is almost-sure (a.s.) nonrandom and coincides with $[0, +\infty)$.

4.2 Self-Averaging of the IDS

• The restriction $h_L^\omega := (-\Delta/2 + v^\omega)_{\Lambda_L,\mathcal{D}}$ has the (*random*) finite-volume *IDS*:

$$\mathcal{N}_L^\omega(E) := \frac{1}{|\Lambda_L|} \max\{k : E_k^\omega(L) < E\}, \ \omega \in \Omega$$

Proposition 4.2 There exists a nonrandom distribution $\mathcal{N}(E)$ (*measure* $\mathcal{N}(dE)$) such that (*a.s.*)

$$\lim_{L\to\infty} \mathcal{N}_L^\omega(E) = \mathcal{N}(E),$$

and $\mathcal{N}(E) = \mathbb{E}\{\mathcal{E}_{h^\omega}(E;0,0)\}$, $\mathcal{E}_{h^\omega}(E;x,y)$ is the *kernel* of the spectral decomposition measure of the *RSO* h^ω. The spectrum $\sigma(h^\omega) = \mathrm{supp}\,\mathcal{N}$ with (*nonrandom*) lower edge $E_0 = 0$.

Proposition 4.3 (*Lifshitz tail*)
The asymptotic behaviour of $\mathcal{N}(E)$ as $E \downarrow 0$:

$$\mathcal{N}(E) \sim \exp\left\{-\tau\,(c_d/E)^{d/2}\right\}\,,$$

with $c_d > 0$.

• N.B. For the free case, $v^\omega = 0$, one has: $\mathcal{N}^{(0)}(E) \sim E^{d/2},\ E \downarrow 0$.

• The *self-averaging* of the limiting IDS is true for the Poisson *point-impurities*: $u(x) = a\,\delta(x), a > 0$.

• It is known *explicitly* for $a \to +\infty$.

4.3 BEC of the Perfect Bose-Gas in RPP

• The *random* finite-volume particle density:

$$\rho_L^{\omega}(\beta,\mu) = \int_0^{\infty} \mathcal{N}_L^{\omega}(dE)\frac{1}{e^{\beta(E-\mu)}-1}$$

for $\beta > 0$, $\mu < 0$ and any realization $\omega \in \Omega$.

• **Proposition 4.4** By Proposition 3.2

$$a.s. - \lim_{L\to\infty} \rho_L^{\omega}(\beta,\mu) =$$
$$-\int_0^{\infty} dE\ \mathcal{N}(E)\partial_E \ \left\{\frac{1}{e^{\beta(E-\mu)}-1}\right\} \equiv \rho(\beta,\mu),$$

uniformly in μ on compacts in $(-\infty,0)$.

• **Corollary 4.5** *Lifshitz tail* implies that $\rho_c(\beta) := \rho(\beta,-0) < \infty$ for $d > 0$, so there is condensation of the Perfect Bose-Gas at *low dimensions* $d = 1,2$.

Proposition 4.6 [LPZ (2004)] Let $\rho \geq \rho_c(\beta)$ and $\mu_L^\omega(\beta, \rho)$ be a unique root of equation $\rho = \rho_L^\omega(\beta, \mu)$ for $\omega \in \Omega$. Then $a.s.-\lim_{L\to\infty} \mu_L^\omega(\beta, \rho) = 0$, and:

$$\lim_{\epsilon \downarrow 0} \left\{ a.s. - \lim_{L\to\infty} \int_0^\epsilon \mathcal{N}_L^\omega(dE) \frac{1}{e^{\beta(E-\mu_L^\omega(\beta,\rho))} - 1} \right\}$$

$$(a.s.) = \rho - \rho_c(\beta) = \rho_0(\beta, \rho) \geq 0 \ .$$

- A.s. *nonrandom* $\rho_0(\beta, \rho)$ is the BEC density.

4.4 BEC in One-Dimensional Random Potential: Poisson Point-Impurities

- For $d = 1$ *Poisson point-impurities*, $a > 0$:

$$v^{\omega}(x) := \int_{\mathbb{R}^1} \mu_{\tau}^{\omega}(dy) a\, \delta(x - y) = \sum_j a\, \delta(x - y_j^{\omega})$$

Proposition 4.7 Let $a = +\infty$. Then $\sigma(h^{\omega})$ is a.s. nonrandom, dense *pure-point* spectrum $\overline{\sigma_{p.p.}(h^{\omega})} = [0, +\infty)$, with IDS

$$\mathcal{N}(E) = \tau \frac{e^{-\pi\tau/\sqrt{2E}}}{1 - e^{-\pi\tau/\sqrt{2E}}} \sim \tau e^{-\pi\tau/\sqrt{2E}}, \; E \downarrow \mathbf{0}$$

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- **Spectrum:**

$$(a.s.) - \sigma(h^\omega) = \bigcup_j \left\{\pi^2 s^2 / 2(L_j^\omega)^2\right\}_{s=1}^{\infty}$$

- Intervals $L_j^\omega = y_j^\omega - y_{j-1}^\omega$ are *i.i.d.r.v.* :

$$dP_{\tau,j_1,\dots,j_k}(L_{j_1},\dots,L_{j_k}) = \tau^k \prod_{s=1}^{k} e^{-\tau L_{j_s}} dL_{j_s}$$

4.5 BEC in One-Dimensional Nonrandom Potential: Point-Impurities(*hierarchical* model [LZ (2007)])

- Let $[0, L] = \bigcup_{j=1}^{n} I_j$, $I_j = [y_{j-1}, y_j]$, $y_0 = 0, y_n = L$ and $v(x) := \sum_{j=0}^{n} a\, \delta(x - y_j)\, , a = +\infty$
- Let $h_0(I_j) := (-\Delta/2)_{I_j, \mathcal{D}}$. The model: $h_L := (-\Delta/2) \dot{+} v(x) = \bigoplus_{j=1}^{n-1} h_0(I_j)$, $L_j = \left| I_j \right|$

$$\sigma(h_L) = \bigcup_{j=1}^{n-1} \left\{ E_s(L_j) \equiv \pi^2 s^2 / 2(L_j)^2 \right\}_{s=1}^{\infty}\, , (p.p.)$$

- Let $L_{j=2,3,\ldots} = (L - L_1)/(n-1) \equiv \tilde{L}$ and $L_1 = f(L) < L$: $\lim_{L\to\infty} f(L)/L = 0$.

- Finite-volume *total* particle density :

$$\rho_L(\beta,\mu) = \frac{1}{L}\sum_{s=1}^{\infty}\left\{e^{\beta(E_s(L_1)-\mu)} - 1\right\}^{-1} +$$
$$\frac{n-1}{L}\sum_{s=1}^{\infty}\left\{e^{\beta(E_s(\tilde{L})-\mu)} - 1\right\}^{-1} \ , \ \ \mu \leq 0$$

- For $\tau = \lim_{n,L\to\infty} n/L = \lim_{n,L\to\infty} \tilde{L}^{-1}$ the *critical* density $\rho_c(\beta) := \lim_{\mu\nearrow 0}\lim_{n,L\to\infty}\rho_L(\beta,\mu)$.

$$\rho_c(\beta) = \tau\sum_{s=1}^{\infty}\left\{e^{\beta E_s(\tau^{-1})} - 1\right\}^{-1} < \infty$$

4.6 BEC in One-Dimensional Nonrandom Potential (I)

- Let $\rho_L(\beta, \mu_L(\beta, \rho)) = \rho < \rho_c(\beta)$. Then $\lim_{L\to\infty} \mu_L(\beta, \rho) = \tilde{\mu}(\beta, \rho) < 0$ and

$$\rho = \tau \sum_{s=1}^{\infty} \left\{ e^{\beta[E_s(\tau^{-1}) - \tilde{\mu}(\beta,\rho)]} - 1 \right\}^{-1}$$

- $L_1 = L^{1/2-\epsilon}$: Let $\rho_L(\beta, \mu_L(\beta, \rho)) = \rho \geq \rho_c(\beta)$ and $L_1 = f(L) = L^{1/2-\epsilon}$, $\epsilon > 0$. Then

$$\mu_L(\beta, \rho) = \pi^2/2L_1^2 - (\beta L(\rho - \rho_c(\beta)))^{-1} + O(L^{-2})$$

- BEC density $\rho_0(\beta, \rho) = \rho - \rho_c(\beta)$:

$$\rho_0(\beta, \rho) = \lim_{n,L\to\infty} \frac{1}{L} \sum_{s=1}^{\infty} \left\{ e^{\beta(E_s(L_1) - \mu_L(\beta,\rho))} - 1 \right\}^{-1}$$

$$= \lim_{n,L\to\infty} \frac{1}{L} \left\{ e^{\beta(E_1(L_1) - \mu_L(\beta,\rho))} - 1 \right\}^{-1}$$

This is the *ground-state* (**type I**) BEC, *localized* in the *largest* box $L_1 \to \infty$.

- **Type II BEC:** $L_1 = L^{1/2}$

Then

$$\mu_L(\beta, \rho) = -A(\beta, \rho)/L + O(L^{-2})$$

and BEC is *fragmented* among **infinitely** many levels in *one largest* box.

4.7 BEC in One-Dimensional Nonrandom Potential (II)

• This is the **type II** *generalized* BEC in the *largest* box, with **infinitely** many (single-particle) levels **macroscopically** occupied:

$$\rho - \rho_c(\beta) = \lim_{n,L\to\infty} \frac{1}{L} \sum_{s=1}^{\infty} \frac{1}{e^{\beta(E_s(L_1)-\mu_L(\beta,\rho))} - 1}$$

$$= \sum_{s=1}^{\infty} \left\{ \beta(\pi^2 s^2/2 + A(\beta,\rho)) \right\}^{-1} , \;\; A(\beta,\rho) > 0$$

• $L_1 = L^{1/2+\epsilon}$: One gets the **type III** *generalized* BEC in the *largest* box: **none** of single-particle levels is **macroscopically** occupied.

- Chemical potential:

$$\mu_L(\beta,\rho) = -B(\beta,\rho)/L^{1-2\epsilon} + O(L^{-1})$$

and

$$\rho - \rho_c(\beta) = \lim_{n,L\to\infty} \frac{1}{L} \sum_{s=1}^{\infty} \frac{1}{e^{\beta(E_s(L_1)-\mu_L(\beta,\rho))} - 1}$$

$$= \frac{1}{\sqrt{2\pi\beta}} \int_0^\infty dt e^{-\beta t B(\beta,\rho)} t^{-1/2} \;, \;\; B(\beta,\rho) > 0$$

4.8 Nonrandom/Random Potential (III)

•*Spatially* fragmented *type III* BEC in the *hierarchical* model splitted between (*infinitely*) many *different* intervals:

$$L_j = \frac{\ln(\lambda L)}{\lambda}, 1 \leq j \leq [\ln(k+1)] =: M_k,$$

$$L_{j>M_k} = \tilde{L}_k := \frac{L - L_1 M_k}{k - M_k}$$

$$\rho_L(\beta,\mu) = \frac{1}{L}\sum_{j=1}^{M_k}\sum_{s=1}^{\infty}\frac{1}{e^{\beta(c^2s^2/L_j^2-\mu)}-1} +$$

$$\frac{k - M_k}{L}\sum_{s=1}^{\infty}\frac{1}{e^{\beta(c^2s^2/\tilde{L}_k^2-\mu)}-1}$$

$\lim_{L\to\infty}\tilde{L}_k = \lim_{L\to\infty} L/(k-M_k) = 1/\lambda$, condensate $\rho - \rho_c(\beta) = \rho_0(\beta,\rho) > 0$ is *equally* splitted between *infinitely* many intervals.

- For **Poisson Point-Impurities** one gets:

$$\mathbb{P}\{\omega : L^{\omega}_{j_1} - L^{\omega}_{j_2} > \delta\} = e^{-\lambda\delta}$$

$\Big\{L^{\omega}_{j_1} \geq L^{\omega}_{j_2} \geq \ldots \geq L^{\omega}_{j_k} : \sum_{s=1}^{k} L^{\omega}_{j_s} = L\Big\}$,

$\mathbb{E}_{\sigma^{>}_{\lambda,k}}(L^{\omega}_{j\,1,2}) = \frac{1}{\lambda}\ln(k) + \frac{1}{\lambda}P_{1,2} + O(1/k)$,

$P_2 = P_1 - 1$.

V Off-Diagonal-Long-Range-Order (ODLRO)

5.1 BEC of the Free Bose-Gas: ODLRO

- *PBG one-body reduced density matrix*:

$$\rho_L(\beta,\mu;x,y) = \sum_{k\geq 1} \frac{1}{e^{\beta(E_k(L)-\mu)}-1}\overline{\psi_{k,L}(x)}\psi_{k,L}(y)$$

Its *diagonal* part is the *local* particle number density.

Proposition 5.1 For the free Bose-gas ($L \to \infty$)

$\rho(\beta,\mu(\beta,\rho);x,y) =$

$$\begin{cases} \sum_{s=1}^{\infty} (2\pi\beta s)^{-d/2} e^{s\beta\mu(\beta,\rho)-\|x-y\|^2/2\beta s}, \rho < \rho_c(\beta) \\ \\ \rho_0(\beta,\rho)\left|\psi_{k,L=1}(0)\right|^2 + \sum_{s=1}^{\infty} \frac{e^{-\|x-y\|^2/2\beta s}}{(2\pi\beta s)^{d/2}}, \rho \geq \rho_c(\beta) \end{cases}$$

Here $\rho_0(\beta, \rho)$ is the condensate density and $\psi_{k=1,L=1}(0)$ is the ground state eigenfunction in domain $\Lambda_{L=1}$ evaluated at the point of dilation $x = 0$.

- **Definition:** The *Off-Diagonal Long-Range Order*:

$$ODLRO(\beta, \rho) := \lim_{\|x-y\|\to\infty} \rho(\beta, \mu(\beta, \rho); x, y)$$

5.2 One-Body Reduced Density Matrix for Random Potentials

- *Space averaged* reduced density matrix

$$\tilde{\rho}_L^\omega(\beta,\mu;x,y) := \frac{1}{|\Lambda_L|}\int_{\Lambda_L} da\,\rho_L^\omega(\beta,\mu;x+a,y+a)$$

- For non-negative measurable ergodic random potentials, any $\mu < 0$ and any fixed $x, y \in \mathbb{R}^d$ one gets *self-averaging* of the *reduced density matrix*:

$$a.s. - \lim_{L\to\infty} \tilde{\rho}_L^\omega(\beta,\mu;x,y) = \tilde{\rho}(\beta,\mu;x-y)$$

Proposition 5.2 Then

$$\rho(\beta, \mu - \tau\tilde{u}; x-y) \leq \tilde{\rho}(\beta, \mu; x-y) \leq \rho(\beta, \mu; x-y),$$

where $\tilde{u} := \int_{\mathbb{R}^1} dx u(x)$.

Proposition 5.3 Let $\mu < 0$. For one-dimensional Poisson potential with $\operatorname{supp} u(x) = [-\delta/2, \delta/2]$

$$\tilde{\rho}(\beta, \mu; x-y) \leq \rho(\beta, \mu; x-y) e^{-\tau\tilde{\gamma}(|x-y|-\delta)},$$

where $\tilde{\gamma} := 1 - e^{-\tilde{u}}$.

Corollary 5.4 If impurity concentration $\tau \downarrow 0$:

$$\lim_{\tau \downarrow 0} \tilde{\rho}(\beta, \mu; x-y) = \rho(\beta, \mu; x-y)$$

VI Kac-Luttinger Conjecture [KL (1973-74]

- *In the case of the one-dimensional random Poisson potential of point impurities the BEC for the PBG is of the type I and it is localized in one "largest box".*

6.1 Statistics of Poisson Intervals:

- Consistent *marginals* in the (thermodynamic) limit $\lambda = \lim_{L\to\infty} n/L$:

$$d\sigma_{\lambda,k}(L_{j_1},\ldots,L_{j_k}) = \lambda^k \prod_{s=1}^{k} e^{-\lambda L_{j_s}} dL_{j_s} \ .$$

- For ordered intervals: $\left\{L^{\omega}_{j_1} \geq L^{\omega}_{j_2} \geq \ldots \geq L^{\omega}_{j_k} : \sum_{s=1}^{k} L^{\omega}_{j_s} = L \simeq k/\lambda\right\}$:

$$d\sigma^{>}_{\lambda,k}(L_{j_1},\ldots,L_{j_k}) := k!\ \theta(L_{j_1}-L_{j_2}) \ldots \theta(L_{j_{k-1}}-L_{j_k})\ d\sigma_{\lambda,k}(L_{j_1},\ldots,L_{j_k}) \ .$$

- $\mathbb{E}_{\sigma_\lambda}(L^{\omega}_{j_s}) = \lambda \int_0^\infty dL\, L\, e^{-\lambda L} = \lambda^{-1}$ and $L^{\omega}_1 \sim \lambda^{-1} \ln(\lambda L)\ ,\ L \to \infty$.
- Probabilities of the "energies repulsions" in different boxes:

$$\mathbb{P}\{\omega : L^{\omega}_{j_1} - L^{\omega}_{j_2} > \delta\} = e^{-\lambda\delta},\ \delta > 0.$$

6.2 Application of the Borel-Cantelli Lemma

- Energies in the *samples* $\left\{|I_j^\omega(k)| = L_j^\omega(k)\right\}_{j=1}^k$:

$$E_s(L_{j_r}^\omega(k)) = \frac{c^2 s^2}{(L_{j_r}^\omega(k))^2} \ , \ r = 1, ..., k \ , \ s = 1, 2, ... \ .$$

- Let the events $(k = 1, 2, \ldots)$

$S_k(a > 0, 0 < \gamma < 1) := \{\omega : E_{s=1}(L_{j_2}^\omega(k)) - E_{s=1}(L_{j_1}^\omega(k)) > \frac{a}{k^{1-\gamma}}\}$

- Since $\lim_{k\to\infty} \mathbb{P}\{S_k(a, 0 < \gamma < 1)\} = 1$, one gets *divergence*

$$\lim_{k\to\infty} \sum_{r=1}^{k} \mathbb{P}\{S_k(a, \gamma)\} = \infty.$$

- Then *independence* of the *events* $\{S_k(a, \gamma)\}_{k=1}^\infty$ and the well-known *Borel-Cantelli* lemma imply:

$$\mathbb{P}\,\{\overline{\lim_{k\to\infty}}\ S_k(a,\gamma)\} = 1,\ \overline{\lim_{k\to\infty}}\ S_k(a,\gamma) = \bigcap_{k=1}^{\infty}\bigcup_{l=k} S_l(a,\gamma)$$

• Notice that the event:

$$\overline{\lim}\ S_k(a,\gamma) := \bigcap_{k=1}^{\infty}\bigcup_{l=k} S_l(a,\gamma)$$

means that *infinitely* many events $\{S_k(a,\gamma)\}_{k\geq 1}$ take place.

• This means (in turn) that with the probability 1 the BEC is localized in the thermodynamic limit $\mathbb{R}$ in a single "*largest box*", and this condensation is of the *type* I.

VII Bose Condensation in "Weak" Harmonic Traps

7.1 Harmonic Traps

- Consider in $\mathfrak{H} := L^2(\mathbb{R}^d)$ a $\kappa-$parameterized family of *self-adjoint* one-particle Hamiltonians

$$h_\kappa := \frac{1}{2}\sum_{j=1}^{d}\left(-\frac{\partial^2}{\partial x_j^2} + \frac{x_j^2}{\kappa^2} - \frac{1}{\kappa}\right)\ ,\ \ \kappa > 0\ ,$$

spec$(h_\kappa) = \{\varepsilon_{s,\,\kappa} := |s|_1/\kappa \,|\, s = (s_1, \cdots, s_d) \in \mathbb{N}^d\}$, $|s|_1 := \sum_{j=1}^d s_j$.

- **Definition 7.1** The **global** particle "density" in the trap $\kappa > 0$:

$$\rho_\kappa(\beta,\mu) := \frac{1}{\kappa^d}\sum_{s\in\mathbb{N}^d}\frac{1}{e^{\beta(\varepsilon_{s,\kappa}-\mu)}-1}\ .$$

Why κ^d (an *effective volume*) ?

- **Ground state**: $\phi_{s=0,\,\kappa}(x) = \frac{1}{(\pi\kappa)^{d/4}}e^{-|x|^2/2\kappa} \Rightarrow \kappa^{d/2}$!

7.2 Weak Harmonic Trap Limit

• **Motivation:** Total density: $\rho_\kappa(\beta,\mu)$ is the *Darboux-Riemann* sum for the integral in the limit $\kappa \to \infty$ of the *weak trap*:

$$\rho(\beta,\mu) = \lim_{\kappa\to\infty} \rho_\kappa(\beta,\mu) = \int_{[0,\infty)^d} dp \, \frac{1}{e^{\beta(|p|_1-\mu)} - 1} = \sum_{n=1}^{\infty} \frac{e^{\beta\mu n}}{(\beta n)^d} \, .$$

• Critical density: $\rho_c(\beta) := \sup_{\mu<0} \rho(\beta,\mu) = \zeta(d)/\beta^d < \infty$, if $d > 1$.

• **Proposition 7.2** Density of states for the *weak-trap* limit:

$$d\mathcal{N}_{wt}(E) = \frac{E^{d-1}}{\Gamma(d)} dE \ , \ \text{and} \ \rho(\beta,\mu) = \int_0^\infty d\mathcal{N}_{wt}(E) \, \frac{1}{e^{\beta(E-\mu)} - 1} \, .$$

• **Remark 7.3** For the *free* boson gas ($\Lambda = \mathbb{R}^d$) the *well-known* results are: $\rho_c(\beta) = \zeta(d/2)/(2\pi\beta)^{d/2} < \infty$, if $d > 2$, and

$$d\mathcal{N}_f(E) = \frac{E^{(d-2)/2}}{(2\pi)^{d/2}\Gamma(d/2)} dE \, , \quad \rho(\beta,\mu) = \int_0^\infty d\mathcal{N}_f(E) \, \frac{1}{e^{\beta(E-\mu)} - 1} \, .$$

7.2 BEC in the Weak Harmonic Trap Limit

- **Corollary 7.4** Since the BEC *critical temperature* is defined by equation $\rho_c(\beta_c(\rho)) = \rho$, then for the BEC density $\rho_0(\beta) := \rho - \rho_c(\beta)$ in the Weak Harmonic Trap Limit one has:

$$\frac{\rho_0(\beta)}{\rho} = 1 - \left(\frac{\beta}{\beta_c}\right)^{d} .$$

- **N.B.** For the *free* boson gas: $\rho_0(\beta)/\rho = 1 - (\beta/\beta_c)^{d/2}$.
- **Local Particle Density:**

$$\rho_{\beta,\mu,\kappa}(x) := \omega_{\beta,\mu,\kappa}(a^*(x)a(x)) = \frac{1}{\kappa^{d/2}} \sum_{s\in\mathbb{N}^d} \frac{|\phi_{s,\kappa=1}(x/\sqrt{\kappa})|^2}{e^{\beta(\varepsilon_{s,\kappa}-\mu)} - 1} .$$

- **Proposition 7.5** For $\rho = \rho_\kappa(\beta, \overline{\mu}_\kappa(\rho))$ and $\rho > \rho_c(\beta)$, $\delta > 0$:

$$\lim_{\kappa\to\infty} \frac{1}{\kappa^{d/2}}\, \rho_{\,\beta,\overline{\mu}_\kappa,\kappa}(|x|^2 < \kappa) = \rho - \rho_c(\beta)\ ;\ \lim_{\kappa\to\infty} \rho_{\,\beta,\overline{\mu}_\kappa,\kappa}(|x|^2 \geq \kappa^{1+\delta}) = 0 .$$

This means that BEC is localized in the ball of the radius $\sim \sqrt{\kappa}$.

7.3 Weak Harmonic Trap Limit $\neq$ Thermodynamic Limit

• **Proposition 7.5** Let open $\Lambda_{L=1} \subset \mathbb{R}^d$ with $|\Lambda_{L=1}| = 1$ and piecewise continuously differentiable boundary $\partial\Lambda_{L=1}$ contain the origin $\{x = 0\}$. Put $\Lambda_L := \{x \in \mathbb{R}^d : L^{-1}x \in \Lambda_{L=1}\}$, $L > 0$.
Let $\{h_L := (-\Delta)_L/2\}_{L>0}$ be self-adjoint one-particle operators with a "non-sticky" (e.g.*Dirichlet*) boundary conditions and denote by $h_{L=\infty}$ its strong resolvent limit, when $L \to \infty$.
Let $h_\kappa \to (-\Delta)_{\kappa=\infty}/2$ (in the strong resolvent sense), for $\kappa \to \infty$, denote the *Weak Harmonic Trap* limit.
Then $h_{L=\infty} = h_{\kappa=\infty} = (-\Delta)/2$, since $C_0^\infty(\mathbb{R}^d)$ is a *form-core* for $(-\Delta)/2$.
• **N.B.** But: $\mathcal{N}_{wt}(E) \neq \mathcal{N}_f(E)$, since κ^d is the *effective volume* for the **global** particle density $\rho_\kappa(\beta, \mu)$.

END

THANK YOU FOR YOUR ATTENTION !

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