

The Bethe Ansatz and the Ising model

Tony C. Dorlas

Dublin Institute for Advanced Studies, School of Theoretical Physics
10 Burlington road, Dublin 04, Ireland

Conference in honour of Prof. Dai Evans, Gregynog, Wales.

28 July 2022

1 The One-Dimensional Ising Model

Ising Hamiltonian:

$$H_N(\{s_i\}_{i=1}^N) = -J \sum_{i=1}^N s_i s_{i+1} - H \sum_{i=1}^N s_i. \quad (1)$$

(Here $s_{N+1} = s_1$.) The **partition function** is defined by

$$Z_N(\beta) = \sum_{s_1, \dots, s_N = \pm 1} e^{-\beta H_N(\{s_i\}_{i=1}^N)}, \quad (2)$$

where $\beta > 0$ is the inverse temperature (setting $k_B = 1$). The thermodynamics of the model in the **thermodynamic limit** is then given by the **free energy density**

$$f(\beta, J, H) = -\frac{1}{\beta} \lim_{N \rightarrow \infty} \frac{1}{N} \ln Z_N(\beta). \quad (3)$$

Algebraic solution

Transfer matrix expression:

$$Z_N = e^{\beta J N} (\cosh \beta H)^N \text{Tr}(A B)^N, \quad (4)$$

where $A = \mathbf{1} + \lambda \sigma^x$ with $\lambda = e^{-2\beta J}$, and $B = \mathbf{1} + u \sigma^z$ with $u = \tanh(\beta H)$.

To see this, note that we can write separately the interaction term and the magnetic field term thus

$$e^{\beta J s_i s_{i+1}} e^{\beta H s_{i+1}}.$$

Then

$$e^{\beta J s_i s_{i+1}} = e^{\beta J} (\delta_{s_i, s_{i+1}} + \lambda \delta_{s_i, -s_{i+1}}) = e^{\beta J} (\mathbf{1} + \lambda \sigma^x)_{s_i, s_{i+1}},$$

where $\lambda = e^{-2\beta J}$, and

$$e^{\beta H s_{i+1}} = \cosh(\beta H) (1 + u s_{i+1}) = \cosh(\beta H) (\mathbf{1} + u \sigma^z)_{s_{i+1}, s_{i+1}}.$$

We put $Z_N = e^{\beta J N} \cosh^N(\beta H) \tilde{Z}_N$ where

$$\tilde{Z}_N = \text{Tr}(A B)^N = \text{Tr}((\mathbf{1} + \lambda \sigma^x)(\mathbf{1} + u \sigma^z))^N. \quad (5)$$

In deriving an expression for \tilde{Z}_N , we now simply use the anti-commutation relations

$$\sigma^x \sigma^z + \sigma^z \sigma^x = 0; (\sigma^x)^2 = (\sigma^z)^2 = \mathbf{1}. \quad (6)$$

We expand the product $(A B)^N$ choosing in each of the factors $A B$ of the product the term $\mathbf{1}$ or at least one σ operator. There must be an even number of factors σ^x because otherwise the diagonal is zero, and there must also be an even number of factors σ^z because otherwise the trace is zero. Therefore let $2k$ be the number of factors where we choose at least one σ operator. From those factors we next choose among those the factors containing a σ^x at positions i_1, \dots, i_{2p} out of the total $2k$.

This yields

$$\begin{aligned}
\text{Trace}(A B)^N &= \sum_{k=0}^{[N/2]} \binom{N}{2k} \sum_{p=0}^k \sum_{1 \leq i_1 < \dots < i_{2p} \leq 2k} \lambda^{2p} u^{2k-2p} \\
&\quad \times \text{Tr} \left((\sigma^z)^{i_1-1} \sigma^x (\mathbf{1} + u\sigma^z) (\sigma^z)^{i_2-i_1-1} \dots (\sigma^z)^{2k-i_{2p}} \right).
\end{aligned} \tag{7}$$

If each second factor $\mathbf{1} + u\sigma^z$ is permuted with the previous factor σ^x it becomes $\mathbf{1} - u\sigma^z$. This can then be combined with the previous factor $\mathbf{1} + u\sigma^z$ to give $(1 - u^2)\mathbf{1}$, which, in all, results in a factor $(1 - u^2)^p$ in front of the trace. The remaining traces are all equal ± 2 .

We finally notice that, if we keep the position of the even-numbered σ^x factors fixed, and move the odd-numbered ones across the σ^z , the sign of the trace alternates. It follows that the sum over the positions of the odd-numbered factors σ^x cancels unless all i_{2j} ($j = 1, \dots, p$) are even, and in that case, the sum equals 2. There are thus $\binom{k}{p}$ possible choices for the even-numbered factors, and the result is

$$\begin{aligned}
\tilde{Z}_N &= 2 \sum_{k=0}^{[N/2]} \binom{N}{2k} \sum_{p=0}^k \binom{k}{p} \lambda^{2p} (1 - u^2)^p u^{2k-2p} \\
&= 2 \sum_{k=0}^{[N/2]} \binom{N}{2k} (u^2 + (1 - u^2)\lambda^2)^k.
\end{aligned} \tag{8}$$

Finally, we have the expansion

$$(1 + \sqrt{x})^N + (1 - \sqrt{x})^N = 2 \sum_{k=0}^{[N/2]} \binom{N}{2k} x^k, \quad (9)$$

so that

$$\tilde{Z}_N = (1 + \sqrt{u^2 + \lambda^2(1 - u^2)})^N + (1 - \sqrt{u^2 + \lambda^2(1 - u^2)})^N. \quad (10)$$

Alternatively, in the thermodynamic limit, we have the variational expression

$$\lim_{N \rightarrow \infty} \frac{1}{N} \ln \tilde{Z}_N = \sup_{x \in [0,1]} \{x \ln(u^2 + (1 - u^2)\lambda^2) - I(2x)\} \quad (11)$$

where $I(x) = x \ln x + (1 - x) \ln(1 - x)$.

The free energy density is

$$\lim_{N \rightarrow \infty} \frac{1}{N} \ln \tilde{Z}_N = \ln(1 + \sqrt{u^2 + (1 - u^2)e^{-4\beta J}}). \quad (12)$$

2 The Ising model on linked chains

In the case of M linked chains, the Hamiltonian reads

$$H_{N,M}(\{s_{i,j}\}_{i=1;j=1}^{N,M}) = -J_1 \sum_{i=1}^N \sum_{j=1}^M s_{i,j} s_{i+1,j} - J_2 \sum_{i=1}^N \sum_{j=1}^M s_{i,j} s_{i,j+1}, \quad (13)$$

where we set $s_{N+1,j} = s_{1,j}$ and $s_{i,M+1} = s_{i,1}$ for periodic boundary conditions. (We take $H = 0$.) The corresponding **partition function** is

$$Z_{N,M}(\beta) = \sum_{\{s_{i,j}\}; s_{i,j}=\pm 1} e^{-\beta H_{N,M}(\{s_{i,j}\})}. \quad (14)$$

The free energy density of the two-dimensional model is given by

$$f(\beta, J, H) = -\frac{1}{\beta} \lim_{N,M \rightarrow \infty} \frac{1}{NM} \ln Z_{N,M}(\beta). \quad (15)$$

Transfer matrix expression

Again, we can write a transfer matrix expression for $Z_{N,M}$ analogous to (4):

$$Z_{N,M}(\beta) = e^{\beta J_1 N M} \cosh(\beta J_2)^{N M} \tilde{Z}_{N,M}(\beta), \text{ with } \tilde{Z}_{N,M} = \text{Tr}(A B)^N, \quad (16)$$

where

$$\begin{aligned} A &= \prod_{j=1}^M (\mathbf{1} + \lambda \sigma_j^x) \text{ and} \\ B &= \prod_{j=1}^M (\mathbf{1} + u \sigma_j^z \otimes \sigma_{j+1}^z). \end{aligned} \quad (17)$$

Here $\sigma_j^x = \mathbf{1} \otimes \cdots \otimes \sigma^x \otimes \cdots \otimes \mathbf{1}$, with σ^x at the j -th position, and similarly, σ_j^z . Moreover, $\lambda = e^{-2\beta J_1}$ and $u = \tanh(\beta J_2)$.

3 The Ising model on a four-stranded chain

The B -operator now reads

$$B = (\mathbf{1} + u\sigma_1^z\sigma_2^z)(\mathbf{1} + u\sigma_2^z\sigma_3^z)(\mathbf{1} + u\sigma_3^z\sigma_4^z)(\mathbf{1} + u\sigma_4^z\sigma_1^z). \quad (18)$$

We consider the two eigenspaces of $\sigma^x \otimes \sigma^x \otimes \sigma^x \otimes \sigma^x$. The eigenspace with eigenvalue +1 now has three invariant subspaces. The relevant symmetric subspace is spanned by $|++++\rangle$, $\frac{1}{2}(|++--\rangle + |-++-\rangle + |--++\rangle + |+- - +\rangle)$, $\frac{1}{\sqrt{2}}(|+-+-\rangle + |-+-+\rangle)$ and $|- - - -\rangle$.

We want to write B as a tensor product on this subspace. We have obtained B as

$$\begin{aligned} \cosh(\beta J_2)^4 B &= \exp[\beta J_2 B_0], \text{ where} \\ B_0 &= \sigma_1^z\sigma_2^z + \sigma_2^z\sigma_3^z + \sigma_3^z\sigma_4^z + \sigma_4^z\sigma_1^z. \end{aligned} \quad (19)$$

On the above 4-dimensional subspace B_0 acts as follows.

$$B_0 = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 2 & 0 & 2\sqrt{2} & 2 \\ 0 & 2\sqrt{2} & 0 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix}.$$

To bring this into the form $B_1 \otimes \mathbf{1} + \mathbf{1} \otimes B_2$ where $B_i = \begin{pmatrix} a_i & b_i \\ b_i & c_i \end{pmatrix}$, using an orthogonal matrix of the form affecting only the states with total spin 0, we write

$$B_1 \otimes \mathbf{1} + \mathbf{1} \otimes B_2 = \begin{pmatrix} a_1 + a_2 & b_2 & b_1 & 0 \\ b_2 & a_1 + c_2 & 0 & b_1 \\ b_1 & 0 & c_1 + a_2 & b_2 \\ 0 & b_1 & b_2 & c_1 + c_2 \end{pmatrix}.$$

It follows that we must diagonalise the centre matrix, i.e. $\begin{pmatrix} 0 & 2\sqrt{2} \\ 2\sqrt{2} & 0 \end{pmatrix}$.

We obtain

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } UB_0U = \begin{pmatrix} 0 & \sqrt{2} & \sqrt{2} & 0 \\ \sqrt{2} & 2\sqrt{2} & 0 & \sqrt{2} \\ \sqrt{2} & 0 & -2\sqrt{2} & \sqrt{2} \\ 0 & \sqrt{2} & \sqrt{2} & 0 \end{pmatrix}$$

and hence

$$B_1 = \sqrt{2}(\sigma^x + \sigma^z) \text{ and } B_2 = \sqrt{2}(\sigma^x - \sigma^z).$$

It follows that

$$U B U = ((1 + u^2)\mathbf{1} + \sqrt{2}\sigma^z + \sqrt{2}\sigma^x) \otimes ((1 + u^2)\mathbf{1} - \sqrt{2}\sigma^z + \sqrt{2}\sigma^x). \quad (20)$$

Note that the matrix of A is unaffected by the transformation U and can be written as

$$A = ((1 + \lambda^2)\mathbf{1} + 2\lambda\sigma^z) \otimes ((1 + \lambda^2)\mathbf{1} + 2\lambda\sigma^z). \quad (21)$$

Each factor can now be diagonalized individually and the result is

$$\begin{aligned} \text{Tr}(A B_{+,even})^N &= \left(\sum_{\pm} \left[(1 + \lambda^2)(1 + u^2) + 2\sqrt{2}u\lambda \pm \sqrt{\Delta(\frac{\pi}{4})} \right]^N \right) \\ &\quad \times \left(\sum_{\pm} \left[(1 + \lambda^2)(1 + u^2) - 2\sqrt{2}u\lambda \pm \sqrt{\Delta(\frac{3\pi}{4})} \right]^N \right), \end{aligned} \quad (22)$$

where

$$\Delta(\frac{\pi}{4}) = [(1 + \lambda^2)(1 + u^2) + 2\sqrt{2}u\lambda]^2 - (1 - u^2)^2(1 - \lambda^2)^2 \quad (23)$$

and

$$\Delta(\frac{3\pi}{4}) = [(1 + \lambda^2)(1 + u^2) - 2\sqrt{2}u\lambda]^2 - (1 - u^2)^2(1 - \lambda^2)^2. \quad (24)$$

4 The two-dimensional Ising model

The general case with M chains is of course equivalent to the 2-dimensional Ising model. To generalize the above approach, we want to transform B into a tensor product of 2-dimensional matrices. Equivalently, since $\cosh(\beta J_2)^M B = \exp(\beta J_2 B_0)$, we need to find a transformation such that B_0 has the form

$$B_0 = \sum_{i=1}^{[M/2]} B_i, \text{ where } B_i = \mathbf{1} \otimes \cdots \otimes \begin{pmatrix} a_i & b_i \\ b_i & -a_i \end{pmatrix} \otimes \cdots \otimes \mathbf{1}.$$

(Here the matrix $\begin{pmatrix} a_i & b_i \\ b_i & -a_i \end{pmatrix}$ is at the i -th position.)

We can subdivide the Hilbert space $\mathcal{H} = \mathbb{C}^{2^M}$ into subspaces \mathcal{H}_n where $\oplus_{i=1}^M \sigma_i^x$ has eigenvalue $M - 2n$ with $n \leq M/2$, i.e. in the representation in which σ^x is diagonal the number of minuses equals n . On the subspace \mathcal{H}_n , A has the eigenvalue $(1 + \lambda)^{M-n}(1 - \lambda)^n$. We can therefore diagonalize the restriction \tilde{B}_0 of B_0 to each \mathcal{H}_n as we did in the case $M = 4$ above. This does not affect the matrix A . Note also that B_0 only connects \mathcal{H}_n with \mathcal{H}_{n-2} and \mathcal{H}_{n+2} .

5 The Bethe Ansatz

The operator \tilde{B}_0 leaves the number of minus signs n invariant. It can be diagonalized using the Bethe Ansatz.

let $\varphi(x_1, \dots, x_n)$ denote the basis vector with minus signs at the positions x_1, \dots, x_n , where $1 \leq x_1 < \dots < x_n \leq M$. We write the eigenvectors as

$$\psi = \sum_{1 \leq x_1 < \dots < x_n \leq M} f(x_1, \dots, x_n) \varphi(x_1, \dots, x_n). \quad (25)$$

where the functions f are assumed to be of the Bethe form:

$$f(x_1, \dots, x_n) = \sum_{P \in \mathcal{S}_n} A(P) \prod_{j=1}^n \omega_{P(j)}^{x_j}. \quad (26)$$

The coefficients $A(P)$ are to be determined as well as the numbers ω_j ($j = 1, \dots, n$).

We first write the general expression for \tilde{B}_0 on the n -particle space:

$$\begin{aligned}
\tilde{B}_n f(x_1, \dots, x_n) = & \sum_{i=1}^n (1 - \delta_{x_i - x_{i-1}, 1}) f(x_1, \dots, x_i - 1, \dots, x_n) \\
& + \sum_{i=1}^n (1 - \delta_{x_{i+1} - x_i, 1}) f(x_1, \dots, x_i + 1, \dots, x_n) \\
& + \delta_{x_1, 1} (1 - \delta_{x_n, M}) f(x_2, \dots, x_n, M) \\
& + \delta_{x_n, M} (1 - \delta_{x_1, 1}) f(1, x_1, \dots, x_{n-1}), \tag{27}
\end{aligned}$$

where we set $x_0 = 0$ and $x_{n+1} = M + 1$.

Inserting into the eigenvalue equation for \tilde{B}_0 , one finds that the eigenfunctions of \tilde{B}_0 are given by

$$f(x_1, \dots, x_n) = \frac{1}{M^{n/2}} \sum_{P \in S_n} (-1)^{|P|} \prod_{j=1}^n \omega_{P(j)}^{x_j}, \quad (28)$$

where the ω_i ($i = 1, \dots, n$) are distinct M -th roots of $(-1)^{n-1}$. The corresponding eigenvalues are

$$\tilde{B}_0 f(x_1, \dots, x_n) = \lambda f(x_1, \dots, x_n); \quad \lambda = \sum_{j=1}^n (\omega_j + \omega_j^{-1}).$$

Next, we need to compute the matrix elements of B_0 connecting \mathcal{H}_n and \mathcal{H}_{n-2} . The corresponding matrix $C_n = P_{n-2}B_0|_{\mathcal{H}_n}$ is given by

$$\begin{aligned} (Cf_n)(x_1, \dots, x_{n-2}) &= \sum_{j=0}^{n-2} \sum_{x=x_j+1}^{x_{j+1}-2} f_n(x_1, \dots, x_j, x, x+1, x_{j+1}, \dots, x_{n-2}) \\ &\quad + f_n(1, x_1, \dots, x_{n-2}, M) (1 - \delta_{x_1, 1})(1 - \delta_{x_{n-2}, M}), \end{aligned} \tag{29}$$

where $x_0 = 0$ and $x_{n-1} = M + 1$. We therefore have to compute $\langle f_{n-2} | Cf_n \rangle$, where f_n is given by (28) and

$$f_{n-2}(x_1, \dots, x_{n-2}) = \frac{1}{M^{(n-2)/2}} \sum_{Q \in S_{n-2}} (-1)^{|Q|} \prod_{i=1}^{n-2} (\omega'_{Q(i)})^{x_i}.$$

The result is that the scalar product $\langle f_{n-2} | C_n f_n \rangle$ equals zero unless among the ω_j ($j = 1, \dots, n$) defining f_n there are $n - 2$ which are equal to the ω'_i defining f_{n-2} , and the remaining two are complex conjugates. In that case, the corresponding matrix element equals $\omega - \bar{\omega}$, where ω and $\bar{\omega}$ are the remaining two ω_j .

The complete matrix for B_0 .

We can thus write the complete matrix for B_0 on the basis of BA eigenvectors of \tilde{B}_0 . For the case that M and n are even, it is

$$\begin{aligned}
\langle f'_{2k} | B_0 f_{2k} \rangle &= \begin{cases} \sum_{p=1}^n (\omega_{j_p} + \omega_{j_p}^{-1}) & \text{if } \omega'_{j_p} = \omega_{j_p} \text{ for all } p = 1, \dots, k; \\ 0 & \text{otherwise;} \end{cases} \\
\langle f'_{2k-2} | B_0 f_{2k} \rangle &= \begin{cases} \omega_j - \omega_j^{-1} & \text{if } \{\omega_{j_p}\}_{p=1}^k = \{\omega'_{j_q}\}_{q=1}^{k-1} \cup \{\omega_j\} \\ 0 & \text{otherwise.} \end{cases} \\
\langle f'_{2k} | B_0 f_{2k-2} \rangle &= \overline{\langle f_{2k-2} | B_0 f'_{2k} \rangle}, \\
\langle f'_{2l} | B_0 f_{2k} \rangle &= 0 \text{ if } |k - l| > 1.
\end{aligned} \tag{30}$$

Relabelling the BA eigenvectors

We now label the vectors f_n defined by $(\omega_{j_1}, \dots, \omega_{j_n})$ such that $j_{n+1-p} = j_p$, ($p = 1, \dots, n$) by a sequence $(s_1, \dots, s_{M/2})$ where s_j is an Ising spin such that $s_j = +1$ if $\omega_j \in \{\omega_{j_1}, \dots, \omega_{j_n}\}$ and $s_j = -1$ if $\omega_j \notin \{\omega_{j_1}, \dots, \omega_{j_n}\}$. We write this vector as $|\{s_j\}_{j=1}^{M/2}\rangle$. On this basis, B_0 has the matrix elements

$$\langle s'_1, \dots, s'_{M/2} | B_0 | s_1, \dots, s_{M/2} \rangle = \begin{cases} 4 \sum_{j=1}^{M/2} \delta_{s'_j, 1} \cos \frac{(2j-1)\pi}{M} & \text{if } s'_j = s_j \text{ for all } j = 1, \dots, M/2; \\ 2 \sin \frac{(2j-1)\pi}{M} & \text{if } s'_j s_j = -1 \text{ and } s'_i = s_i \text{ for } i \neq j; \\ 0 & \text{otherwise.} \end{cases} \quad (31)$$

This is just the matrix

$$\overline{B_0} = \sum_{j=1}^{M/2} (\mathbf{1} \otimes \cdots \otimes B_j \otimes \cdots \otimes \mathbf{1}), \quad (32)$$

where the factor B_j appears in the j -th position and equals

$$\begin{aligned} B_j &= 2 \begin{pmatrix} \cos \frac{(2j-1)\pi}{M} & \sin \frac{(2j-1)\pi}{M} \\ \sin \frac{(2j-1)\pi}{M} & -\cos \frac{(2j-1)\pi}{M} \end{pmatrix} \\ &= 2 \cos \theta_{2j-1} \sigma^z + 2 \sin \theta_{2j-1} \sigma^x, \end{aligned} \quad (33)$$

where

$$\theta_r = \frac{r\pi}{M}. \quad (34)$$

This reduces the problem to the diagonalization of 2×2 matrices. The resulting contribution to $\tilde{Z}_{N,M}$:

$$\tilde{Z}_{\max,+} = \prod_{j=1}^{M/2} (\zeta_{2j-1,+}^N + \zeta_{2j-1,-}^N), \quad (35)$$

where

$$\begin{cases} \zeta_{r,\pm} = (1 + \lambda^2)(1 + u^2) - 4u\lambda \cos \frac{r\pi}{M} \pm \sqrt{\Delta_r}, & \text{where} \\ \Delta_r = \left[(1 + \lambda^2)(1 + u^2) - 4u\lambda \cos \frac{r\pi}{M} \right]^2 - (1 - \lambda^2)^2(1 - u^2)^2. \end{cases} \quad (36)$$

6 Thermodynamic limit

The thermodynamic limit is given by

$$\begin{aligned}
\lim_{N, M \rightarrow \infty} \frac{1}{NM} \ln \tilde{Z}_{N, M} &= \lim_{M \rightarrow \infty} \frac{1}{M} \max \left\{ \sum_{j=1}^{[M/2]} \ln \zeta_{2j-1, +}, \sum_{j=1}^{[M/2]} \ln \zeta_{2j, +} \right\} \\
&= \frac{1}{2\pi} \int_0^\pi d\theta \ln \zeta(\lambda, u; \theta),
\end{aligned} \tag{37}$$

where

$$\begin{aligned}
\zeta(\lambda, u; \theta) &= (1 + \lambda^2)(1 + u^2) - 4u\lambda \cos \theta \\
&\quad + \sqrt{[(1 + \lambda^2)(1 + u^2) - 4u\lambda \cos \theta]^2 - (1 - \lambda^2)^2(1 - u^2)^2}.
\end{aligned} \tag{38}$$