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Lectures on Pirogov-Sinai Theory

By

T. C. Dorlas

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Preface

In 2017, I gave a series of 4 lectures on Pirogov-Sinai theory, following the book by Sinai[1] which is based on the two ground-breaking articles by Pirogov and Sinai[2]. At the time I was unhappy that there were certain details that I did not fully understand. During the time in lockdown due to the COVID-19 epidemic in 2020, I therefore decided to write up more detailed notes of these lectures. In doing so, I discovered that there were in fact more aspects of the arguments which were rather sketchy. Elaborating these details further, these notes will hopefully be useful for others who are trying to understand this important development in the theory of phase transitions. It should be noted that a slightly different approach to the theory was developed by Zahradnik[3]. There are also extensions to certain quantum lattice models[4]

[1]Ya. G. Sinai: *Theory of Phase Transitions: Rigorous Results.* Oxford etc.: Pergamon Press, 1982

[2]S. A. Pirogov and Ya. G. Sinai: Phase Diagrams of Classical Lattice Systems I, II, *Teor. Mat. Fiz.* **25** 358–69 (1975) and *Teor. Mat. Fiz.* **26**, 61–76 (1976). (In Russian)

[3]M. Zahradnik: An Alternative Version of Pirogov-Sinai Theory. *Commun. Math. Phys.* **93**, 559–81 (1984).

[4]See C. Borgs, R. Kotecky and D. Ueltschi: Low Temperature Phase Diagrams for Quantum Perturbations of Classical Spin Systems. *Commun. Math. Phys.* **181** 409–46 (1996) and N. Datta, R. Fernandez and J. Fröhlich: Low-temperature phase diagrams of quantum lattice systems. I. Stability for quantum perturbations of classical systems with finitely many ground states, *J. Stat. Phys.* **84**, 455–534 (1996).

1 The Peierls Argument

Pirogov-Sinai theory is a generalisation of the Peierls argument to classical lattice spin models without symmetry. We therefore begin by reviewing the Peierls argument for the Ising model.

The Ising model is a classical spin system with spins $s_x = \pm 1$ on $\mathbb{Z}^d$ with nearest-neighbour interaction given by a Hamiltonian

$$\mathcal{H}_\Lambda(s_\Lambda) = -J \sum_{\{x,y\}\subset\Lambda:\,|x-y|=1} s_x s_y - h \sum_{x\in\Lambda} s_x, \tag{1.1}$$

where Λ is a (large) finite subset of $\mathbb{Z}^d$, and s_Λ denotes the spin configuration $\{s_x\}_{x\in\Lambda}$ on Λ. h is an external magnetic field. The Gibbs measure μ_β at inverse temperature β corresponding to this Hamiltonian is defined by

$$\mu_\Lambda^\beta(A) = \frac{1}{Z_\Lambda(\beta)} \sum_{s_\Lambda\in A} e^{-\beta\mathcal{H}_\Lambda(s_\Lambda)}, \tag{1.2}$$

for any subset $A \subset \Omega(\Lambda) = \{-1,+1\}^\Lambda$, where

$$Z_\Lambda(\beta) = \sum_{s_\Lambda\in\Omega(\Lambda)} e^{-\beta\mathcal{H}_\Lambda(s_\Lambda)} \tag{1.3}$$

is a normalisation factor called the **partition function**.

Peierls[5] showed that in dimensions $d \geq 2$, this model has a phase transition for $h = 0$ at low temperatures. To be exact, he showed that there is spontaneous magnetization for low temperatures, in the sense that if we assume positive boundary conditions, changing the Hamiltonian to

$$\mathcal{H}_\Lambda(s_\Lambda\,|\,+) = -J \sum_{\{x,y\}\subset\Lambda:\,|x-y|=1} s_x s_y - J \sum_{x\in\partial\Lambda} \sum_{y\in\Lambda^c:\,|x-y|=1} s_x,$$

then the expected value of the spin s_0 (where $0 \in \Lambda$) w.r.t. the Gibbs measure $\mu_\Lambda^{\beta,+}$ is bounded below by a positive constant $m_0(\beta) > 0$ as Λ increases in all directions.

[5] R. Peierls: On the Ising model of ferromagnetism. *Proc. Cambridge Phil. Soc.* **32**, 477–81 (1936). See also R. B. Griffiths: Peierls proof of spontaneous magnetization in a two-dimensional Ising ferromagnet, *Phys. Rev.* A**136**, 437–9 (1964), and R. L. Dobrushin: Existence of a phase transition in the two-dimensional and three-dimensional Ising models, *Sov. Phys. Doklady* **10**, 111–3 (1965).

In fact, it is known (by a simple compactness argument) that there exist so-called **limit-Gibbs measures** μ^β on $\{-1,+1\}^{\mathbb{Z}^d}$ given by limits of Gibbs measures $\mu^\beta_{\Lambda_l}$ where $(\Lambda_l)_{l\in\mathbb{N}}$ is a sequence of subsets $\Lambda_l \subset \mathbb{Z}^d$ tending to $\mathbb{Z}^d$ uniformly in all directions. One can then formulate Peierls' result in the following way. There exists $\beta_0 > 0$ such that for $\beta > \beta_0$ there exist two translationally-invariant limit-Gibbs states $\mu^\beta_\pm$ such that

$$\mu^\beta_\pm(s_0) = \pm m_0 \text{ where } m_0 > 0.$$

Peierls' proof depends on a reformulation of the Ising model in terms of **contours**.

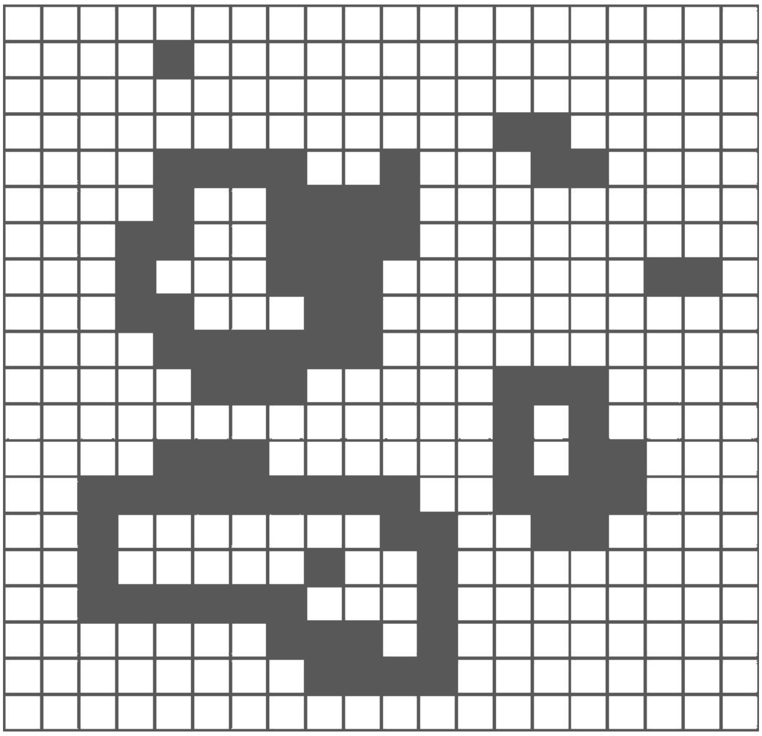

Figure 1 Ising model configuration:
Black squares represent minus spins, white ones plus spins.

A configuration of spins on the lattice can equivalently be described by a collection of contours composed of lines or plaquettes between lattice points separating opposite spins. A contour is defined as a maximal collection of connected plaquettes which does not self-intersect. Given positive (or negative) boundary conditions, these lines/plaquettes form closed curves/surfaces.

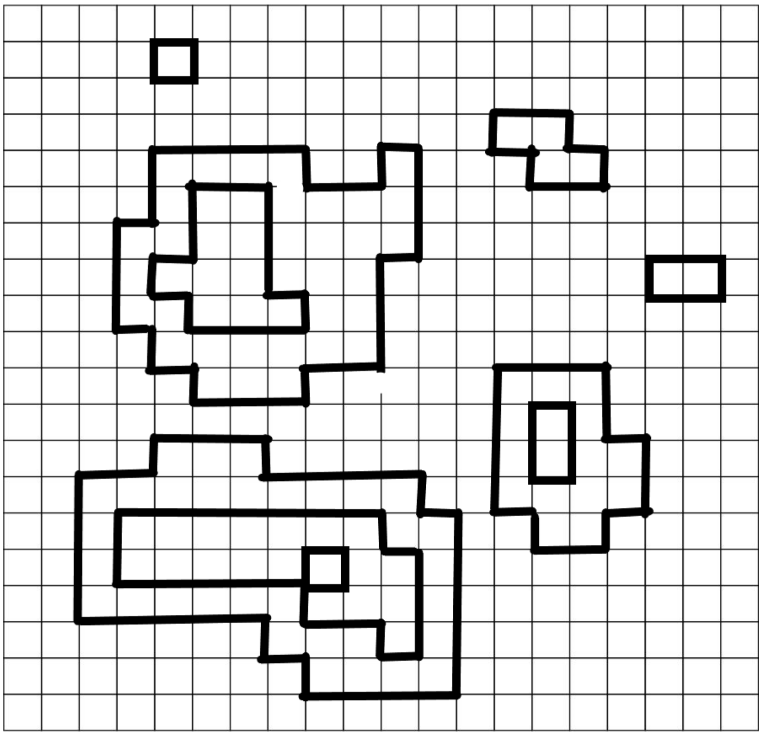

Figure 2 Contours for the above Ising configuration.

We denote the contours of a configuration s_Λ on Λ by $\partial(s_\Lambda)$. The energy function is then given by

$$\mathcal{H}_\Lambda^+ = -dJ|\Lambda| + 2J \sum_{\gamma \in \partial(s_\Lambda)} |\gamma| + h\Big(\sum_{\gamma \in \partial^+(s_\Lambda)} |\mathrm{Int}(\gamma)| - \sum_{\gamma \in \partial^-(s_\Lambda)} |\mathrm{Int}(\gamma)|\Big), \quad (1.4)$$

where $|\gamma|$ is the length/area of the contour γ, $\partial^\pm(s_\Lambda)$ denotes the contours with outer boundary equal to ± 1, and $|\mathrm{Int}(\gamma)|$ is the total number of lattice sites enclosed by the contour. (We use the fact that the total number of pairs $\{x, y\}$ with $\{x, y\} \cap \Lambda \neq \emptyset$ equals $2|\Lambda|$.)

Now suppose that $h = 0$. The corresponding Gibbs measure in Λ is then given by

$$\mu_\Lambda^+(s_\Lambda) = \frac{1}{Z_\Lambda^+(\beta)} \exp\left\{ \beta J \sum_{\{x,y\} \cap \Lambda \neq \emptyset :\, ||x-y||=1} s_x s_y \right\}, \quad (1.5)$$

where $s_x = +1$ if $x \notin \Lambda$.

We claim:

Lemma 1.1 (Peierls) *Let γ be a given contour. Then*

$$\mu_\Lambda^+(\{s_\Lambda :\ \gamma \in \partial(s_\Lambda)\}) \le e^{-2\beta J|\gamma|}, \tag{1.6}$$

where $|\gamma| = \ell(\gamma)$ denotes the total length/area of the contour γ.

Proof. Set $|\partial(s_\Lambda)| = \sum_{\gamma \in \partial} |\gamma|$. Then

$$\mu_\Lambda^+(\{s_\Lambda :\ \gamma \in \partial(s_\Lambda)\}) = \frac{\sum_{s_\Lambda :\, \gamma \in \partial(s_\Lambda)} e^{-2\beta J|\partial(s_\Lambda)|}}{\sum_{s_\Lambda} e^{-2\beta J|\partial(s_\Lambda)|}}. \tag{1.7}$$

Denote the set of configurations on Λ by $\Omega(\Lambda) = \{-1, +1\}^\Lambda$, and let $\Omega_\gamma(\Lambda) \subset \Omega(\Lambda)$ be the subset of configurations s_Λ such that $\gamma \in \partial(s_\Lambda)$. Then there is a bijective map π_γ between $\Omega_\gamma(\Lambda)$ and its complement defined by flipping all spins inside γ. Given $s_\Lambda \in \Omega_\gamma(\Lambda)$, obviously $|\partial(s_\Lambda)| = |\partial(\pi_\gamma(s_\Lambda))| + |\gamma|$, since in $\pi_\gamma(s_\Lambda)$ the contour γ is missing. We therefore have

$$\mu_\Lambda^+(\{s_\Lambda :\ \gamma \in \partial(s_\Lambda)\}) = e^{-2\beta J|\gamma|} \frac{\sum_{s_\Lambda \in \Omega_\gamma(\Lambda)^c} e^{-2\beta J|\partial(s_\Lambda)|}}{\sum_{s_\Lambda} e^{-2\beta J|\partial(s_\Lambda)|}} \le e^{-2\beta J|\gamma|}. \tag{1.8}$$

∎

The existence of spontaneous magnetization now follows by counting contours:

Theorem 1.1 (Peierls) *There exists $\beta_0 > 0$ independent of Λ such that for all $\beta > \beta_0$ there is $m_0(\beta) > 0$ such that*

$$\mu_\Lambda^+(\{s_\Lambda \in \Omega(\Lambda) :\ s_0 = -1\}) \le \frac{1}{2}(1 - m_0(\beta)). \tag{1.9}$$

Proof. If $s_0 = -1$ then the origin must be surrounded by at least one contour γ. By the lemma, we therefore have

$$\mu_\Lambda^+(\{s_\Lambda :\ s_0 = -1\}) \le \sum_{\gamma :\, 0 \in \mathrm{Int}(\gamma)} e^{-2\beta J|\gamma|}.$$

Clearly, the smallest possible contour surrounding 0 has area $2d$, so if C_n is the number of contours of area n then

$$\mu_\Lambda^+(\{s_\Lambda :\ s_0 = -1\}) \le \sum_{n=2d}^{\infty} C_n\, e^{-2\beta J n}.$$

It is easy to estimate the number C_n of contours. There is a plaquette perpendicular to the first coordinate axis crossing that axis at minimal positive distance r_1 from the origin. Obviously, $2(d-1)r_1 < n$. Starting from that plaquette, we can build up the contour by choosing a direction at every edge of subsequent plaquettes. There are 3 possible directions in which to place the next plaquette. Therefore $C_n \le \frac{n}{2(d-1)} 3^n$. It follows that

$$\mu_\Lambda^+(\{s_\Lambda :\ s_0 = -1\}) \le \sum_{n=2d}^{\infty} \frac{n}{2(d-1)} 3^n\, e^{-2\beta J n}.$$

Clearly, the right-hand side converges for $\beta J > \frac{1}{2}\ln(3)$ and tends to 0 for $\beta \to \infty$, uniformly in Λ. ∎

At low temperatures the contours are likely to be small:

Corollary 1.1 *There is $\beta_0 > 0$ such that for $\beta > \beta_0$ there is a constant $c(\beta) > 0$ such that $\lim_{\beta\to+\infty} c(\beta) = 0$ and*

$$\lim_{\Lambda\to\mathbb{Z}^d} \mu_\Lambda^+\big[\exists\gamma \in \partial(s_\Lambda) :\ |\gamma| > c(\beta)\ln|\Lambda|\big] = 0.$$

Proof. As in the above proof, we have

$$\begin{aligned}
&\mu_\Lambda^+\left[\exists\gamma \in \partial(s_\Lambda) :\ |\gamma| > c(\beta)\ln|\Lambda|\right] \\
&\qquad \le |\Lambda| \sum_{n > c(\beta)\ln|\Lambda|} \frac{n}{2(d-1)} 3^n e^{-2\beta J n} \\
&\qquad \le \frac{c(\beta)}{2(d-1)} |\Lambda| \ln(|\Lambda|) \frac{e^{(\ln 3 - 2\beta J)c(\beta)\ln|\Lambda|}}{(1-3e^{-2\beta J})^2} \to 0
\end{aligned}$$

if $(2\beta J - \ln 3)c(\beta) > 1$. ∎

Note that in Peierls' proof the *spin-flip invariance* of the model is an essential ingredient. Pirogov-Sinai theory is an extension of Peierls' idea of writing the Hamiltonian in terms of contours to more general classical spin systems. However, because in general the number of spin values is greater than 2 and the spin-flip symmetry is absent, the phase transition(s) in general do not occur at zero external field(s). The challenge is then to determine the critical values of these fields. These can only be obtained in the thermodynamic limit. We therefore consider first the thermodynamic limit of classical spin models in general.

2 Classical Spin Systems

2.1 Invariant states

Pirogov-Sinai theory holds for periodic states, but for simplicity we only consider translation-invariant Hamiltonians and states here. Some of the proofs are omitted: see for example Israel or Hugenholtz[6]. Let the spins take values in a finite set $S = \{1, \ldots, q\}$ and denote $\Omega = S^{\mathbb{Z}^d}$. The set of translation-invariant probability measures on Ω is a compact convex set (w.r.t. the topology of weak convergence). We denote it by $\mathcal{P}_I$. We can characterize its extremal points as follows.

Theorem 2.1 *An invariant probability measure $\mu \in \mathcal{P}_I$ is extremal invariant if and only if the (orthogonal) projection P_μ on the set of invariant functions $f \in L^2(\Omega, \mu)$ is 1-dimensional, i.e.*

$$P_\mu f = \left(\int f \, d\mu \right) 1_\Omega.$$

Proof. Suppose that μ is extremal invariant. Then any operator $B \in \mathcal{B}(L^2(\mu))$ commuting with the multiplication operators $\mathcal{M} = \{M_f;\, f \in \mathcal{C}(\Omega)\}$ and the translations must be a multiple of the identity. For, if B is not a multiple of the identity, it has a non-trivial spectral projection P, which also commutes with $\mathcal{M}$ and the translations. In that can we can define translation-invariant probability measures μ_1 and μ_2 by

$$\int f \, d\mu_1 = \frac{\int |P1|^2 f \, d\mu}{\int |P1|^2 \, d\mu} \text{ and } \int f \, d\mu_2 = \frac{\int |(\mathbf{1} - P)1|^2 f \, d\mu}{\int |(\mathbf{1} - P)1|^2 \, d\mu}$$

so that $\mu = c\mu_1 + (1-c)\mu_2$ with $c = \int |P1|^2 \, d\mu = ||P1||^2 < 1$.

Now consider the space $P_\mu(L^2(\mu))$. This space is translation-invariant because P_μ commutes with translations. There are therefore translation-

[6] R. B. Israel: *Convexity in the Theory of Lattice Gases*, Princeton University Press, 1979, and N. M. Hugenholtz: *C^*-algebras and Statistical Mechanics.* In: *Operator Algebras and Applications* (Proc. Symp. Pure Math. **38**, Part 2. pp. 407–65. R. V. Kadison (ed). Providence RI: American Mathematical Society).

invariant functions $f_n \in \mathcal{C}(\Omega)$ converging to $\psi \in P_\mu(L^2(\mu))$. Now M_{f_n} commutes with $\mathcal{M}$ and with translations, so must be a multiple of the identity. Hence $f_n = \lambda_n$ is a constant and it follows that $\psi = \lambda$ is also a constant. Consequently, $P_\mu f = \lambda(f)$. Since $\langle P_\mu f \mid P_\mu f \rangle = \langle f \mid P_\mu f \rangle$, we have $\lambda(f)^2 = \lambda(f) \int f \, d\mu$.

Conversely, suppose that P_μ is one-dimensional: $P_\mu f = \int f \, d\mu$ and $\mu = \lambda\mu_1 + (1-\lambda)\mu_2$ for $\mu_1, \mu_2 \in \mathcal{P}_I$ and $\lambda \in (0,1)$. Assuming that $\mu_1 \neq \mu_2$ the measure $\nu = \mu_1 - \mu_2$ is neither positive nor negative. By the Hahn decomposition there exist subset $\Omega_+, \Omega_- \subset \Omega$ such that $\nu|_{\Omega_+}$ is positive and $\nu|_{\Omega_-}$ is negative. Moreover, since ν is translation-invariant, $\Omega_\pm$ are also translation-invariant. We can now write $\mu = \mu(\Omega_+)\mu_+ + \mu(\Omega_-)\mu_-$ where $\mu_\pm = 1_{\Omega_\pm}\mu/\mu(\Omega_\pm)$. Now the functions $1_{\Omega_\pm}$ are translation-invariant and $P_\mu = P_+ + P_-$, where $P_\pm = M_{1_{\Omega_\pm}}$ are orthogonal projections, i.e. P_μ is not 1-dimensional. ∎

There is another characterization of extremal invariant states as ergodic states.

First we define an **M-net** as follows. It is a net (h_α) of functions $h_\alpha : \mathbb{Z}^d \to [0, +\infty)$ such that

$$\sum_{x \in \mathbb{Z}^d} h_\alpha(x) = 1 \text{ and } \lim_\alpha \sum_{x \in \mathbb{Z}^d} |h_\alpha(x+y) - h_\alpha(x)| = 0$$

for all $y \in \mathbb{Z}^d$. An example is $h_n(x) = \frac{1}{n^d}\mathbf{1}_{K_n(x_0)}$ where

$$K_n(x_0) = \{x \in \mathbb{Z}^d : -\tfrac{1}{2}n < x_i - x_{0,i} \leq \tfrac{1}{2}n \ (i = 1, \ldots, d)\}$$

is a cube centred at x_0 of side n. The mean ergodic theorem states:

Theorem 2.2 (mean ergodic theorem) *If $x \mapsto U(x)$ is a unitary representation of $\mathbb{Z}^d$ on a Hilbert space $\mathfrak{H}$ then*

$$\lim_\alpha \sum_{x \in \mathbb{Z}^d} h_\alpha(x) U(x) = P,$$

where P is the projection onto the $\mathbb{Z}^d$-invariant vectors. Here the convergence is in the strong operator topology.

We have

Theorem 2.3 *An invariant probability measure $\mu \in \mathcal{P}_I$ is extremal invariant if and only if it is* ***weakly clustering****, i.e. for all $f, g \in C(\Omega)$ and all M-nets (h_α),*

$$\lim_\alpha \sum_{x \in \mathbb{Z}^d} h_\alpha(x) \int (\tau_x f)\, g\, d\mu = \int f\, d\mu \int g\, d\mu,$$

where τ_x is the translation operator, $(\tau_x f)(y) = f(y - x)$.

Proof. Suppose first that μ is weakly clustering. Then by the mean ergodic theorem,

$$\lim_\alpha \sum_{x \in \mathbb{Z}^d} h_\alpha(x) \int (\tau_x f)\, g\, d\mu = \int (P_\mu f)\, g\, d\mu = \int f\, d\mu \int g\, d\mu.$$

Since this holds for all $f, g \in \mathcal{C}(\Omega)$, it follows that $P_\mu f = (\int f\, d\mu) 1_\Omega$.

Conversely, suppose that μ is extremal invariant. Then $P_\mu f = \int f\, d\mu$ and we have

$$\begin{aligned} \lim_\alpha \sum_{x \in \mathbb{Z}^d} h_\alpha(x) \int (\tau_x f)\, g\, d\mu &= \lim_\alpha \sum_{x \in \mathbb{Z}^d} h_\alpha(x) \langle \overline{g} \,|\, U(x) f \rangle \\ &= \langle \overline{g} \,|\, P_\mu f \rangle = \langle \overline{g} \,|\, 1 \rangle \int f\, d\mu \\ &= \int g\, d\mu \int f\, d\mu. \end{aligned}$$

∎

A stronger version of this property is the following. An invariant measure $\mu \in \mathcal{P}_I$ is called **strongly clustering** if for all $f, g \in \mathcal{C}(\Omega)$,

$$\lim_{|x| \to \infty} \int (\tau_x f)\, g\, d\mu = \int f\, d\mu \int g\, d\mu. \tag{2.1}$$

It is easy to see that this implies weak clustering. It is this property that we shall use to prove the existence of pure phases.

EXAMPLE 1.1 *Example of a mixed state*

Consider the following mixture of two measures: $\mu = \frac{1}{2}(\mu_+ + \mu_-)$ where $\mu_\pm$ are product measures given by $\mu_\pm(d\underline{s}) = \prod_{x\in\mathbb{Z}^d} \mu_\epsilon^\pm(ds_x)$ and $\mu_\epsilon^\pm$ is a measure on $\{-1,+1\}$ given by

$$\mu_\epsilon^\pm(s_x = s) = \tfrac{1}{2} \pm \tfrac{1}{2}(1-2\epsilon)s = \begin{cases} 1-\epsilon & \text{if } s = \pm 1; \\ \epsilon & \text{if } s = \mp 1. \end{cases}$$

where we assume $\epsilon \ll 1$. The measures $\mu_\pm$ are extremal invariant. Indeed, we may assume that $f(\underline{s})$ and $g(\underline{s})$ only depend on finite sets of spins, i.e. $f(\underline{s}) = f(s_\Lambda)$ and $g(\underline{s}) = g(s_{\Lambda'})$ (so-called cylinder functions). This is because the cylinder functions are dense in $\mathcal{C}(\Omega)$ and

$$\left|\int (\tau_x f)\, g\, d\mu\right| \leq ||f||\,||g||.$$

But if f only depends on s_Λ then for $|x|$ large enough, $\tau_x f$ depends on $s_{\Lambda+x}$ where $(\Lambda+x)\cap\Lambda' = \emptyset$. Hence

$$\begin{aligned}
&\int (\tau_x f)(s_\Lambda) g(s_{\Lambda'})\, \mu^+(d\underline{s}) \\
&= \sum_{\underline{s}\in\Omega} f(s_{\Lambda+x}) g(s_{\Lambda'}) \prod_{y\in\mathbb{Z}^d} \mu_\epsilon^+(s_y) \\
&= \sum_{s_{\Lambda+x}} f(s_{\Lambda+x}) \prod_{y\in\Lambda+x} \mu_\epsilon^+(s_y) \sum_{s_{\Lambda'}} g(s_{\Lambda'}) \prod_{y\in\Lambda'} \mu_\epsilon^+(s_y) \\
&= \int f(\underline{s})\, \mu_+(d\underline{s}) \int g(\underline{s}) \mu_+(d\underline{s})
\end{aligned}$$

and similarly for μ_-. Note that the measure $\mu_\pm$ is concentrated on the subset $\Omega_\pm$ of Ω given by the configurations $\underline{s}$ such that the (average) magnetization equals

$$\lim_\alpha \sum_{x\in\mathbb{Z}^d} h_\alpha(x) s_x = \pm(1-2\epsilon).$$

These are disjoint translation-invariant sets, so P_μ is the projection onto the two-dimensional subspace spanned by 1_{Ω_+} and 1_{Ω_-}. The measure μ is not clustering. For example, for $z \neq 0$, $\mathbb{E}^\mu(s_0 s_z) = \frac{1}{2}\int s_0 s_z\, d(\mu_+ + \mu_-) = (1-2\epsilon)^2$ whereas $\mathbb{E}^\mu(s_0) = 0 = \mathbb{E}^\mu(s_z)$.

2.2 Thermodynamic functions

The existence of thermodynamic functions for classical spin systems is quite standard, so it suffices to be brief. We consider a Hamiltonian given by

$$\mathcal{H}_\Lambda(\Phi)(s_\Lambda) = \sum_{X \subset \Lambda} \Phi_X(s_X), \tag{2.2}$$

where Φ_X are a potential functions, $\Phi_X : \Omega_X \to \mathbb{R}$ (where $\Omega_X = S^X$) which we assume to be translation-invariant and to satisfy

$$||\Phi||_1 = \sum_{X \subset \mathbb{Z}^d :\, 0 \in X} \frac{||\Phi_X||}{|X|} < +\infty.$$

(Here $||\Phi_X|| = \max_{s_X \in \Omega_X} |\Phi_X(s_X)|$.) The Banach space of such potential is denoted $\mathcal{B}_1$. For $\Phi \in \mathcal{B}_1$ we have the useful bound

Lemma 2.1 $||\mathcal{H}_\Lambda(\Phi)|| \leq |\Lambda| \, ||\Phi||_1$.

Later we need more restricted classes of potentials, for example $\mathcal{B}_{\exp}$ given by

$$||\Phi||_{\exp} = \sum_{X \subset \mathbb{Z}^d} \exp[|X|] \, ||\Phi_X||$$

and in particular the finite-range potentials $\mathcal{B}_0$ defined by

$$|\Delta_\Phi| < +\infty, \text{ where } \Delta_\Phi = \bigcup \{X \subset \mathbb{Z}^d : \, 0 \in X, \, \Phi_X \neq 0\}.$$

These are obviously dense in $\mathcal{B}_1$.

The free energy for $\mathcal{H}_\Lambda$ is defined by

$$F_\Lambda(\beta, \Phi) = -\frac{1}{\beta} \ln \sum_{s_\Lambda \in \Omega_\Lambda} e^{-\beta \mathcal{H}_\Lambda(\Phi)(s_\Lambda)}. \tag{2.3}$$

It follows from lemma 2.1 that F_Λ is continuous.

Proposition 2.1 $|F_\Lambda(\beta, \Phi) - F_\Lambda(\beta, \Phi')| \leq |\Lambda| \, ||\Phi - \Phi'||_1$.

This implies in particular,

$$\left|\frac{1}{|\Lambda|}F_\Lambda(\beta,\Phi)+\ln q\right|\le ||\Phi||_1. \tag{2.4}$$

One can prove the existence of the free energy density in the thermodynamic limit[7]:

Theorem 2.4 *If $\Phi\in\mathcal{B}_1$ then the thermodynamic limit*

$$\lim_{n\to\infty}\frac{1}{|\Lambda_n|}F_{\Lambda_n}(\beta,\Phi)=f(\beta,\Phi) \tag{2.5}$$

exists if $\Lambda_n\to\mathbb{Z}^d$ in the sense of Van Hove. Moreover, f is a concave function of Φ and

$$|f(\beta,\Phi)-f(\beta,\Phi')|\le ||\Phi-\Phi'||_1.$$

A similar result holds for the entropy density. Let $\mu\in\mathcal{P}_I$ be an invariant measure. The restriction of μ to $\Omega(\Lambda)=S^\Lambda$ will be denoted μ_Λ. The **local entropy** is defined by

$$S_\Lambda(\mu)=-\sum_{s_\Lambda\in\Omega_\Lambda}\mu_\Lambda(s_\Lambda)\ln\mu_\Lambda(s_\Lambda). \tag{2.6}$$

It has the following basic properties.

Proposition 2.2 *$S_\Lambda(\mu)$ for $\mu\in\mathcal{P}_I$ satisfies*

1. *$0\le S_\Lambda(\mu)\le|\Lambda|\ \ln(q)$;*
2. *$S_\Lambda(\mu)$ is a concave function of μ;*
3. *If $\Lambda_1\cap\Lambda_2=\emptyset$ then $S_{\Lambda_1\cup\Lambda_2}(\mu)\le S_{\Lambda_1}(\mu)+S_{\Lambda_2}(\mu)$ (subadditivity);*
4. *If $\Lambda\subset\Lambda'$ then $S_{\Lambda'}(\mu)-S_\Lambda(\mu)\le(|\Lambda'|-|\Lambda|)\ln(q)$.*

[7] See R. B. Israel *loc. cit.* and Hugenholtz *loc. cit.* or T. C. Dorlas: *Statistical Mechanics: Fundamentals and Model Solutions* 2nd edn., Taylor & Francis, 2021.

5. *For arbitrary finite* $\Lambda_1, \Lambda_2 \subset \mathbb{Z}^d$,

$$S_{\Lambda_1 \cup \Lambda_2}(\mu) - S_{\Lambda_1}(\mu) - S_{\Lambda_2}(\mu) + S_{\Lambda_1 \cap \Lambda_2}(\mu) \leq 0. \tag{2.7}$$

(This is called strong subadditivity.)

Using the (strong) subadditivity property, one can prove that the thermodynamic limit exists:

Theorem 2.5 *If* $\mu \in \mathcal{P}_I$ *then the entropy density defined by*

$$\lim_{n\to\infty} \frac{1}{|\Lambda_n|} S_{\Lambda_n}(\mu) = s(\mu)$$

exists if $\Lambda_n \to \mathbb{Z}^d$ *in the sense of Van Hove. Moreover, it satisfies*

1. $0 \leq s(\mu) \leq \ln(q)$;
2. *The map* $\mu \mapsto s(\mu)$ *is affine and upper semi-continuous on* $\mathcal{P}_I$.

The affine property of $s(\mu)$ is proved as follows. By concavity,

$$\begin{aligned}
&\lambda S_\Lambda(\mu_1) + (1-\lambda) S_\Lambda(\mu_2) \\
&\quad\leq\; S_\Lambda(\lambda\mu_1 + (1-\lambda)\mu_2) \\
&\quad=\; -\lambda \sum_{s_\Lambda} \mu_1(s_\Lambda) \ln(\lambda\mu_1(s_\Lambda) + (1-\lambda)\mu_2(s_\Lambda)) \\
&\qquad\qquad -(1-\lambda) \sum_{s_\Lambda} \mu_2(s_\Lambda) \ln(\lambda\mu_1(s_\Lambda) + (1-\lambda)\mu_2(s_\Lambda)) \\
&\quad\leq\; -\lambda \sum_{s_\Lambda} \mu_1(s_\Lambda) \ln(\lambda\mu_1(s_\Lambda)) - (1-\lambda) \sum_{s_\Lambda} \mu_2(s_\Lambda) \ln((1-\lambda)\mu_2(s_\Lambda)) \\
&\quad=\; \lambda S_\Lambda(\mu_1) + (1-\lambda) S_\Lambda(\mu_2) - \lambda \ln\lambda - (1-\lambda)\ln(1-\lambda).
\end{aligned}$$

Using the simple bound $-\lambda\ln\lambda - (1-\lambda)\ln(1-\lambda) \leq \ln 2$ we obtain, after dividing by $|\Lambda|$ and taking the thermodynamic limit,

$$s(\lambda\mu_1 + (1-\lambda)\mu_2) = \lambda s(\mu_1) + (1-\lambda) s(\mu_2). \tag{2.8}$$

Finally, we consider the energy density:

Theorem 2.6 *If $\mu \in \mathcal{P}_I$ and $\Phi \in \mathcal{B}_1$ then*

$$\lim_{n\to\infty} \frac{1}{|\Lambda_n|} \int \mathcal{H}_{\Lambda_n}(\Phi)\, d\mu = \int A_\Phi d\mu, \tag{2.9}$$

where A_Φ is defined by

$$A_\Phi(\underline{s}) = \sum_{X \subset \mathbb{Z}^d :\, 0 \in X} \frac{\Phi_X(s_X)}{|X|}. \tag{2.10}$$

This follows from the identity

$$\frac{1}{|\Lambda|} \int \mathcal{H}_\Lambda(\Phi) d\mu = \frac{1}{|\Lambda|} \sum_{X \subset \Lambda} \int \Phi_X d\mu = \frac{1}{|\Lambda|} \sum_{x \in \Lambda} \sum_{X \subset \Lambda :\, x \in X} \int \frac{\Phi_X}{|X|} d\mu.$$

2.3 Translation-invariant equilibrium states

We want to define translation-invariant equilibrium states as minimisers of the free energy. First consider finite volume.

Proposition 2.3 *For any finite $\Lambda \subset \mathbb{Z}^d$,*

$$F_\Lambda(\beta, \Phi) \le \int \mathcal{H}_\Lambda(\Phi) d\mu - \frac{1}{\beta} S_\Lambda(\mu).$$

Moreover, there is precisely one probability measure μ for which the equality holds, namely the Gibbs measure

$$\mu_\Lambda^\Phi(s_\Lambda) = \frac{1}{Z_\Lambda(\beta\Phi)} \exp[-\beta \mathcal{H}_\Lambda(\Phi)(s_\Lambda)], \tag{2.11}$$

where $Z_\Lambda(\beta\Phi)$ is the partition function

$$Z_\Lambda(\beta\Phi) = \sum_{s_\Lambda \in \Omega_\Lambda} e^{-\beta \mathcal{H}_\Lambda(\Phi)(s_\Lambda)}. \tag{2.12}$$

This is a straightforward calculation. The uniqueness follows from the fact that the entropy $S_\Lambda(\mu)$ is a *strictly* concave function.

It follows that in the thermodynamic limit,

$$f(\beta, \Phi) \le \int A_\Phi d\mu - \frac{1}{\beta} s(\mu) \tag{2.13}$$

for $\mu \in \mathcal{P}_I$ and $\Phi \in \mathcal{B}_1$. We want to prove that there are measures $\mu \in \mathcal{P}_I$ for which equality holds. For this we consider large cubes of side n: $K_n(na)$ for $a \in \mathbb{Z}^d$. These cubes are disjoint and constitute a covering of $\mathbb{Z}^d$. We define a product measure

$$\tilde{\mu}_n = \prod_{a\in\mathbb{Z}^d} \mu^\Phi_{K_n(na)} \tag{2.14}$$

and average over translates to obtain

$$\mu_n = \frac{1}{n^d} \sum_{x\in K_n(0)} \tilde{\mu}_n \circ \tau_x. \tag{2.15}$$

We now compute the corresponding entropy and energy densities. Since $s(\mu)$ is affine, we have, writing $K_n = K_n(0)$,

$$\begin{aligned} s(\mu_n) &= \frac{1}{|K_n|} \sum_{x\in K_n} s(\tilde{\mu}_n \circ \tau_x) \\ &= s(\tilde{\mu}_n) = \lim_{m\to\infty} \frac{S_{K_{mn}}(\tilde{\mu}_n)}{|K_{mn}|}. \end{aligned}$$

But, since $K_{mn} = \bigcup_{a\in K_m} K_n(na)$,

$$\begin{aligned} S_{K_{mn}}(\tilde{\mu}_n) &= -\sum_{s_{K_{mn}}} \prod_{a\in K_m} \mu^\Phi_{K_n(na)}(s_{K_n(na)}) \ln \prod_{a\in K_m} \mu^\Phi_{K_n(na)}(s_{K_n(na)}) \\ &= -m^d \sum_{s_{K_n}} \mu^\Phi_{K_n}(s_{K_n}) \ln \mu^\Phi_{K_n}(s_{K_n}) \end{aligned}$$

and hence

$$s(\mu_n) = \frac{1}{|K_n|} S_{K_n}(\mu^\Phi_{K_n}). \tag{2.16}$$

Next we estimate the energy density of μ_n. We have

$$\begin{aligned} \int A_\Phi d\mu_n &= \frac{1}{|K_n|} \sum_{x\in K_n} \int \tau_x(A_\Phi)\, d\tilde{\mu}_n \\ &= \frac{1}{|K_n|} \sum_{x\in K_n} \sum_{X\subset\mathbb{Z}^d:\, x\in X} \frac{1}{|X|} \int \Phi_X d\tilde{\mu}_n \\ &= \frac{1}{|K_n|} \sum_{x\in K_n} \sum_{X\subset K_n:\, x\in X} \frac{1}{|X|} \int \Phi_X d\tilde{\mu}_n \\ &\quad + \frac{1}{|K_n|} \sum_{x\in K_n} \sum_{\substack{X\subset\mathbb{Z}^d:\\ x\in X;\, X\cap K_n^c\neq\emptyset}} \frac{1}{|X|} \int \Phi_X d\tilde{\mu}_n. \end{aligned}$$

The first term equals $\frac{1}{|K_n|}\sum_{x\in K_n}\int \mathcal{H}_{K_n}(\Phi)d\tilde{\mu}_n$. The second term is bounded by

$$\frac{1}{|K_n|}\sum_{x\in K_n}\sum_{X\subset\mathbb{Z}^d:\, x\in X; X\cap K_n^c\neq\emptyset}\frac{||\Phi_X||}{|X|}$$

and tends to zero as $n\to\infty$ if $\Phi\in\mathcal{B}_1$ because it is a boundary term. Given $\epsilon>0$ therefore, if n is large enough,

$$\left|\int A_\Phi d\mu_n - \frac{1}{|K_n|}\int \mathcal{H}_{K_n}(\Phi)d\tilde{\mu}_n\right| < \epsilon. \tag{2.17}$$

Finally, we know that $\lim_{n\to\infty}\frac{1}{|K_n|}F_{K_n}(\beta,\Phi)=f(\beta,\Phi)$, so for n large enough, also

$$\left|\frac{1}{|K_n|}F_{K_n}(\beta,\Phi)-f(\beta,\Phi)\right| < \epsilon. \tag{2.18}$$

By proposition 2.3 we have

$$F_{K_n}(\beta,\Phi)=\int \mathcal{H}_{K_n}(\Phi)d\mu_n - \frac{1}{\beta}S_{K_n}(\mu^\Phi_{K_n}),$$

and combining this with (2.16), (2.17) and (2.18) we conclude that

$$f(\beta,\Phi) > \int A_\Phi\, d\mu_n - \frac{1}{\beta}s(\mu_n) - 2\epsilon.$$

Now taking $\epsilon\to 0$, we obtain

$$f(\beta,\Phi)=\inf_{\mu\in\mathcal{P}_I}\left[\int A_\Phi d\mu - \frac{1}{\beta}s(\mu)\right]. \tag{2.19}$$

Together with the fact that $s(\mu)$ is lower semicontinuous, it follows from (2.19) that the infimum is attained for at least one μ. Moreover, since s is affine, the set of minimisers is convex. Thus we have proved

Theorem 2.7 *If $\Phi\in\mathcal{B}_1$, then*

$$f(\beta,\Phi)=\min_{\mu\in\mathcal{P}_I}\left[\int A_\Phi d\mu - \beta^{-1}s(\mu)\right].$$

The invariant measures for which the minimum is attained are called the ***invariant equilibrium states*** *for the interaction potential Φ at inverse temperature β, and denoted $\mathcal{G}_I(\Phi,\beta)$. The set $\mathcal{G}_I(\Phi,\beta)$ is a non-empty closed convex set.*

The extremal points of the set $\mathcal{G}_I(\Phi, \beta)$ are the **pure states** or **pure phases**. Clearly, an extremal point of $\mathcal{P}_I$ belonging to $\mathcal{G}_I(\Phi, \beta)$ is also an extremal point of $\mathcal{G}_I(\Phi, \beta)$. The converse is also true:

Theorem 2.8 $\mathrm{Ext}(\mathcal{G}_I(\Phi, \beta)) = \mathrm{Ext}(\mathcal{P}_I)$.

Proof. Let $\mu \in \mathrm{Ext}(\mathcal{G}_I(\Phi, \beta))$ and suppose $\mu \notin \mathrm{Ext}(\mathcal{P}_I)$. Then there are $\mu_1, \mu_2 \in \mathcal{P}_I$ and $\lambda \in (0, 1)$ such that $\mu_1 \neq \mu_2$ and $\mu = \lambda\mu_1 + (1-\lambda)\mu_2$. At least one of μ_1 or μ_2 does not belong to $\mathcal{G}_I(\Phi, \beta)$. Assume $\mu_1 \notin \mathcal{G}_I(\Phi, \beta)$. Then $f(\beta, \Phi) = \int A_\Phi d\mu - \beta^{-1}s(\mu) = \lambda(\int A_\Phi d\mu_1 - \beta^{-1}s(\mu_1)) + (1-\lambda)(\int A_\Phi d\mu_2 - \beta^{-1}s(\mu_2)) > f(\beta, \Phi)$, a contradiction. ∎

2.4 The DLR condition

General, not necessarily translation-invariant equilibrium states are defined by the **Dobrushin-Lanford-Ruelle (DLR) condition**[8], as follows. Let $\mathcal{B} \subset \mathcal{B}_\infty$ be the Banach space of potentials Φ such that the norm $||\Phi|| < +\infty$ where

$$||\Phi|| = \sum_{X \subset \mathbb{Z}^d :\, 0 \in X} ||\Phi_X||.$$

Definition 2.1 (DLR condition) *Let $\Phi \in \mathcal{B}$ be a potential function. A probability measure μ on Ω is an* ***equilibrium state*** *for Φ at inverse temperature $\beta > 0$ if for all finite subsets $\Lambda \subset \mathbb{Z}^d$, and all boundary conditions $\tilde{s}_{\Lambda^c}$ on the compliment of Λ, the conditional measure is given by*

$$\mu(s_\Lambda \,|\, \tilde{s}_{\Lambda^c}) = \frac{1}{Z_\Lambda(\beta\Phi \,|\, \tilde{s})} \exp\left[-\beta \sum_{X \subset \mathbb{Z}^d :\, X \cap \Lambda \neq \emptyset} \Phi_X(s_{X\cap\Lambda}, \tilde{s}_{X \setminus \Lambda})\right], \qquad (2.20)$$

[8] R. L Dobrushin: The description of a random field by its conditional distributions and its regularity conditions. *Theor. Veroyatn. Primen.* **13**, 201–29 (1971), and O. E. Lanford and D. Ruelle: Observables at infinity and states with short-range correlations. *Commun. Math. Phys.* **13**, 194–215 (1969).

where

$$Z_\Lambda(\beta\Phi \,|\, \tilde{s}) = \sum_{s_\Lambda \in \Omega_\Lambda} \exp\left[-\beta \sum_{X \subset \mathbb{Z}^d :\, X\cap\Lambda \neq \emptyset} \Phi_X(s_{X\cap\Lambda}, \tilde{s}_{X\setminus\Lambda})\right]. \tag{2.21}$$

Here, the conditional measure is defined by the relations

$$\int_B \mu(s_\Lambda \,|\, \tilde{s}_{\Lambda^c})\, \mu_{\Lambda^c}(d\tilde{s}_{\Lambda^c}) = \mu(\{s_\Lambda\} \times B), \tag{2.22}$$

where μ_{Λ^c} is the marginal distribution on $\Omega)\Lambda^c)$, $B \in \mathcal{B}(\Omega(\Lambda^c))$ is a Borel set, and we have written $\mu(s_\Lambda \,|\, \tilde{s}_{\Lambda^c}) = \mu(\{s_\Lambda\} \,|\, \tilde{s}_{\Lambda^c})$.

The DLR condition can be justified as follows. Let Λ_n be a sequence of increasing finite regions of $\mathbb{Z}^d$ and suppose that the finite-volume Gibbs measures $\mu^\Phi_{\Lambda_n}$ on Ω_{Λ_n} converge to an equilibrium measure μ on Ω. If $\Lambda \subset \Lambda_n$ is a given region then the conditional distribution of $\mu^\Phi_{\Lambda_n}$ on Λ is given by

$$\mu^\Phi_\Lambda(\overline{s}_\Lambda \,|\, \tilde{s}_{\Lambda_n\setminus\Lambda}) = \frac{\mu^\Phi_{\Lambda_n}(\overline{s}_\Lambda \times \tilde{s}_{\Lambda_n\setminus\Lambda})}{\sum_{s_\Lambda\in\Omega_\Lambda} \mu^\Phi_{\Lambda_n}(s_\Lambda \times \tilde{s}_{\Lambda_n\setminus\Lambda})}. \tag{2.23}$$

(This follows from $\mathbb{P}(A \,|\, B) = \mathbb{P}(A \cap B)/\mathbb{P}(B)$ with $A = \{s_{\Lambda_n} : s_\Lambda = \overline{s}_\Lambda\}$ and $B = \{s_{\Lambda_n} : s_{\Lambda_n\setminus\Lambda} = \tilde{s}_{\Lambda_n\setminus\Lambda}\}$.) Inserting the Gibbs distribution

$$\mu^\Phi_{\Lambda_n}(s_{\Lambda_n}) = \frac{1}{Z_{\Lambda_n}} e^{-\beta\sum_{X\subset\Lambda_n}\Phi_X(s_X)}$$

we get

$$\begin{aligned}
\mu^\Phi_\Lambda(\overline{s}_\Lambda \,|\, \tilde{s}_{\Lambda_n\setminus\Lambda}) &= \frac{e^{-\beta\sum_{X\subset\Lambda_n}\Phi_X(\overline{s}_{X\cap\Lambda},\tilde{s}_{X\setminus\Lambda})}}{\sum_{s_\Lambda\in\Omega_\Lambda} e^{-\beta\sum_{X\subset\Lambda_n}\Phi_X(s_{X\cap\Lambda},\tilde{s}_{X\setminus\Lambda})}} \\
&= \frac{e^{-\beta\sum_{X\subset\Lambda_n,\, X\cap\Lambda\neq\emptyset}\Phi_X(\overline{s}_{X\cap\Lambda},\tilde{s}_{X\setminus\Lambda})}}{\sum_{s_\Lambda\in\Omega_\Lambda} e^{-\beta\sum_{X\subset\Lambda_n,\, X\cap\Lambda\neq\emptyset}\Phi_X(s_{X\cap\Lambda},\tilde{s}_{X\setminus\Lambda})}}.
\end{aligned} \tag{2.24}$$

As $n \to \infty$, the left-hand side tends to $\mu(\overline{s}_\Lambda, \tilde{s}_{\Lambda^c})$. This follows from rewriting (2.23) as

$$\mu^\Phi_\Lambda(\overline{s}_\Lambda \,|\, \tilde{s}_{\Lambda_n\setminus\Lambda})\mu^\Phi_{\Lambda_n\setminus\Lambda}(\tilde{s}_{\Lambda_n\setminus\Lambda}) = \mu^\Phi_{\Lambda_n}(\overline{s}_\Lambda \times \tilde{s}_{\Lambda_n\setminus\Lambda})$$

and summing over a cylinder set $\tilde{s}_{\Lambda^c} \in B'$, $B' \subset \Omega_{\Lambda'}$, with $\Lambda' \subset \Lambda^c$ finite. This yields the relation (2.22). In the right-hand side,

$$\sum_{X\subset\Lambda_n,\, X\cap\Lambda\neq\emptyset} \Phi_X(\overline{s}_{X\cap\Lambda}, \tilde{s}_{X\setminus\Lambda}) \to \sum_{X\subset\mathbb{Z}^d,\, X\cap\Lambda\neq\emptyset} \Phi_X(\overline{s}_{X\cap\Lambda}, \tilde{s}_{X\setminus\Lambda})$$

because $\Phi \in \mathcal{B}$. For, if $\sum_{X \ni 0} ||\Phi_X|| < +\infty$ then for all $\epsilon > 0$ there exists Λ_0 such that $\sum_{X:\, 0 \in X, X \cap \Lambda_0 \neq \emptyset} ||\Phi_X|| < \epsilon$. Thus, if n is so large that $\tau_x(\Lambda_0) \subset \Lambda_n$ for all $x \in \Lambda$ then $\sum_{X:\, X \cap \Lambda \neq \emptyset, X \cap \Lambda_n \neq \emptyset} ||\Phi_X|| < |\Lambda|\, \epsilon$.

We next prove that the DLR condition and the variational condition in Theorem 2.7 are equivalent for translation-invariant measures $\mu \in \mathcal{P}_I$.

Theorem 2.9 *Suppose that $\Phi \in \mathcal{B}$. Then a measure $\mu \in \mathcal{P}_I$ satisfies the DLR condition if and only if it is an invariant equilibrium state in the sense of Theorem 2.7.*

Proof. Suppose first that μ satisfies the DLR condition. Fix $\Lambda \subset \mathbb{Z}^d$ finite. Then

$$\mu(s_\Lambda \,|\, \tilde{s}_{\Lambda^c}) = \frac{e^{-\beta \sum_{X \cap \Lambda \neq \emptyset} \Phi_X(s_{X \cap \Lambda}, \tilde{s}_{X \cap \Lambda^c})}}{Z_\Lambda(\beta\Phi \,|\, \tilde{s}_{\Lambda^c})}. \tag{2.25}$$

We want to compare the restriction of μ to Ω_Λ with

$$\mu_\Lambda^\Phi(s_\Lambda) = \frac{e^{-\beta \sum_{X \subset \Lambda} \Phi_X(s_X)}}{Z_\Lambda(\beta\Phi)}. \tag{2.26}$$

The following identity holds:

$$\begin{aligned} \ln \mu(s_\Lambda \,|\, \tilde{s}_{\Lambda^c}) - \ln \mu_\Lambda^\Phi(s_\Lambda) &= -\beta \sum_{\substack{X \cap \Lambda \neq \emptyset \\ X \cap \Lambda^c \neq \emptyset}} \Phi_X(s_{X \cap \Lambda}, \tilde{s}_{X \cap \Lambda^c}) \\ &\quad - \ln Z_\Lambda(\beta\Phi \,|\, \tilde{s}_{\Lambda^c}) + \ln Z_\Lambda(\beta\Phi), \end{aligned} \tag{2.27}$$

and we have

$$\left| \frac{Z_\Lambda(\beta\Phi, \tilde{s}_{\Lambda^c})}{Z_\Lambda(\beta\Phi)} \right| \leq \exp\big[\beta \sum_{\substack{X \cap \Lambda \neq \emptyset \\ X \cap \Lambda^c \neq \emptyset}} ||\Phi_X||\big]. \tag{2.28}$$

We now use properties of the **relative entropy** defined by

$$\begin{aligned} D(\nu_1 \,|\, \nu_2) &= \sum_{s_\Lambda} \nu_1(s_\Lambda) \left[\ln \nu_1(s_\Lambda) - \ln \nu_2(s_\Lambda)\right] \\ &= -S(\nu_1) - \sum_{s_\Lambda} \nu_1(s_\Lambda) \ln \nu_2(s_\Lambda). \end{aligned} \tag{2.29}$$

for two probability measures on Ω_Λ. It satisfies $D(\nu_1 \,|\, \nu_2) \geq 0$ and it is a jointly convex function of its arguments.

With $\nu_1 = \mu_\Lambda(\cdot \,|\, \tilde{s}_{\Lambda^c})$ and $\nu_2 = \mu_\Lambda^\Phi$, we have by (2.27) and (2.28),

$$|D(\mu_\Lambda(\cdot \,|\, \tilde{s}_{\Lambda^c}) \,|\, \mu_\Lambda^\Phi)| \leq 2\beta \sum_{\substack{X\cap\Lambda\neq\emptyset \\ X\cap\Lambda^c\neq\emptyset}} ||\Phi_X||.$$

By definition,

$$\int \mu_\Lambda(s_\Lambda \,|\, \tilde{s}_{\Lambda^c})\, \mu_{\Lambda^c}(d\tilde{s}_{\Lambda^c}) = \mu_\Lambda(s_\Lambda).$$

Therefore, using convexity of the relative entropy,

$$\begin{aligned} 0 &\leq D(\mu_\Lambda \,|\, \mu_\Lambda^\Phi) \leq \int D(\mu_\Lambda(\cdot \,|\, \tilde{s}_{\Lambda^c}) \,|\, \mu_\Lambda^\Phi)\, \mu_{\Lambda^c}(d\tilde{s}_{\Lambda^c}) \\ &\leq 2\beta \sum_{\substack{X\cap\Lambda\neq\emptyset \\ X\cap\Lambda^c\neq\emptyset}} ||\Phi_X||. \end{aligned}$$

If $\Phi \in \mathcal{B}$ then

$$\frac{1}{|\Lambda|} \sum_{\substack{X\cap\Lambda\neq\emptyset \\ X\cap\Lambda^c\neq\emptyset}} ||\Phi_X|| \to 0.$$

Inserting the definition (2.26) of μ_Λ^Φ and taking the logarithm, we have $\int \ln(\mu_\Lambda^\Phi)\, d\mu_\Lambda = -\beta \int \mathcal{H}_\Lambda(\Phi)\, d\mu_\Lambda - \ln Z_\Lambda(\beta\Phi)$ and therefore

$$0 \leq -S_\Lambda(\mu_\Lambda) + \beta \int \mathcal{H}_\Lambda(\Phi)\, d\mu_\Lambda + \ln Z_\Lambda(\beta\Phi) \leq \epsilon|\Lambda|$$

for $|\Lambda|$ large enough. Dividing by $\beta|\Lambda|$ and taking the thermodynamic limit this becomes

$$0 \leq -\frac{1}{\beta} s(\mu) + \int A_\Phi d\mu - f(\beta, \Phi) \leq \epsilon/\beta.$$

Then taking $\epsilon \to 0$ we conclude that the variational principle holds.

Conversely, suppose that μ satisfies the variational principle. To prove that μ satisfies the DLR condition, we need the following proposition, which follows from an identification of equilibrium states with tangent planes to the graph of $f(\beta, \Phi)$, but which we shall not prove here.

Proposition 2.4 *If $\mu \in \mathcal{P}_I$ is an equilibrium state for $\Phi \in \mathcal{B}_1$ (in then sense of Theorem 2.7) then μ is contained in the (weakly) closed convex hull of the set of equilibrium states ν such that $f(\beta, \Phi)$ is differentiable at Φ with derivative equal to the map $\Phi \mapsto \int A_\Phi \, d\nu$.*

Note that a continuous linear form α on $\mathcal{B}_1$ is the derivative of a concave function f_β at Φ if and only if

$$f_\beta(\Phi + \Psi) - f_\beta(\Phi) \leq \alpha(\Psi) \text{ for all } \Psi \in \mathcal{B}.$$

(The tangent lies above the graph. We have written f_β for the functional $f_\beta(\Phi) = f(\beta, \Phi)$.) Inserting the variational equality, $f(\beta, \Phi) = \int A_\Phi d\mu - \beta^{-1} s(\mu)$ of Theorem 2.7, we have

$$f_\beta(\Phi + \Psi) - f_\beta(\Phi) = \int A_\Psi \, d\mu$$

for an invariant equilibrium state μ. Therefore if f_β is differentiable at Φ, its derivative at Φ must equal $\int A_\Phi \, d\mu$.

In the proof of Theorem 2.7 we showed that the measures μ_n converge to an equilibrium state μ. If f_β is differentiable at Φ then μ is the only equilibrium state and we conclude that $\mu = \lim_{n\to\infty} \mu_n$, where μ_n is given by (2.15). Now consider a fixed finite subset $\Lambda \subset \mathbb{Z}^d$. For n large enough, $\Lambda \subset K_n = K_n(0)$.

If $\Lambda \subset K_n$ then, since $\tilde{\mu}_n$ is a product measure, $\tilde{\mu}_n(s_{K_n} \,|\, \tilde{s}_{K_n^c}) = \mu^\Phi_{K_n}(s_{K_n})$ and

$$\begin{aligned} \tilde{\mu}_n(s_\Lambda \,|\, \tilde{s}_{K_n \setminus \Lambda}) &= \mu^\Phi_{K_n}(s_\Lambda \,|\, \tilde{s}_{K_n \setminus \Lambda}) \\ &= \frac{e^{-\beta \mathcal{H}_\Lambda(s_\Lambda) - \beta W_{\Lambda, K_n}(s_\Lambda, \tilde{s}_{K_n \setminus \Lambda})}}{\sum_{\bar{s}_\Lambda} e^{-\beta \mathcal{H}_\Lambda(\bar{s}_\Lambda) - \beta W_{\Lambda, K_n}(\bar{s}_\Lambda, \tilde{s}_{K_n \setminus \Lambda})}}, \end{aligned}$$

where

$$W_{\Lambda, K_n}(s_\Lambda, \tilde{s}_{K_n \setminus \Lambda}) = \sum_{\substack{X \subset K_n: \\ X \cap \Lambda \neq \emptyset, X \cap K_n \setminus \Lambda \neq \emptyset}} \Phi_X(s_{X \cap \Lambda}, \tilde{s}_{X \cap K_n \setminus \Lambda}). \tag{2.30}$$

Hence, if we define

$$\nu_\Lambda(s_\Lambda \,|\, \tilde{s}_{\Lambda^c}) = \frac{e^{-\beta \mathcal{H}_\Lambda(s_\Lambda) - \beta W_\Lambda(s_\Lambda, \tilde{s}_{\Lambda^c})}}{\sum_{\bar{s}_\Lambda} e^{-\beta \mathcal{H}_\Lambda(\bar{s}_\Lambda) - \beta W_\Lambda(\bar{s}_\Lambda, \tilde{s}_{\Lambda^c})}},$$

where

$$W_\Lambda(s_\Lambda, \tilde{s}_{\Lambda^c} = \sum_{\substack{X \subset \mathbb{Z}^d: \\ X\cap\Lambda\neq\emptyset, X\cap\Lambda^c\neq\emptyset}} \Phi_X(s_{X\cap\Lambda}, \tilde{s}_{X\cap\Lambda^c})$$

then

$$||\tilde{\mu}_n(\cdot \mid \tilde{s}_{K_n\setminus\Lambda}) - \nu_\Lambda(\cdot \mid \tilde{s}_{\Lambda^c})|| \leq e^{2\beta\sum_{\substack{X\subset\mathbb{Z}^d:\, X\cap\Lambda\neq\emptyset \\ X\cap K_n^c\neq\emptyset}} ||\Phi_X||} - 1. \tag{2.31}$$

Assuming $\Phi \in \mathcal{B}$ and choosing $\Lambda_0 \supset \Lambda$ so large that

$$\sum_{\substack{X\subset\mathbb{Z}^d:\, X\cap\Lambda\neq\emptyset \\ X\cap\Lambda_0^c\neq\emptyset}} ||\Phi_X|| < \epsilon,$$

we have, for $x \in K_n$ such that $\tau_x\Lambda_0 \subset K_n$,

$$||\tilde{\mu}_n(\cdot \mid \tilde{s}_{K_n\setminus\tau_x\Lambda}) - \nu_\Lambda(\cdot \mid \tilde{s}_{\tau_x\Lambda^c})|| < e^{2\beta\epsilon} - 1. \tag{2.32}$$

On the other hand,

$$\frac{1}{|K_n|}\#\{x \in K_n \,:\, \tau_x\Lambda_0 \neq \emptyset\} \to 0.$$

It follows that

$$||\mu_n(\cdot \mid \tilde{s}_{\Lambda^c}) - \nu_\Lambda(\cdot \mid \tilde{s}_{\tau_x\Lambda^c})|| \to 0 \text{ as } n \to \infty,$$

which proves that

$$\mu(\cdot \mid \tilde{s}_{\Lambda^c}) = \nu_\Lambda(\cdot \mid \tilde{s}_{\Lambda^c}).$$

This is the DLR condition. By the above proposition, an arbitrary translation-invariant equilibrium state is a limit of convex combinations of states at which $f_\beta(\Phi)$ and by continuity they also satisfy the DLR condition. ∎

3 Contours for the Ising model and the Potts model

Pirogov-Sinai theory goes beyond the existence of spontaneous magnetization, proving that for low temperatures the phase diagram is a continuous deformation of the zero-temperature phase diagram. In particular, it is shown that the extremal equilibrium states at low temperature are close to the zero-temperature ground states in the sense that the large majority of spins is in the ground state. To show that an equilibrium measure is extremal it is shown to be clustering. To show clustering of the measure one uses the fact that the external contours cluster. To be precise, for some $\tau > 0$,

$$|\mu(\Theta_1 \cup \Theta_2 \subset \Theta(\underline{s})) - \mu(\Theta_1 \subset \Theta(\underline{s}))\mu(\Theta_2 \subset \Theta(\underline{s}))| \leq e^{-\tau(|\Theta_1|+|\Theta_2|+d(\Theta_1,\Theta_2))}, \tag{3.1}$$

where $\Theta(\underline{s})$ is the collection of outer contours of the configuration $\underline{s}$, and $d(\Theta_1, \Theta_2)$ is the distance between two sets of (external) contours Θ_1 and Θ_2.

3.1 The Ising model

We define the correlation functions for external contours Θ by

$$\tilde{\rho}_\Lambda(\Theta) = \mu(\Theta \subset \Theta(\underline{s})). \tag{3.2}$$

They satisfy a set of implicit equations proved in Lemma 4.4 which implies the existence of the thermodynamic limit and the estimate (3.1) given in Theorem 4.2 and its corollary. The basic idea is that by the corollary of Theorem 1.1, the contours are small and rare, so can be considered a gas of 'particles' in a sea of plus or minus spins according to the choice of boundary condition. As in Peierls' argument, we have

$$\tilde{\rho}(\Theta) \leq e^{-\beta J|\Theta|}. \tag{3.3}$$

We can write equation (3.1) as follows

$$|\tilde{\rho}(\Theta_1 \cup \Theta_2) - \tilde{\rho}(\Theta_1)\tilde{\rho}(\Theta_2)| \leq e^{-\tau(|\Theta_1|+|\Theta_2|+d(\Theta_1,\Theta_2))}.$$

Now consider the case of positive boundary conditions. To see that equation (3.1) implies that the measure μ_+ is clustering, consider the example of two points 0 and z, i.e. let us show that $\mathbb{E}^{\mu_+}(s_0 s_z) - \mathbb{E}^{\mu_+}(s_0)\mathbb{E}^{\mu_+}(s_z) \to 0$ when $|z| \to \infty$. As $|z| \to \infty$, the likelihood is that there is no single contour enclosing both 0 and z. Let us denote

$$\mathcal{C}_x = \{\gamma : \, x \in \mathrm{Int}(\gamma)\}, \tag{3.4}$$

the set of contours which contain x in their interior. If an exterior boundary Θ' contains both points 0 and z then there is a minimal exterior boundary $\Theta \subset \Theta'$ consisting of one contour containing both points or two separate contours containing one each. Similarly, if Θ' contains one point but not the other then there is $\gamma \in \Theta' \cap \mathcal{C}_0$ but $\Theta' \cap \mathcal{C}_z = \emptyset$ or vice versa. Finally, if Θ' does not contain either point then $\Theta \cap (\mathcal{C}_0 \cup \mathcal{C}_z) = \emptyset$. Thus we have (writing $\tilde{\rho}(\gamma) = \tilde{\rho}(\{\gamma\})$,

$$\begin{aligned}
\mathbb{E}^{\mu_+}(s_0 s_z) \;=\; & \sum_{\Theta = (\Theta \cap \mathcal{C}_0) \cup (\Theta \cap \mathcal{C}_z)} \tilde{\rho}(\Theta) \prod_{\gamma \in \Theta} \mathbb{E}^{\mu_-}_{\gamma}(s_0\, s_z) \\
& + \sum_{\gamma_0 \in \mathcal{C}_0 \setminus \mathcal{C}_z} \left(\tilde{\rho}(\gamma_0) - \sum_{\Theta = \{\gamma_0, \gamma_z\} :\, \gamma_z \in \mathcal{C}_z} \tilde{\rho}(\Theta) \right) \mathbb{E}^{\mu_-}_{\gamma_0}(s_0) \\
& + \sum_{\gamma_z \in \mathcal{C}_z \setminus \mathcal{C}_0} \left(\tilde{\rho}(\gamma_z) - \sum_{\Theta = \{\gamma_0, \gamma_z\} :\, \gamma_0 \in \mathcal{C}_0} \tilde{\rho}(\Theta) \right) \mathbb{E}^{\mu_-}_{\gamma_z}(s_z) \\
& + \; 1 - \sum_{\Theta = (\Theta \cap \mathcal{C}_0) \cup (\Theta \cap \mathcal{C}_z)} \tilde{\rho}(\Theta) \\
& - \sum_{\gamma_0 \in \mathcal{C}_0 \setminus \mathcal{C}_z} \left(\tilde{\rho}(\gamma_0) - \sum_{\Theta = \{\gamma_0, \gamma_z\} :\, \gamma_z \in \mathcal{C}_z} \tilde{\rho}(\Theta) \right) \\
& - \sum_{\gamma_z \in \mathcal{C}_z \setminus \mathcal{C}_0} \left(\tilde{\rho}(\gamma_z) - \sum_{\Theta = \{\gamma_0, \gamma_z\} :\, \gamma_0 \in \mathcal{C}_0} \tilde{\rho}(\Theta) \right)
\end{aligned} \tag{3.5}$$

(Here $\mathbb{E}^{\mu_-}_{\gamma}$ is the expectation w.r.t. the conditional measure inside the contour γ with boundary condition $-$ inside γ. Note that the spin flips from $+$ to $-$ across the exterior contour γ.) Similarly,

$$\mathbb{E}^{\mu_+}(s_0) = \sum_{\gamma \in \mathcal{C}_0} \tilde{\rho}(\gamma) \mathbb{E}^{\mu_-}_{\gamma}(s_0) + 1 - \sum_{\gamma \in \mathcal{C}_0} \tilde{\rho}(\gamma) \tag{3.6}$$

and

$$\mathbb{E}^{\mu_+}(s_z) = \sum_{\gamma \in \mathcal{C}_z} \tilde{\rho}(\gamma) \mathbb{E}^{\mu_-}_{\gamma}(s_z) + 1 - \sum_{\gamma \in \mathcal{C}_z} \tilde{\rho}(\gamma). \tag{3.7}$$

Therefore

$$\begin{aligned}
&\mathbb{E}^{\mu_+}(s_0 s_z) - \mathbb{E}^{\mu_+}(s_0)\mathbb{E}^{\mu_+}(s_z) \qquad (3.8)\\
&\quad = \sum_{\gamma \in \mathcal{C}_0 \cap \mathcal{C}_z} \tilde{\rho}(\gamma)(\mathbb{E}^{\mu_-}_{\gamma}(s_0\, s_z) - 1) \\
&\qquad + \sum_{\substack{\Theta = \{\gamma_0, \gamma_z\}; \\ \gamma_0 \in \mathcal{C}_0, \gamma_z \in \mathcal{C}_z}} \tilde{\rho}(\Theta)(\mathbb{E}^{\mu_-}_{\gamma_0}(s_0)\mathbb{E}^{\mu_-}_{\gamma_z}(s_z) - \mathbb{E}^{\mu_-}_{\gamma_0}(s_0) - \mathbb{E}^{\mu_-}_{\gamma_z}(s_z) + 1) \\
&\qquad - \sum_{\gamma_0 \in \mathcal{C}_0 \setminus \mathcal{C}_z} \tilde{\rho}(\gamma_0)\,(1 - \mathbb{E}^{\mu_-}_{\gamma_0}(s_0)) - \sum_{\gamma_z \in \mathcal{C}_z \setminus \mathcal{C}_0} \tilde{\rho}(\gamma_z)\,(1 - \mathbb{E}^{\mu_-}_{\gamma_z}(s_z)) \\
&\qquad + \sum_{\gamma_0 \in \mathcal{C}_0} \tilde{\rho}(\gamma_0)\,(1 - \mathbb{E}^{\mu_-}_{\gamma_0}(s_0)) + \sum_{\gamma_z \in \mathcal{C}_z} \tilde{\rho}(\gamma_z)\,(1 - \mathbb{E}^{\mu_-}_{\gamma_z}(s_z)) \\
&\qquad - \sum_{\gamma_0 \in \mathcal{C}_0} \sum_{\gamma_z \in \mathcal{C}_z} \tilde{\rho}(\gamma_0)\tilde{\rho}(\gamma_z)(\mathbb{E}^{\mu_-}_{\gamma_0}(s_0)\mathbb{E}^{\mu_-}_{\gamma_z}(s_z) - \mathbb{E}^{\mu_-}_{\gamma_0}(s_0) - \mathbb{E}^{\mu_-}_{\gamma_z}(s_z) + 1) \\
&\quad = \sum_{\gamma \in \mathcal{C}_0 \cap \mathcal{C}_z} \tilde{\rho}(\gamma)\,(1 + \mathbb{E}^{\mu_-}_{\gamma}(s_0 s_z) - \mathbb{E}^{\mu_-}_{\gamma}(s_0) - \mathbb{E}^{\mu_-}_{\gamma}(s_z)) \\
&\qquad - \sum_{\gamma_0 \in \mathcal{C}_0} \sum_{\substack{\gamma_z \in \mathcal{C}_z \\ \mathrm{Int}(\gamma_0) \cap \mathrm{Int}(\gamma_z) \neq \emptyset}} \tilde{\rho}(\gamma_0)\tilde{\rho}(\gamma_z)(1 - \mathbb{E}^{\mu_-}_{\gamma_0}(s_0))(1 - \mathbb{E}^{\mu_-}_{\gamma_z}(s_z)). \qquad (3.9)
\end{aligned}$$

In the first term $\ell(\gamma) > 2|z|$, and by (3.3) $\tilde{\rho}(\gamma) \to 0$ as $|z| \to \infty$. Similarly, in the second term either γ_0 or γ_z has length (area) greater than $|z|/2$ and this term also tends to 0.

3.2 The inhomogeneous Potts model

Let us now consider more general lattice spin systems. Pirogov and Sinai considered general lattice spin systems with periodic finite-range interaction, i.e. $\Phi \in \mathcal{B}_0$ and periodic states. But the main ideas can be explained in the case of nearest-neighbour interaction and translation-invariant states. We therefore consider here models of the Potts type. We assume the spin space to be a finite set $\{1, \ldots, q\}$, and

$$\Phi_X = 0 \text{ unless } |X| = 1, 2 \text{ and if } |X| = 2 \text{ then } X = \{x, y\},\ |x - y| = 1.$$

The 1-point interaction is an external field as in the Ising model, and we write it explicitly if non-zero. For $X = \{x, y\}$ we put

$$\Phi_{\{x,y\}}(s_x, s_y) = \Phi_{x-y}(s_x, s_y).$$

In the Potts model,

$$\Phi_{x-y}(s_x, s_y) = -J\,\delta_{s_x,s_y} \qquad (J > 0). \tag{3.10}$$

A more general interaction is

$$\begin{aligned}\Phi_{x-y}(s_x, s_y) &= \sum_{1\le r<r'\le q} J_{r,r'}(\delta_{s_x,r} - \delta_{s_y,r})^2(\delta_{s_x,r'} - \delta_{s_y,r'})^2 \\ &= \sum_{1\le r<r'\le q} J_{r,r'}(\delta_{s_x,r}\delta_{s_y,r'} + \delta_{s_x,r'}\delta_{s_y,r}).\end{aligned} \tag{3.11}$$

Note that in the homogeneous case,

$$J \sum_{1\le r<r'\le q} (\delta_{s_x,r}\delta_{s_y,r'} + \delta_{s_x,r'}\delta_{s_y,r}) = J\left(1 - \delta_{s_x,s_y}\right),$$

which differs by an irrelevant constant from (3.10). In all cases $\Phi_{x-y}(s_x, s_y) = 0$ if $s_x = s_y$. By adding a constant, we made sure that these ground states have energy 0. In general, a **ground state** is defined as a minimiser of the specific energy A_Φ given by (2.10). For the generalized Potts model,

$$A_\Phi(\underline{s}) = \frac{1}{2}\sum_{x\colon |x|=1}\sum_{1\le r<r'\le q} J_{r,r'}(\delta_{s_0,r}\delta_{s_x,r'} + \delta_{s_0,r'}\delta_{s_x,r}). \tag{3.12}$$

The ground states of the Hamiltonian are therefore given by the constant configurations. The corresponding Hamiltonian is

$$\mathcal{H}_\Lambda(s_\Lambda) = \sum_{\substack{\{x,y\}\subset\Lambda \\ |x-y|=1}} \Phi_{x-y}(s_x, s_y). \tag{3.13}$$

Introducing external fields

$$\tilde{\Phi}_x(s_x) = -\sum_{r=1}^{q} h_r\delta_{s_x,r}, \tag{3.14}$$

some of these ground states are lifted in energy and others are lowered. Define $h_{\min} = \min_{r=1}^q h_r$. If $h_r > h_{\min}$ then r is not a ground state for the interaction $\Phi + \tilde{\Phi}$. Hence, writing $t_r = h_r - h_{\min}$, $(r = 1, \ldots, q)$, the map $(h_2 - h_1, \ldots, h_q - h_1) \mapsto (t_1, \ldots, t_q)$ is a homeomorphism of $\mathbb{R}^{q-1}$ onto the boundary of the positive r-dimensional octant,

$$O_q = \{(b_1, \ldots, b_q) : \min_{r=1}^{q} b_r = 0\}$$

in such a way that the number of ground states equals the number of r such that $t_r = 0$. The Hamiltonian including external fields will be denoted $\widehat{\mathcal{H}}_\Lambda$:

$$\widehat{\mathcal{H}}_\Lambda(s_\Lambda) = \mathcal{H}_\Lambda(s_\Lambda) + \sum_{x \in \Lambda} \tilde{\Phi}_x(s_x). \tag{3.15}$$

Pirogov and Sinai showed that this map is continuously deformed for low temperatures.

We now need to generalize the definition of contour. In the case of nearest-neighbour interaction, we define the **boundary** of a given configuration $\underline{s}$ on $\mathbb{Z}^d$ by

$$\partial(\underline{s}) = \bigcup_{\substack{x \in \mathbb{Z}^d: \\ s_{K_2(x)} \text{ not constant}}} K_2(x),$$

where, as before $K_2(x) = \{z \in \mathbb{Z}^d : x_i \le z_i \le x_i + 1, \, (i = 1, \ldots, d)\}$. (Note that these cubes overlap!) A **contour** for $\underline{s}$ is then defined as a pair (Γ, s_Γ), where Γ is a minimal connected subset of $\partial(\underline{s})$ such that the restriction s_Γ of the configuration $\underline{s}$ to Γ is constant on the edges of Γ. (See Figures 3 and 4.) These will be called a **boundary values** of the contour. Clearly, the set of contours determines the configuration uniquely. Conversely, a given set of contours corresponds to a configuration $\underline{s}$ if they are **compatible**. Two adjoining contours can be compatible if they are disjoint and their configurations agree on neighbouring edges, that is, if along each connected path from one contour to another, not intersecting a third, the configuration is constant.

As in Peierls' argument, it follows that for large β, contours are finite with probability 1. The **exterior** $\mathrm{Ext}(\Gamma)$ of the contour Γ, consisting of all

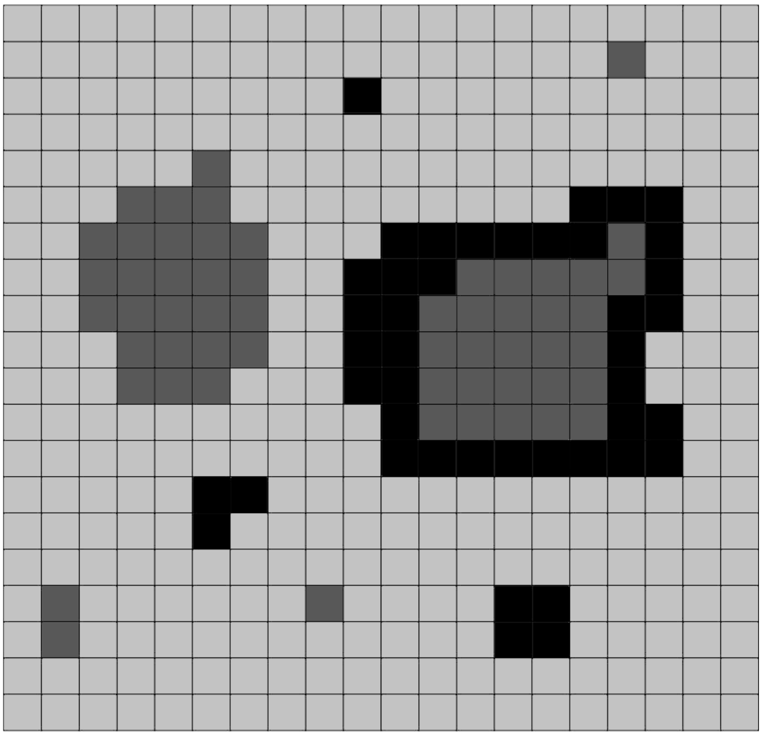

Figure 3. A Potts model configuration with $q = 3$:
The colours represent the spin values.

lattice points outside Γ is then unique. The **interior** $\mathrm{Int}(\Gamma)$ consisting of all points not in Γ but surrounded by Γ can be empty, or it can consist of several disconnected regions. On each connected part of $\mathrm{Int}(\Gamma)$ the configuration is constant, and the union of interior regions where the configuration is r will be denoted $\mathrm{Int}_r(\Gamma)$.

It terms of the contours we can write the Hamiltonian in a finite region Λ as

$$\widehat{\mathcal{H}}_\Lambda(s_\Lambda) \quad = \quad \sum_{\substack{\Gamma \subset \partial(s_\Lambda): \\ \Gamma \,\text{connected comp.}}} \left(\mathcal{H}_\Gamma(s_\Gamma) + \sum_{x \in \Gamma} \tilde{\Phi}_x(s_x) \right) + \sum_{x \in \Lambda \setminus \partial(\underline{s})} \tilde{\Phi}_x(s_x), \tag{3.16}$$

where the last term breaks down into terms corresponding to regions where $s_x = r$, where $\Phi_x(s_x) = -h_r$.

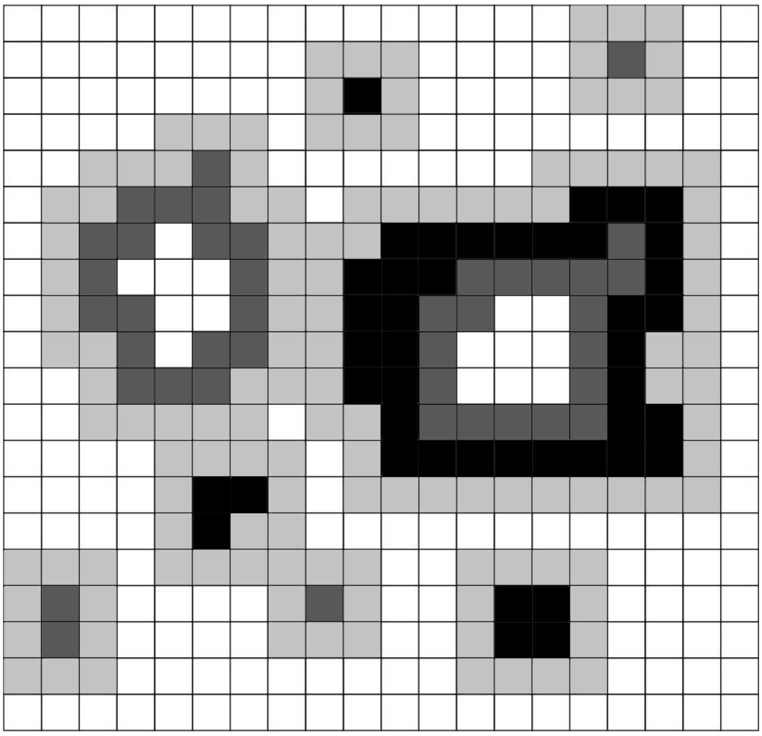

Figure 4. Contours for the above Potts model configuration.

EXAMPLE 3.1. *The 3-state inhomogeneous Potts model.*
Let us consider the special case of the 3-state Potts model with Hamiltonian given by (3.11),

$$\Phi_{x-y}(s_x, s_y) = \sum_{1 \le r < r' \le 3} J_{r,r'}(\delta_{s_x,r}\delta_{s_y,r'} + \delta_{s_x,r'}\delta_{s_y,r}). \tag{3.17}$$

In particular, assume that $J_{12} = J < \tilde{J} = J_{13} = J_{23}$. The states $s = 1$ and $s = 2$ are then obviously equivalent. We show that for non-zero temperature, these states are favoured over the state $s = 3$. For low temperatures, the free energy will be minimal for configurations near one of the ground states.

Consider first the case of the ground state $s = 1$ ($s = 2$ is equivalent). The low excitations are then single impurities with $s_x = 2, 3$. Since they do not have the same energy, they need to be considered separately. Let ρ_2 be the density of impurities $s_x = 2$ and ρ_3 that for $s_x = 3$. The entropy is then

$$\begin{aligned} s(\rho_2, \rho_3) &= -\rho_2 \ln \rho_2 - \rho_3 \ln \rho_3 - (1 - \rho_2 - \rho_3) \ln(1 - \rho_2 - \rho_3) \\ &\approx -\rho_2(\ln \rho_2 - 1) - \rho_3(\ln \rho_3 - 1), \end{aligned}$$

assuming that ρ_2 and ρ_3 are small. Hence

$$f_1(\beta, \rho_2, \rho_3) \approx 2dJ\rho_2 + 2d\tilde{J}\rho_3 - \frac{1}{\beta} s(\rho_2, \rho_3).$$

($2dJ$ and $2d\tilde{J}$ are the energies due to the links on either side of the impurity in each coordinate direction.) Minimising over ρ_2 and ρ_3 yields $\rho_2 \approx e^{-2\beta dJ}$ and $\rho_3 \approx e^{-2d\beta\tilde{J}}$ and thus

$$f_{1,\min}(\beta) \approx -\frac{1}{\beta}(e^{-2\beta dJ} + e^{-2\beta d\tilde{J}}).$$

Next consider the case of the ground state $s = 3$. Assume that excitations are again given by a small density of single sites with spins $s_x = 1, 2$. If this density is ρ, then this gives rise to an entropy $s(\rho) + \rho \ln 2$, where $s(\rho) = -\rho \ln \rho - (1 - \rho) \ln(1 - \rho)$ is the usual entropy per site for a density ρ of impurities and the term $\rho \ln 2$ is due to the choice of $s_x = 1, 2$ at each of the impurity sites. The free energy density thus becomes

$$f_3(\beta, \rho) \approx 2d\rho\tilde{J} - \frac{1}{\beta} s(\rho) - \frac{\rho}{\beta} \ln 2.$$

(Here the energy is $2d\tilde{J}$ for both types of impurity.) For large β, ρ is again small and we can approximate $s(\rho)$ by $s(\rho) \approx -\rho(\ln \rho - 1)$. f_3 is then minimized for $\rho = 2e^{-2d\beta\tilde{J}}$. Thus

$$f_{3,\min}(\beta) \approx -\frac{2}{\beta} e^{-2d\beta\tilde{J}}.$$

Clearly, $f_{1,\min} < f_{3,\min}$ and the state $s = 1$ is favoured over the state $s = 3$. In order to restore the $s = 3$ state to be in equilibrium with the states $s = 1, 2$, we need to lower its energy by introducing a field $h_3 > 0$. Then f_3 changes to

$$f_3(\beta, \rho, h_3) \approx 2d\tilde{J}\rho - h_3(1 - \rho) - \frac{1}{\beta} s(\rho) - \frac{\rho}{\beta} \ln 2.$$

Minimising,

$$f_{3,\min}(\beta, h_3) \approx -h_3 - \frac{2}{\beta} e^{-2\beta\tilde{J}-\beta h_3}.$$

Also,

$$f_1(\beta, \rho_2, \rho_3, h_3) \approx 2dJ\rho_2 - h_3\rho_3 + 2d\tilde{J}\rho_3 - \frac{1}{\beta} s(\rho_2, \rho_3).$$

Minimising,

$$f_{1,\min}(\beta, h_3) \approx -\frac{1}{\beta}(e^{-2\beta dJ} + e^{-\beta(2d\tilde{J}-h_3)}).$$

The two expressions are equal if

$$\begin{aligned} h_3 &= \frac{1}{\beta}\left(e^{-2\beta dJ} + e^{-\beta(2d\tilde{J}-h_3)} - 2e^{-\beta(2d\tilde{J}+h_3)}\right) \\ &\approx \frac{1}{\beta}(e^{-2\beta dJ} - e^{-2\beta d\tilde{J}}) + 3e^{-2\beta d\tilde{J}} h_3, \end{aligned}$$

or

$$h_3 \approx \frac{1}{\beta} \frac{e^{-2\beta dJ} - e^{-2\beta d\tilde{J}}}{1 - 3e^{-2\beta d\tilde{J}}}.$$

On the other hand, if we keep $h_3 = 0$, but impose boundary conditions $s_x = 3$ for $x \in \Lambda^c$, then it is advantageous to have a large contour Γ along the boundary of Λ, so that $s_x = 1$ (or $s_x = 2$) for $x \in \mathrm{Int}(\Gamma)$ except for a small density of sites where $s_x = 2(1), 3$. The large contour yields an additive constant to the free energy given by $2d\tilde{J}|\partial\Lambda|$ which is negligible in the limit $\Lambda \to \mathbb{Z}^d$. Since $s_x = 1$ for most sites, the magnetization is (almost) equal 1, and the resulting state will be the same as the equilibrium state with boundary condition 1 (or 2).

Obviously, the above is only an approximate analysis. General contours can be more complicated even if rare. To include this possibility we introduce general **contour models**, which represent single pure phases.

4 Contour models

4.1 Definitions

We define an *abstract* **contour model** to be given by a functional F on a set of contours $\mathcal{C}$, which are now considered simply as decorated connected subsets of $\mathbb{Z}^d$, unions of cubes of side 2, where F can depend on the decoration. We assume that F is translation-invariant and satisfies the property that there is a constant $\tau > 0$ (independent of the decoration) such that for every $\Gamma \in \mathcal{C}$,

$$F(\Gamma) \geq \tau\,|\Gamma|. \tag{4.1}$$

This is called the **Peierls condition**. In the following the decoration plays no role other than determining the functional F and will not be included in the notation. Contours Γ and Γ' will now be called **compatible** if they are disjoint. A collection of compatible contours will be denoted ∂ and is called a **boundary**. The set of boundaries is denoted $\mathcal{D}$. The set of contours compatible with $\partial \in \mathcal{D}$ we write as $\mathrm{Cp}(\partial)$. For $\partial \in \mathcal{D}$, we set $F(\partial) = \sum_{\Gamma\in\partial} F(\Gamma)$. A contour is called an **external contour** of ∂ if there is no contour surrounding it, i.e. for all $\Gamma' \in \partial$ not equal Γ, $\Gamma \in \mathrm{Ext}(\Gamma')$. We denote $|\partial| = \#\{\Gamma \in \partial\}$, the number of contours in ∂, and $||\partial|| = \sum_{\Gamma\in\partial} |\Gamma|$, the total area/volume of the contours of ∂.

As in the Peierls argument, we have

Lemma 4.1 *There exists a constant $c_d > 0$ such that the number of contours Γ with area/volume $|\Gamma| = n$ containing a given point, say $0 \in \Gamma \cup \mathrm{Int}(\Gamma)$ is bounded by $e^{c_d n}$.*

Proof. We follow Dorlas[9]. This is similar to counting the number of contours containing 0 in Peierls' argument (cf. the proof of Theorem 1.1). Let x be the point on the x_1-axis with smallest norm, belonging to Γ. If $|\Gamma| = n$ the number of possibilities for x is obviously $\leq n$. We number the sites of Γ as follows. First fix an ordering of the unit vectors of $\mathbb{Z}^d$ from 1 to $2d$. Then

[9] T. C. Dorlas *loc. cit.*

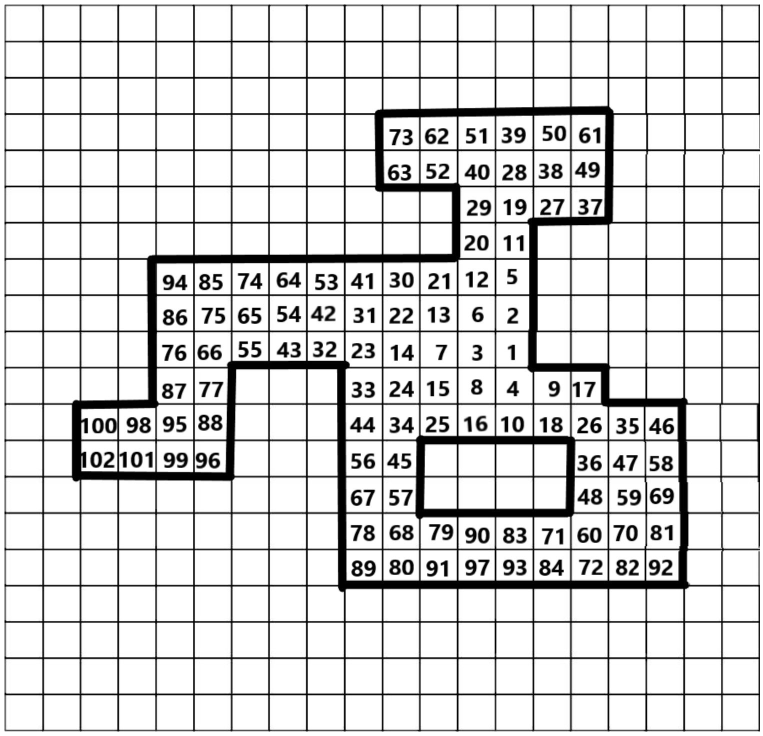

Figure 5. Numbering of the sites of a contour.

let x have number 1 and suppose that the first k points of Γ have already been numbered. Choose the already numbered site of lowest assigned number which has still got a neighbour in Γ which has not been numbered. Assign the number $k+1$ to its unnumbered neighbour with difference vector of lowest order. This defines a unique map from the sets Γ containing x with n sites to numberings of sites in $\mathbb{Z}^d$. It is illustrated in Figure 5.

Conversely, suppose the neighbours of the first $k-1$ points of Γ have already been determined. Then k has at most $2d-1$ unfilled neighbours left, for which there are $2^{2d-1}-1$ possible fillings. Therefore the possible number of choices is certainly bounded by $2^{(2d-1)(n-1)}$. The total number of sets Γ is therefore bounded by $n2^{(2d-1)(n-1)} \leq 2^{2dn}$. For each site in the set, the number of possible spin values is $\leq q$, so we find that $c_d \leq 2d\ln 2 + \ln q$. ∎

In the following we put $c_d = 2d\ln 2 + \ln q$ for definiteness.

We now define partition functions as follows

Definition 4.1 *If F is a contour functional then we define the* ***crystal partition function*** *by*

$$\Xi(\Gamma \,|\, F) = e^{-F(\Gamma)} \sum_{\partial \subset \mathrm{Int}(\Gamma)} e^{-F(\partial)}, \tag{4.2}$$

for a contour $\Gamma \in \mathcal{C}$, and the ***dilute partition function*** *for any finite $\Lambda \subset \mathbb{Z}^d$ by*

$$\Xi_\Lambda(F) = \sum_{\partial \in \mathcal{D}(\Lambda)} e^{-F(\partial)}, \tag{4.3}$$

where $\mathcal{D}(\Lambda)$ is the set of boundaries $\partial \subset \Lambda$.

We also define the corresponding probability measure

$$\mu_\Lambda(\partial) = \mu_\Lambda(\partial \,|\, F) = \frac{e^{-F(\partial)}}{\Xi_\Lambda(F)} \tag{4.4}$$

for $\partial \in \mathcal{D}(\Lambda)$, and the finite-volume correlation functions $\rho_\Lambda(\partial)$ by

$$\rho_\Lambda(\partial) = \rho_\Lambda(\partial \,|\, F) = \sum_{\tilde{\partial} \in \mathcal{D}(\Lambda) : \, \partial \subset \tilde{\partial}} \mu_\Lambda(\tilde{\partial}). \tag{4.5}$$

4.2 Thermodynamic limit of correlation functions

We want to prove that the infinite-volume limit of the correlation functions exists.

Lemma 4.2 *The correlation functions $\rho_\Lambda(\partial)$ satisfy the condition*

$$\rho_\Lambda(\partial) \le e^{-F(\partial)}$$

and the ***Mayer-Montroll equations***

$$\rho_\Lambda(\partial) = \chi_\Lambda(\partial) e^{-F(\partial)} \left[1 + \sum_{\substack{\partial' \in \mathcal{D}(\Lambda): \\ \partial' \subset \mathrm{Cp}(\partial)^c, \, \partial' \neq \emptyset}} (-1)^{|\partial'|} \rho_\Lambda(\partial') \right] \tag{4.6}$$

where $\chi_\Lambda(\partial) = 1$ if $\partial \subset \Lambda$ and $= 0$ otherwise.

Proof. We have by definition, for $\partial \in \mathcal{D}$,

$$\begin{aligned}
\rho_\Lambda(\partial) &= \chi_\Lambda(\partial) e^{-F(\partial)} \sum_{\partial' \in \mathcal{D}(\Lambda):\, \partial' \cup \partial \in \mathcal{D}} \mu_\Lambda(\partial') \\
&= \chi_\Lambda(\partial) e^{-F(\partial)} \left[1 - \sum_{\partial' \in \mathcal{D}(\Lambda):\, \partial' \cup \partial \notin \mathcal{D}} \mu_\Lambda(\partial') \right].
\end{aligned}$$

It follows immediately that $\rho_\Lambda(\partial) \leq e^{-F(\partial)}$.

Defining $\mathcal{D}_\Gamma(\Lambda) = \{\partial \in \mathcal{D}(\Lambda) : \Gamma \in \partial\}$, we have

$$\{\partial' \in \mathcal{D}(\Lambda) : \partial' \cup \partial \notin \mathcal{D}\} = \bigcup_{\Gamma \in \mathcal{C} : \Gamma \in \mathrm{Cp}(\partial)^c} \mathcal{D}_\Gamma(\Lambda).$$

By the **inclusion-exclusion principle**,

$$\mathbb{P}\left(\bigcup_{k=1}^{n} A_k \right) = \sum_{I \subset \{1,\ldots,n\};\, I \neq \emptyset} (-1)^{|I|-1} \mathbb{P}\left(\bigcap_{k \in I} A_k \right). \tag{4.7}$$

Applying this in our case we have

$$\begin{aligned}
\sum_{\partial' \in \mathcal{D}(\Lambda):\, \partial' \cup \partial \notin \mathcal{D}} \mu_\Lambda(\partial') &= \mu_\Lambda \left(\bigcup_{\Gamma \in \mathcal{C} : \Gamma \in \mathrm{Cp}(\partial)^c} \mathcal{D}_\Gamma(\Lambda) \right) \\
&= - \sum_{\partial' \in \mathcal{D}(\Lambda):\, \partial' \subset \mathrm{Cp}(\partial)^c,\, \partial' \neq \emptyset} (-1)^{|\partial'|} \mu_\Lambda \left(\bigcap_{\Gamma \in \partial'} \mathcal{D}_\Gamma(\Lambda) \right) \\
&= - \sum_{\partial' \in \mathcal{D}(\Lambda):\, \partial' \subset \mathrm{Cp}(\partial)^c,\, \partial' \neq \emptyset} (-1)^{|\partial'|} \rho_\Lambda(\partial').
\end{aligned}$$

∎

We now define an operator A acting on boundary functionals ξ as follows.

$$(A\xi)(\partial) = e^{-F(\partial)} \sum_{\partial' \subset \mathrm{Cp}(\partial)^c,\, \partial' \neq \emptyset} (-1)^{|\partial'|} \xi(\partial'). \tag{4.8}$$

By the above lemma, ρ_Λ satisfies the following equation,

$$\xi = \chi_\Lambda e^{-F} + \chi_\Lambda A \chi_\Lambda \xi. \tag{4.9}$$

We may then expect that in the thermodynamic limit, the correlation functions satisfy

$$\xi = e^{-F} + A\xi. \tag{4.10}$$

Obviously, for this to hold the operator A must be small in some sense. We therefore define a norm on the space of boundary functionals as follows.

$$||\xi||_\Lambda = \sup_{\partial \in \mathcal{D}(\Lambda)} |\xi(\partial)|\, e^{F(\partial) - c_d ||\partial|| + (\tau - c_d) d(\partial, \Lambda^c)}. \tag{4.11}$$

Here we assume $\tau > c_d$ and we denote $||\partial|| = \sum_{\Gamma \in \partial} |\Gamma|$. We denote the corresponding Banach space of boundary functionals by $\mathcal{E}_\Lambda$. In particular, if $\Lambda = \emptyset$,

$$||\xi||_\emptyset = \sup_{\partial \in \mathcal{D}} |\xi(\partial)|\, e^{F(\partial) - c_d ||\partial||}.$$

Lemma 4.3 *If $\tau \geq 3c_d$ then $||A||_\Lambda \leq e^{-c_d}$ for all finite $\Lambda \subset \mathbb{Z}^d$.*

Proof. Assume $||\xi||_\Lambda \leq 1$. Then, by the definition of A and the norm, and Peierls' bound (4.1),

$$|(A\xi)(\partial)| \leq e^{-F(\partial)} \sum_{\partial' \subset \mathrm{Cp}(\partial)^c,\, \partial' \neq \emptyset} e^{(c_d - \tau)(||\partial'|| + d(\partial', \Lambda^c))}.$$

We first argue that $d(\partial', \Lambda^c) \geq d(\partial, \Lambda^c) - \frac{4}{9}||\partial'||$. Indeed, there exist $\Gamma' \in \partial'$ such that $d(\Gamma', \Lambda^c) = d(\partial', \Lambda^c)$, and there exists $\Gamma \in \partial$ such that $d(\Gamma, \Gamma') \leq 1$, and therefore

$$\begin{aligned} d(\partial, \Lambda^c) &\leq d(\Gamma, \Lambda^c) \leq d(\Gamma, \Gamma') + \mathrm{diam}(\Gamma') + d(\Gamma', \Lambda^c) \\ &\leq 1 + \mathrm{diam}(\Gamma') + d(\partial', \Lambda^c). \end{aligned}$$

Since Γ' has walls of thickness at least 2, $\mathrm{diam}(\Gamma') \leq \frac{1}{3}|\Gamma'| \leq \frac{1}{3}||\partial'||$ and of course $||\partial'|| \geq 3^d \geq 9$. (The largest diameter is obtained if Γ is a rectangle of sides $3 \times L$.) We therefore have

$$|(A\xi)(\partial)| \leq e^{-F(\partial) + (c_d - \tau) d(\partial, \Lambda^c)} \sum_{\partial' \subset \mathrm{Cp}(\partial)^c\, \partial' \neq \emptyset} e^{\frac{5}{9}(c_d - \tau)||\partial'||}.$$

To estimate the latter sum we need to estimate the number of possible boundaries ∂' with total area $||\partial'|| = n$ incompatible with ∂. Now ∂' consists of a number k of contours $\Gamma'_1, \ldots, \Gamma'_k$ all at distances at least 2 from one another. For each of these contours Γ'_i to be incompatible with ∂, there must be a point x_i in one of the contours of ∂ such that $d(x_i, \Gamma'_i) \leq 1$. Hence

$$\begin{aligned}
\sum_{\partial' \subset \mathrm{Cp}(\partial)^c\, \partial' \neq \emptyset} e^{\frac{5}{9}(c_d - \tau)||\partial'||} &\leq \sum_{k=1}^{||\partial||} \binom{||\partial||}{k} \prod_{i=1}^{k} \left(\sum_{\Gamma'_i \colon d(x_i, \Gamma'_i) \leq 1} e^{\frac{5}{9}(c_d - \tau)|\Gamma'_i|} \right) \\
&\leq \sum_{k=0}^{||\partial||} \binom{||\partial||}{k} \left(\sum_{n=1}^{\infty} e^{c_d n} e^{\frac{5}{9}(c_d - \tau) n} \right)^k \\
&= \left(1 - e^{(14 c_d - 5\tau)/9}\right)^{-||\partial||}.
\end{aligned}$$

We conclude that

$$|(A\xi)(\partial)| \leq e^{c_d ||\partial|| - F(\partial) + (c_d - \tau) d(\partial, \Lambda^c)} \left(\frac{e^{-c_d}}{1 - e^{(14 c_d - 5\tau)/9}} \right)^{||\partial||}. \tag{4.12}$$

If $\tau \geq 3c_d$ then the last factor is less than $e^{-\tau}$. Indeed, both e^{-c_d} and $(1 - e^{(14c_d - 5\tau)/9})^{-1}$ are decreasing in c_d, so we only need to check this for $c_d = 4 \ln 2$. ∎

We can now conclude that the thermodynamic limit $\lim_{\Lambda \to \mathbb{Z}^d} \mu_\Lambda(\cdot \mid F)$ exists.

Theorem 4.1 *Let F be a contour functional satisfying the Peierls condition with constant $\tau \geq 3c_d$. Then, for every finite boundary $\partial \in \mathcal{D}$, the thermodynamic limit of the correlation function $\rho(\partial) = \lim_{\Lambda \to \mathbb{Z}^d} \rho_\Lambda(\partial)$ exists and satisfies the inequalities $\rho(\partial) \leq e^{-F(\partial)}$ and*

$$|\rho(\partial) - \rho_\Lambda(\partial)| \leq e^{c_d ||\partial|| - F(\partial) + (c_d - \tau) d(\partial, \Lambda^c))} \text{ if } \partial \subset \Lambda. \tag{4.13}$$

Proof. We expect ρ to satisfy the identity $\rho = e^{-F} + A\rho$, so we can define ρ by

$$\rho = \sum_{n=0}^{\infty} A^n e^{-F}.$$

Clearly, $||e^{-F}||_\emptyset \leq 1$, so this is well-defined as an element of $\mathcal{E}_\emptyset$ by the lemma.

Now, if $\partial \subset \Lambda$, then $\chi_\Lambda(\partial)e^{-F(\partial)} = e^{-F(\partial)}$, so

$$\begin{aligned}\rho(\partial) - \rho_\Lambda(\partial) &= \chi_\Lambda(\partial)((A\rho)(\partial) - (A\rho_\Lambda)(\partial)) \\ &= \chi_\Lambda(\partial)A(\rho - \chi_\Lambda\rho)(\partial) + \chi_\Lambda(\partial)A(\chi_\Lambda\rho - \rho_\Lambda)(\partial).\end{aligned}$$

Writing $\eta_\Lambda = \rho - \chi_\Lambda\rho$ we can write this as

$$\chi_\Lambda\rho - \rho_\Lambda = \chi_\Lambda A\eta_\Lambda + \chi_\Lambda A(\chi_\Lambda\rho - \rho_\Lambda).$$

This equation has the solution

$$\chi_\Lambda\rho - \rho_\Lambda = \sum_{n=1}^{\infty}(\chi_\Lambda A)^n\eta_\Lambda \in \mathcal{E}_\Lambda$$

provided the series converges in $\mathcal{E}_\Lambda$. This is the case if $||\chi_\Lambda A\eta_\Lambda||_\Lambda < +\infty$ since by the lemma, $||A||_\Lambda \leq e^{-c_d}$. However, if $\partial \subset \Lambda$, then the inequality (4.12) still holds even if $\chi_\Lambda\xi \neq 0$. Indeed, the estimates in the proof of the previous lemma remain valid if $d(\partial', \Lambda^c) = 0$. However, $d(\partial', \Lambda^c) = 0$ implies that $||\eta_\Lambda||_\Lambda = ||\eta_\Lambda||_\emptyset \leq ||\rho||_\emptyset \leq ||e^{-F}||_\emptyset(1 - ||A||_\emptyset)^{-1} < \frac{e^{-c_d}}{1-e^{-c_d}} < 1$. It follows that

$$||\chi_\Lambda\rho - \rho_\Lambda||_\Lambda \leq \frac{||A||_\Lambda}{1 - ||A||_\Lambda}||\eta_\Lambda||_\Lambda < 1.$$

This is just equation (4.13) and it implies the existence of the thermodynamic limit. ∎

Corollary 4.1 *If ∂_1 and ∂_2 are compatible boundaries then*

$$\begin{aligned}&|\rho(\partial_1 \cup \partial_2) - \rho(\partial_1)\rho(\partial_2)| \\ \leq\ & e^{c_d(||\partial_1||+||\partial_2||)-F(\partial_1)-F(\partial_2)+(c_d-\tau)d(\partial_1,\partial_2)}.\end{aligned} \tag{4.14}$$

Proof. To prove this mixing condition, let $\Lambda \subset \mathbb{Z}^d$ be finite and suppose $\partial_1, \partial_2 \subset \Lambda$. Set $M_2 = \{x \in \Lambda : d(x, \Gamma) > 1\, \forall \Gamma \in \partial_2\}$. If $\partial_1 \cup \partial_2 \subset \Lambda$ then

$$\frac{\rho_\Lambda(\partial_1 \cup \partial_2)}{\rho_\Lambda(\partial_2)} = \rho_\Lambda(\partial_1 \,|\, \partial_2) = \rho_{\Lambda\setminus M_2}(\partial_1).$$

By the inequality (4.13), we have

$$|\rho_{\Lambda\setminus M_2}(\partial_1) - \rho(\partial_1)| \leq e^{c_d||\partial_1||-F(\partial_1)+(c_d-\tau)d(\partial_1,\Lambda^c\cup M_2)}.$$

Inserting this, and using the fact that $|\rho_\Lambda(\partial_2)| \leq e^{-F(\partial_2)}$, we get

$$|\rho_\Lambda(\partial_1 \cup \partial_2) - \rho(\partial_1)\rho_\Lambda(\partial_2)| \leq e^{c_d(||\partial_1||+||\partial_2||)-F(\partial_1)-F(\partial_2)+(c_d-\tau)d(\partial_1,\Lambda^c\cup M_2)}.$$

Taking $\Lambda \to \mathbb{Z}^d$ results in the mixing property (4.14). ∎

4.3 External boundaries

As remarked, a contour model is an abstract concept independent of the spin models considered. We have seen in the case of the Ising model that we need to consider boundary conditions, and contours with different boundary conditions can be incompatible even if they are further than 1 apart. However, *exterior* contours with the same external boundary conditions are compatible if they are disjoint. We therefore want to relate the probability distribution of the external contours of a spin model with that of a contour model. We now prove the clustering property (3.1) for the external contours of a contour model, similar to (4.14).

Given a boundary $\partial \in \mathcal{D}$ we denote the set of external contours of ∂ by $\Theta(\partial)$. The set of **external boundaries** $\Theta \subset \Lambda$, i.e. a boundaries consisting of contours which are all external to each other, will be denoted $\tilde{\mathcal{D}}$. For $\Theta \in \tilde{\mathcal{D}}(\Lambda)$, we define the correlation functions $\tilde{\rho}_\Lambda(\Theta)$ and $\tilde{\rho}(\Theta)$ by

$$\tilde{\rho}_\Lambda(\Theta) = \mu_\Lambda(\{\partial \in \mathcal{D}(\Lambda) : \Theta \subset \Theta(\partial)\})$$

and

$$\tilde{\rho}(\Theta) = \mu(\{\partial \in \mathcal{D} : \Theta \subset \Theta(\partial)\}).$$

They satisfy similar equations to ρ_Λ and ρ:

Lemma 4.4 *The correlation functions $\tilde{\rho}_\Lambda(\Theta)$ satisfy the condition*

$$\tilde{\rho}_\Lambda(\Theta) \leq e^{-F(\Theta)}$$

and the Mayer-Montroll equations

$$\tilde{\rho}_\Lambda(\Theta) = \chi_\Lambda(\Theta) e^{-F(\Theta)} \left[1 + \sum_{\substack{\Theta' \in \tilde{\mathcal{D}}(\Lambda) : \, \Theta' \neq \emptyset, \\ \Theta' \subset \mathrm{Cp}(\Theta)^c \cup [\Theta]_\Lambda}} (-1)^{|\Theta'|} \tilde{\rho}_\Lambda(\Theta') \right]. \qquad (4.15)$$

Here $\mathrm{Cp}(\Theta)$ *is defined as the set of contours* Γ *such that* $\{\Gamma\} \cup \Theta$ *is an external boundary, and* $[\Theta]_\Lambda = [\Theta] \cap \mathcal{C}(\Lambda)$, *where* $[\Theta] = \{\Gamma \in \mathcal{C} : (\exists \tilde{\Gamma} \in \Theta)\, \tilde{\Gamma} \in \mathrm{Int}(\Gamma)\}$.

Proof. As in Lemma 4.2, we have

$$\begin{aligned}
\tilde{\rho}_\Lambda(\Theta) &= \mu_\Lambda(\{\partial \in \mathcal{D}(\Lambda) : \Theta \subset \Theta(\partial)\}) \\
&= e^{-F(\Theta)} \sum_{\substack{\partial' \in \mathcal{D}(\Lambda): \\ \partial' \subset \mathrm{Cp}(\Theta),\, \Theta \subset \Theta(\partial' \cup \Theta)}} \mu_\Lambda(\partial') \\
&= e^{-F(\Theta)} \left[1 - \mu_\Lambda\Big(\bigcup_{\Gamma \in \mathrm{Cp}(\Theta)^c \cup [\Theta]_\Lambda} \tilde{\mathcal{D}}_\Gamma(\Lambda)\Big) \right]
\end{aligned}$$

where $\tilde{\mathcal{D}}_\Gamma(\Lambda) = \{\partial \in \mathcal{D}(\Lambda) : \Gamma \in \Theta(\partial)\}$. Indeed, the complement of the set of $\partial' \in \mathcal{D}(\Lambda)$ such that $\partial' \subset \mathrm{Cp}(\Theta)$ and $\Theta \subset \Theta(\partial' \cup \Theta)$ is the set of $\partial' \in \mathcal{D}(\Lambda)$ such that $\partial' \not\subset \mathrm{Cp}(\Theta)$ or $\Theta \not\subset \Theta(\partial' \cup \Theta)$, i.e. those $\partial' \in \mathcal{D}(\Lambda)$ for which there exists $\Gamma' \in \partial'$ such that $\Gamma' \in \mathrm{Cp}(\Theta)^c$ or there exists $\tilde{\Gamma} \in \Theta$ such that $\tilde{\Gamma} \in \mathrm{Int}(\Gamma')$. But if there exists $\Gamma' \in \partial' \cap \mathrm{Cp}(\Theta)^c$ then there exists $\tilde{\Gamma} \in \Theta$ such that $\Gamma' \in \mathrm{Cp}(\tilde{\Gamma})^c \cap \partial'$, which implies that there exists $\Gamma'' \in \Theta(\partial')$ such that $\Gamma'' \in \mathrm{Cp}(\tilde{\Gamma})^c$ or $\tilde{\Gamma} \subset \mathrm{Int}(\Gamma'')$.

The statements of the lemma follow from this and the inclusion-exclusion principle (4.7). ∎

Lemma 4.5 *Define the operator B on external boundary functionals by*

$$(B\xi)(\Theta) = e^{-F(\Theta)} \left[1 + \sum_{\Theta' \subset \mathrm{Cp}(\Theta)^c \cup [\Theta] (-1)^{|\Theta'|} \Theta' \neq \emptyset} \xi(\Theta') \right]. \tag{4.16}$$

If $\tau \geq 3c_d$ then $||B||_\Lambda \leq e^{-c_d}$ for all finite $\Lambda \subset \mathbb{Z}^d$.

Proof. Note that there are now two ways in which a contour $\Gamma' \in \Theta'$ can be incompatible with Θ: either $d(\Gamma', \Theta) \leq 1$ or there is a $\Gamma \in \Theta$ such that $\Gamma \in \mathrm{Int}(\Gamma')$. In the former case, we have as before $d(\Theta', \Lambda^c) \geq d(\Theta, \Lambda^c) - \frac{4}{9}||\Theta'||$. In the latter case, $d(\Theta', \Lambda^c) \geq d(\Theta, \Lambda^c) - \frac{1}{3}||\Theta'||$ because Γ' surrounds Γ and has thickness at least 2, so $|\Gamma'| > 3d(\Gamma, \Gamma')$. As in Lemma 4.3, it therefore suffices to estimate the sum

$$\sum_{\Theta' \subset \mathrm{Cp}(\Theta)^c} e^{\frac{5}{9}(c_d - \tau)||\Theta'||}.$$

To estimate the latter sum we need to estimate the number of possible external boundaries Θ' with total area n incompatible with Θ. Now Θ' consists of a number k of external contours $\Gamma'_1, \ldots, \Gamma'_k$ but in this case, the contours $\Gamma'_i \in \Theta'$ are not necessarily at distance ≤ 1 from a contour $\Gamma_i \in \Theta$. There is also the possibility that Γ'_i surrounds a $\Gamma_i \in \Theta$. In both cases however, there is a point $x_i \in \Gamma_i \cup \mathrm{Int}(\Gamma_i)$. Choosing points $x_1, \ldots, x_k \in \bigcup_{\Gamma \in \Theta}(\Gamma \cup \mathrm{Int}(\Gamma))$ and contours surrounding these points therefore certainly exhausts all possible $\Theta' \in \mathrm{Cp}(\Theta)^c$ with $|\Theta'| = k$. The estimate

$$\sum_{\Theta' \subset \mathrm{Cp}(\Theta)^c} e^{\frac{5}{9}(c_d - \tau)||\Theta'||} \leq \left(1 - e^{(14c_d - 5\tau)/9}\right)^{-||\Theta||}. \tag{4.17}$$

therefore remains unchanged since the number of such contours of length n is still bounded by $e^{c_d n}$. ∎

As before, we conclude that the thermodynamic limit of the correlation functions $\tilde{\rho}_\Lambda$ exists.

Theorem 4.2 *Let F be a contour functional satisfying the Peierls condition with constant $\tau \geq 3c_d$. Then, for almost every $\partial \in \mathcal{D}$ the set $\Theta(\partial)$ is complete in the sense that every $\Gamma \in \partial \backslash \Theta(\partial)$ is surrounded by a unique external contour $\tilde{\Gamma} \in \Theta(\partial)$, i.e. $\Gamma \subset \mathrm{Int}(\tilde{\Gamma})$. Moreover, for every bounded external boundary Θ, $\lim_{\Lambda \to \mathbb{Z}^d} \tilde{\rho}_\Lambda(\Theta) = \tilde{\rho}(\Theta)$ exists and satisfies $\tilde{\rho}(\Theta) \leq e^{-F(\Theta)}$ and*

$$|\tilde{\rho}_\Lambda(\Theta) - \tilde{\rho}(\Theta)| \leq e^{(c_d - \tau)(||\Theta|| + d(\Theta, \Lambda^c))} \tag{4.18}$$

if $\Theta \subset \Lambda$.

Proof. The first statement follows from the Borel-Cantelli lemma, using the fact that if there exists and infinite sequence of contours $\Gamma_k \in \partial$ such that $\Gamma_k \subset \mathrm{Int}(\Gamma_{k+1})$, then

$$\sum_{n=1}^{\infty} \mu(\Gamma \subset \mathrm{Int}(\Gamma_n)) \leq \sum_{n=1}^{\infty} e^{(c_d - \tau)|\Gamma_n|} < +\infty$$

since $|\Gamma_n| > n$.

We define $\tilde{\rho}$ by the solution of $\tilde{\rho} = e^{-F} + B\tilde{\rho}$ as before, i.e.

$$\tilde{\rho} = \sum_{n=0}^{\infty} B^n e^{-F}.$$

This is well-defined as a series in the Banach space with norm $||\cdot||_{\emptyset}$. As in the proof of Theorem 4.1 we conclude using the above lemma that (4.18) holds. ∎

As for the correlations ρ we have a mixing condition.

Corollary 4.2 *$\tilde{\rho}$ satisfies the following exponential mixing condition. If Θ_1 and Θ_2 are compatible external boundaries then*

$$\begin{aligned} &|\tilde{\rho}(\Theta_1 \cup \Theta_2)) - \tilde{\rho}(\Theta_1)\tilde{\rho}(\Theta_2)| \\ &\quad \leq \quad e^{c_d(||\Theta_1||+||\Theta_2||)-F(\Theta_1)-F(\Theta_2)+(c_d-\tau)d(\Theta_1,\Theta_2)}. \end{aligned} \tag{4.19}$$

Proof. To prove the mixing condition, let $M = \bigcup_{\Gamma\in\Theta_2}(\Gamma \cup \mathrm{Int}(\Gamma))$. If $\Theta_1 \cup \Theta_2 \subset \Lambda$ then

$$\frac{\tilde{\rho}_\Lambda(\Theta_1 \cup \Theta_2)}{\tilde{\rho}_\Lambda(\Theta_2)} = \tilde{\rho}_\Lambda(\Theta_1 \,|\, \Theta_2) = \tilde{\rho}_{\Lambda\setminus M}(\Theta_1).$$

By the inequality (4.18), we have

$$|\tilde{\rho}_{\Lambda\setminus M}(\Theta_1) - \tilde{\rho}(\Theta_1)| \leq e^{c_d||\Theta_1||-F(\Theta_1)+(c_d-\tau)d(\Theta_1,\Lambda^c\cup M)}.$$

Inserting this, and using the fact that $|\rho_\Lambda(\Theta_2)| \leq e^{-F(\Theta_2)}$, we get

$$\begin{aligned} &|\tilde{\rho}_\Lambda(\Theta_1 \cup \Theta_2)) - \tilde{\rho}(\Theta_1)\tilde{\rho}_\Lambda(\Theta_2)| \\ &\qquad\qquad \leq e^{c_d(||\Theta_1||+||\Theta_2||)-F(\Theta_1)-F(\Theta_2)+(c_d-\tau)d(\Theta_1,\Lambda^c\cup M)}. \end{aligned}$$

Taking $\Lambda \to \mathbb{Z}^d$ results in the mixing property (4.19). ∎

4.4 Pressure and surface tension

We define the **pressure** of a contour model with contour functional F by

$$P(F) = \lim_{\Lambda\to\mathbb{Z}^d} \frac{1}{|\Lambda|} \ln \Xi_\Lambda(F). \tag{4.20}$$

The finite-volume correction will be denoted Δ:

$$\Delta_\Lambda(F) = \ln \Xi_\Lambda(F) - |\Lambda|\, P(F). \tag{4.21}$$

We now prove that the pressure exists and that Δ is of order $|\partial\Lambda|$.

Theorem 4.3 *Assume that F satisfies the Peierls condition with constant $\tau > 3c_d$. Then the pressure (4.20) exists as $\Lambda \to \mathbb{Z}^d$ in the sense of Van Hove, and is given by the formula*

$$P(F) = \int_1^\infty \sum_{\Gamma\in\mathcal{C}:\, 0\in\Gamma} \frac{F(\Gamma)}{|\Gamma|} \rho(\Gamma \mid \lambda F)\, d\lambda, \tag{4.22}$$

where $\rho(\Gamma \mid \lambda F)$ is the correlation function with contour functional λF. Moreover, $0 \le P(F) \le e^{-\tau}$ and $|\Delta_\Lambda(F)| \le e^{-\tau}|\partial\Lambda|$.

Proof. The proof is similar to that of the existence of the energy density in Theorem 2.6. Let $\rho_\Lambda(\cdot \mid F)$ denote the correlation function with contour functional F. Replacing F by a multiple λF and differentiating w.r.t. λ, we have

$$\frac{d}{d\lambda} \ln \Xi_\Lambda(\lambda F) = -\sum_{\Gamma\subset\Lambda} F(\Gamma)\, \rho_\Lambda(\Gamma \mid \lambda F),$$

and since $\lim_{\lambda\to+\infty} \Xi_\Lambda(\lambda F) = 1$ (only the empty boundary survives),

$$\ln \Xi_\Lambda(F) = \int_1^\infty \sum_{\Gamma\subset\Lambda} F(\Gamma)\, \rho_\Lambda(\Gamma \mid \lambda F)\, d\lambda. \tag{4.23}$$

Inserting the formula (4.22) and using translation invariance, the difference $\Delta_\Lambda(F) = \ln \Xi_\Lambda(F) - |\Lambda|\, P(F)$ is therefore given by

$$\begin{aligned} \Delta_\Lambda(F) \;\; = \;\; & \int_1^\infty \sum_{x\in\Lambda} \sum_{\Gamma\subset\Lambda:\, x\in\Gamma} \frac{F(\Gamma)}{|\Gamma|} \left[\rho_\Lambda(\Gamma \mid \lambda F) - \rho(\Gamma \mid \lambda F)\right] d\lambda \\ & - \int_1^\infty \sum_{x\in\Lambda} \sum_{\substack{\Gamma\in\mathcal{C}: \\ x\in\Gamma, \Gamma\cap\Lambda^c\neq\emptyset}} \frac{F(\Gamma)}{|\Gamma|}\, \rho(\Gamma \mid \lambda F)\, d\lambda. \end{aligned} \tag{4.24}$$

Using the inequality (4.13), the first term is bounded by

$$
\begin{aligned}
\int_1^\infty \sum_{\Gamma\subset\Lambda} & F(\Gamma)\,|\rho(\Gamma\,|\,\lambda F) - \rho_\Lambda(\Gamma\,|\,\lambda F)|\,d\lambda \\
&\le \int_1^\infty \sum_{\Gamma\subset\Lambda} F(\Gamma) e^{c_d|\Gamma| - \lambda F(\Gamma) + (c_d - \lambda\tau) d(\Gamma,\Lambda^c)} d\lambda \\
&= \sum_{\Gamma\subset\Lambda} \frac{F(\Gamma)}{F(\Gamma) + \tau d(\Gamma,\Lambda^c)} e^{c_d|\Gamma| - F(\Gamma) + (c_d-\tau) d(\Gamma,\Lambda^c)} \\
&\le \sum_{\Gamma\subset\Lambda} e^{(c_d-\tau)(|\Gamma| + d(\Gamma,\Lambda^c))} \\
&\le |\partial\Lambda| \sum_{k=0}^\infty \sum_{n=3^d}^\infty e^{c_d n} e^{(c_d-\tau)(n+k)} \\
&\le |\partial\Lambda| \frac{e^{3^d(2c_d-\tau)}}{(1-e^{c_d-\tau})(1-e^{2c_d-\tau})} < \frac{1}{2} e^{-\tau} |\partial\Lambda|
\end{aligned}
$$

if $\tau > 3c_d$ and $c_d > 1$. The second term is bounded by

$$
\begin{aligned}
\int_1^\infty \sum_{x\in\Lambda} & \sum_{\substack{\Gamma\in\mathcal{C}: \\ x\in\Gamma, \Gamma\cap\Lambda^c\neq\emptyset}} \frac{F(\Gamma)}{|\Gamma|}\,\rho(\Gamma\,|\,\lambda F)\,d\lambda \\
&\le \int_1^\infty \sum_{x\in\Lambda} \sum_{\substack{\Gamma\in\mathcal{C}: \\ x\in\Gamma, \Gamma\cap\Lambda^c\neq\emptyset}} \frac{F(\Gamma)}{|\Gamma|}\, e^{-\lambda F(\Gamma)}\,d\lambda \\
&= \sum_{x\in\Lambda} \sum_{\substack{\Gamma\in\mathcal{C}: \\ x\in\Gamma, \Gamma\cap\Lambda^c\neq\emptyset}} \frac{1}{|\Gamma|}\, e^{-F(\Gamma)} \\
&\le \sum_{\substack{\Gamma\in\mathcal{C}: \\ \Gamma\cap\Lambda\neq\emptyset, \Gamma\cap\Lambda^c\neq\emptyset}} e^{-\tau|\Gamma|} \\
&\le |\partial\Lambda| \sum_{n=3^d}^\infty e^{(c_d-\tau)n} < \frac{1}{2} e^{-\tau} |\partial\Lambda|.
\end{aligned}
$$

It remains to prove the bound on $P(F)$. Obviously $P(F) \ge 0$ since $\Xi_\Lambda(F) \ge 1$. On the other hand by the fact that $\rho(\Gamma\,|\,F) \le e^{-F(\Gamma)}$,

$$
\begin{aligned}
P(F) &\le \int_1^\infty \sum_{\Gamma\in\mathcal{C}:\, 0\in\Gamma} \frac{F(\Gamma)}{|\Gamma|} e^{-\lambda F(\Gamma)} d\lambda \\
&= \sum_{\Gamma\in\mathcal{C}:\, 0\in\Gamma} \frac{1}{|\Gamma|} e^{-F(\Gamma)} \le \sum_{n=3^d}^\infty \frac{1}{n} e^{(c_d-\tau)n} < e^{-\tau}. \qquad \blacksquare
\end{aligned}
$$

We need continuity properties w.r.t. the functional F. For this we introduce a norm on contour functionals. In fact, we define two norms on the contour functionals. The natural norm is given by

$$||F|| = \sup_{\Gamma\in\mathcal{C}} \frac{|F(\Gamma)|}{|\Gamma|}, \tag{4.25}$$

and we denote the corresponding space of contour functionals by $\mathcal{F}$. We also need a weaker norm-topology defined by

$$||F||_w = \sup_{\Gamma\in\mathcal{C}} \frac{|F(\Gamma)|}{(|\Gamma| + |\mathrm{Int}(\Gamma)|)e^{a\,\delta(\Gamma)}}. \tag{4.26}$$

Here $a \geq 0$ is a constant which we set equal to 2, and we put $\delta(\Gamma) = \mathrm{diam}(\mathrm{Int}(\Gamma))$. The corresponding space of contour functionals we denote by $\mathcal{F}_w$. Clearly, $\mathcal{F} \subset \mathcal{F}_w$.

Theorem 4.4 *If F and F' are contour functionals in $\mathcal{F}_w$ satisfying the Peierls condition with constant $\tau > 3c_d$ then*

$$|P(F) - P(F')| \leq e^{-\tau}\, ||F - F'||_w.$$

Proof. This is similar to the bound on $P(F)$. We define an interpolation $F_t = tF + (1-t)F'$ and write

$$\frac{d}{dt}\ln \Xi_\Lambda(F_t) = \sum_{\Gamma\subset\Lambda}(F(\Gamma) - F'(\Gamma))\,\rho_\Lambda(\Gamma\,|\,F_t).$$

Since $|\mathrm{Int}(\Gamma)| \leq |\Gamma|^{d/(d-1)} \leq |\Gamma|^2$ and $\delta(\Gamma) \leq |\Gamma|$,

$$\begin{aligned}
&|\ln \Xi_\Lambda(F) - \ln \Xi_\Lambda(F')| \\
&\quad\leq \sum_{\Gamma\in\Lambda} e^{-\tau|\Gamma|}|F(\Gamma) - F'(\Gamma)| \\
&\quad\leq ||F - F'||_w \sum_{\Gamma\subset\Lambda} e^{-\tau|\Gamma|}(|\Gamma| + |\Gamma|^2)e^{a\,|\Gamma|} \\
&\quad\leq |\Lambda|\,||F - F'||_w \sum_{n=3^d}^{\infty}(n + n^2)\,e^{(c_d-\tau)n}e^{2n} \\
&\quad\leq |\Lambda|\,||F - F'||_w \frac{e^{(4+c_d-\tau)3^d}}{1 - e^{4+c_d-\tau}} \\
&\quad\leq |\Lambda|\,||F - F'||_w\, e^{-\tau}.
\end{aligned}$$

because $\tau > 3c_d$, $e^{-c_d} < 0.25$ and $3^d > 9$. (Note that $1 < \frac{3}{2}\ln 2$ so $4 \leq \frac{3}{2}c_d$.) ∎

4.5 Parametric contour model

Definition 4.2 *Let F be a contour functional satisfying the Peierls condition with constant τ. If $b \geq 0$, we define the* ***parametric partition function*** *by*

$$\Xi_\Lambda(F,b) = \sum_{\partial \subset \Lambda} e^{-F(\partial)} \prod_{\Gamma \in \Theta(\partial)} e^{b\,|\mathrm{Int}(\Gamma)|}.$$

Clearly,

$$\Xi_\Lambda(F) \leq \Xi_\Lambda(F,b) \leq \Xi_\Lambda(F) e^{b\,|\Lambda|}.$$

Hence, by Theorem 4.3,

$$-b|\Lambda| - e^{-\tau}|\partial\Lambda| \leq \ln \Xi_\Lambda(F,b) - (P(F)+b)|\Lambda| \leq e^{-\tau}|\partial\Lambda|. \tag{4.27}$$

We define analogous to $\Delta_\Lambda(F)$,

$$\Delta_\Lambda(F,b) = \ln \Xi_\Lambda(F,b) - (P(F)+b)|\Lambda|.$$

It is also continuous in the following sense.

Theorem 4.5 *Suppose that F and F' are contour functionals with Peierls constant $\tau > 3c_d$, and $b, b' \geq 0$. Then*

$$\begin{aligned} |\Delta_\Lambda(F',b') - \Delta_\Lambda(F,b)| &\leq 2\,|b'-b|\,|\Lambda| \\ &\quad + (\frac{1}{6} e^{a\,\mathrm{diam}(\Lambda)} + e^{-\tau})|\Lambda|\,||F-F'||_w. \end{aligned} \tag{4.28}$$

Proof. The dependence on b follows immediately from

$$\left| \frac{\partial \ln \Xi_\Lambda(F,b)}{\partial b} \right| \leq |\Lambda|.$$

For fixed b we have

$$\begin{aligned} |\Delta_\Lambda(F',b) - \Delta_\Lambda(F,b)| &\leq |P(F') - P(F)|\,|\Lambda| \\ &\quad + |\ln \Xi_\Lambda(F',b) - \ln \Xi_\Lambda(F,b)|. \end{aligned}$$

By the previous theorem it now remains to prove that

$$|\ln \Xi_\Lambda(F',b) - \ln \Xi_\Lambda(F,b)| \leq \frac{1}{6} e^{a\,\mathrm{diam}(\Lambda)} |\Lambda|\,||F-F'||_w.$$

For this we define the parametric contour model by

$$\mu_\Lambda(\partial \,|\, F, b) = \frac{1}{\Xi_\Lambda(F,b)} e^{-F(\partial)} \prod_{\Gamma \in \Theta(\partial)} e^{b\,|\mathrm{Int}(\Gamma)|}.$$

Introducing the interpolation $F_t = tF + (1-t)F'$ as before, we have

$$\frac{d}{dt} \ln \Xi_\Lambda(F_t, b) = \sum_{\Gamma \subset \Lambda} (F'(\Gamma) - F(\Gamma)) \rho_\Lambda(\Gamma \,|\, F_t, b),$$

where

$$\rho_\Lambda(\partial \,|\, F_t, b) = \sum_{\partial' \subset \Lambda :\, \partial \subset \partial'} \mu_\Lambda(\partial' \,|\, F_t, b)$$

is the parametric correlation function. It follows that, for some $t \in (0,1)$,

$$\begin{aligned}
&|\ln \Xi_\Lambda(F,b) - \ln \Xi_\Lambda(F',b)| \\
&\leq \sum_{\Gamma \subset \Lambda} \rho_\Lambda(\Gamma \,|\, F_t, b)\, |F(\Gamma) - F'(\Gamma)| \\
&\leq ||F - F'||_w \sum_{\partial \subset \Lambda} \mu_\Lambda(\partial \,|\, F_t, b) \sum_{\Gamma \in \partial} (|\Gamma| + |\mathrm{Int}(\Gamma)|) e^{a\,\delta(\Gamma)} \\
&\leq ||F - F'||_w \max_{\partial \subset \Lambda} \sum_{\Gamma \in \partial} (|\Gamma| + |\mathrm{Int}(\Gamma)|) e^{a\,\delta(\Gamma)}.
\end{aligned}$$

Set

$$\alpha(\Lambda) = \max_{\partial \subset \Lambda} \frac{1}{|\Lambda|} \sum_{\Gamma \in \partial} (|\Gamma| + |\mathrm{Int}(\Gamma)|) e^{a\,\delta(\Gamma)}$$

and define $\gamma(n) = \max_{\Lambda :\, \mathrm{diam}(\Lambda) \leq n} \alpha(\Lambda)$. If $\Gamma \subset \Lambda$ then $\delta(\Gamma) \leq \mathrm{diam}(\Lambda) - 1$. Therefore,

$$\begin{aligned}
\sum_{\Gamma \in \partial} (|\Gamma| + |\mathrm{Int}(\Gamma)|) e^{a\,\delta(\Gamma)} &\leq \sum_{\Gamma \in \Theta(\partial)} (|\Gamma| + |\mathrm{Int}(\Gamma)|) e^{a\delta(\Gamma)} \\
&\quad + \sum_{\Gamma \in \Theta(\partial)} |\mathrm{Int}(\Gamma)|\, \alpha(\mathrm{Int}(\Gamma)) \\
&\leq |\Lambda|\, [e^{a(\mathrm{diam}(\Lambda)-1)} + \gamma(\mathrm{diam}(\Lambda) - 1)].
\end{aligned}$$

Hence $\alpha(\Lambda) \leq e^{a(\mathrm{diam}(\Lambda)-1)} + \gamma(\mathrm{diam}(\lambda) - 1)$ and $\gamma(n) \leq e^{a(n-1)} + \gamma(n-1)$, so $\gamma(n) \leq \frac{e^{an}-1}{e^a - 1} < \frac{1}{6} e^{an}$ since $e^2 > 7$. ∎

For a parametric contour model there is likely a large contour inside Λ: see Figure 6. We prove here that the total volume of contours is large. We say that Λ has a **regular boundary** if there is a constant $K > 1$ such that $|\partial\Lambda| \leq K\,|\Lambda|^\delta$, where $\delta = 1 - 1/d$. (Note that there is also a constant $\kappa_d > 1$ such that $|\partial\Lambda| \geq \kappa_d\,|\Lambda|^\delta$.)

Lemma 4.6 *Given a parametric contour model on a large region Λ with regular boundary, with contour functional F and parameter $b > 0$, and given $\epsilon \in (0, 1/d)$, let $\mathcal{A}_\Lambda(\epsilon)$ be the set*

$$\mathcal{A}_\Lambda(\epsilon) = \{\partial \in \mathcal{D}(\Lambda) : \sum_{\Gamma\in\Theta(\partial)} |\mathrm{Int}(\Gamma)| \geq |\Lambda| - |\Lambda|^{1-\epsilon}\}.$$

Then, if $||F|| < +\infty$ and Λ is large enough, $\mu_\Lambda\left(\mathcal{A}_\Lambda(\epsilon)^c \,|\, F, b\right) < e^{-\tau|\partial\Lambda|}$.

Proof. The probability is given by

$$\mu_\Lambda\left(\mathcal{A}_\Lambda(\epsilon)^c \,|\, F, b\right) = \frac{\sum_{\partial\in\mathcal{A}_\Lambda(\epsilon)^c} e^{-F(\partial)} \prod_{\Gamma\in\Theta(\partial)} e^{b\,|\mathrm{Int}(\Gamma)|}}{\sum_{\partial\in\mathcal{D}(\Lambda)} e^{-F(\partial)} \prod_{\Gamma\in\Theta(\partial)} e^{b\,|\mathrm{Int}(\Gamma)|}}. \tag{4.29}$$

The denominator is of course $\Xi_\Lambda(F, b)$. The numerator can be bounded simply by

$$\begin{aligned}\sum_{\partial\in\mathcal{A}_\Lambda(\epsilon)^c} e^{-F(\partial)} \prod_{\Gamma\in\Theta(\partial)} e^{b\,|\mathrm{Int}(\Gamma)|} &\leq e^{b(|\Lambda|-|\Lambda|^{1-\epsilon})} \sum_{\partial\in\mathcal{D}(\Lambda)} e^{-F(\partial)} \\ &= e^{b(|\Lambda|-|\Lambda|^{1-\epsilon})}\Xi_\Lambda(F).\end{aligned} \tag{4.30}$$

In the denominator, we bound $\Xi_\Lambda(F, b)$ by the terms where $\Theta(\partial)$ consists of the largest contour $\Gamma_{\max}$ consisting of the inner boundary of Λ as above. Then

$$\begin{aligned}\Xi_\Lambda(F, b) &= \sum_{\partial\subset\Lambda} e^{-F(\partial)} \prod_{\Gamma\in\Theta(\partial)} e^{b|\mathrm{Int}(\Gamma)|} \\ &\geq \sum_{\partial\subset\Lambda_1} e^{-F(\partial)} e^{-F(\Gamma_{\max})} e^{b(|\Lambda|-|\partial\Lambda|)},\end{aligned}$$

where $\Lambda_1 = \Lambda \setminus \Gamma_{\max}$.

The factor $\sum_{\partial\in\mathcal{D}(\Lambda_1)} e^{-F(\partial)} = \Xi_{\Lambda_1}(F)$ is close to $\Xi_\Lambda(F)$:

$$\Xi_\Lambda(F) = \sum_{\partial\in\mathcal{D}(\Lambda_1)} e^{-F(\partial)} \left(1 + \sum_{\substack{\partial\in\mathcal{D}(\Lambda): \\ (\forall\Gamma\in\partial)\,\Gamma\cap\Lambda\setminus\Lambda_1\neq\emptyset}} e^{-F(\partial)}\right), \tag{4.31}$$

where the expression in brackets is bounded by

$$
\begin{aligned}
1 + & \sum_{\substack{\partial \in \mathcal{D}(\Lambda): \\ (\forall \Gamma \in \partial)\, \Gamma \cap \Lambda \setminus \Lambda_1 \neq \emptyset}} e^{-F(\partial)} \\
\leq & \sum_{r=0}^{\infty} \frac{1}{r!} \left(\sum_{\Gamma \subset \Lambda, \Gamma \cap \Lambda \setminus \Lambda_1 \neq \emptyset} e^{-F(\Gamma)} \right)^r \\
\leq & \sum_{r=0}^{\infty} \frac{1}{r!} \left(|\partial \Lambda| \sum_{\Gamma \in \mathcal{C}:\, 0 \in \Gamma} e^{-F(\Gamma)} \right)^r \\
\leq & \exp \left(|\partial \Lambda| \sum_{n=9}^{\infty} e^{(c_d - \tau) n} \right) \\
= & \exp \left(|\partial \Lambda| \frac{e^{9(c_d - \tau)}}{1 - e^{c_d - \tau}} \right). \qquad (4.32)
\end{aligned}
$$

It follows that, writing $\gamma = \frac{e^{9(c_d-\tau)}}{1-e^{c_d-\tau}}$,

$$
\mu_\Lambda(\mathcal{A}_\Lambda(c)^c \,|\, F, b) \leq e^{-b\,|\Lambda|^{1-\epsilon}} e^{F(\Gamma_{\max})} e^{b\,|\partial\Lambda|} e^{\gamma|\partial\Lambda|}) < e^{-\tau|\partial\Lambda|} \qquad (4.33)
$$

provided $b|\Lambda|^{1-\epsilon} > (\tau + ||F|| + b + \gamma)|\partial\Lambda|$. ∎

We can strengthen this to show that there is a single large contour if we assume in addition that F satisfies a generalized Peierls estimate in the following sense: whenever Γ is a minimal contour given $\partial\Gamma$, i.e. it consists of the cubes of side 2 adjoining the boundary of Γ, then for any contour Γ' with $|\Gamma'| > |\Gamma|$, $F(\Gamma') - F(\Gamma) > \tau(|\Gamma'| - |\Gamma|)$.

Lemma 4.7 *Let a parametric contour model on a large region Λ be given by a contour functional F and a parameter $b > 0$. Let $\epsilon \in (0, 1/d)$ be given. Assume that Λ has a regular boundary and, moreover, that F satisfies the generalized Peierls condition. Define*

$$
\mathcal{B}_\Lambda(\epsilon) = \{\Theta \in \tilde{\mathcal{D}}(\Lambda) : (\exists \Gamma \in \Theta)\, |\mathrm{Int}(\Gamma)| \geq |\Lambda| - |\Lambda|^{1-\epsilon} \textit{ and } |\partial\Gamma| \leq K|V(\Gamma)|^\delta\},
$$

where $V(\Gamma) = |\Gamma| + |\mathrm{Int}(\Gamma)|$. Then, if $||F|| < +\infty$ and $\eta > 0$, for Λ is large enough, $\mu_\Lambda\left(\{\partial \in \mathcal{D}(\Lambda) : \Theta(\partial \in \mathcal{B}_\Lambda(\epsilon)^c \,|\, F, b) < \eta\right.$.

Proof. Choose first $\epsilon' \in (\epsilon, 1/d)$. By the previous lemma we can assume that $\sum_{\Gamma\in\Theta(\partial)} |\mathrm{Int}(\Gamma)| > |\Lambda| - |\Lambda|^{1-\epsilon'}$. We first note that we may assume that there exists $\Theta' \subset \Theta$ such that for $\Gamma \in \Theta'$, $|\Gamma| > \ln|\Lambda|$ and $\sum_{\Gamma\in\Theta'} |\mathrm{Int}(\Gamma)| > |\Lambda| - |\Lambda|^{1-\epsilon}$. Indeed, if $|\Gamma| \le \ln|\Lambda|$, then $|\mathrm{Int}(\Gamma)| \le (\kappa_d^{-1}|\Gamma|)^{d/(d-1)}$ and hence

$$\begin{aligned}\sum_{\Gamma\in\Theta:\,|\Gamma|\le\ln|\Lambda|} |\mathrm{Int}(\Gamma)| &\le \kappa_d^{-1}\sum_{k=3^d}^{\ln|\Lambda|} k^{d/(d-1)} n_k \\ &\le \kappa_d^{-1}(\ln|\Lambda|)^{1/(d-1)}\sum_{k=3^d}^{\ln|\Lambda|} k\, n_k \\ &\le \kappa_d^{-1}(\ln|\Lambda|)^{1/(d-1)}|\Lambda|^{1-\epsilon'} \le |\Lambda|^{1-\epsilon}\end{aligned}$$

for $|\Lambda|$ large enough. (Here n_k is the number of external contours of size k.)

We enumerate contours of size $k > \ln|\Lambda|$ by selecting a random distribution of n_k points in Λ and then estimating the number of contours containing one of these points by $e^{c_d k}$. This overestimates the number of contours because we are counting each contour k times and we are disregarding the fact that contours cannot overlap. The number of possible choices of these points is bounded as follows.

$$\begin{aligned}\frac{|\Lambda|!}{\left(\prod_{k=\ln(|\Lambda|)+1}^{|\Lambda|^{1-\epsilon}} n_k!\right)(|\Lambda| - \sum_k n_k)!} &\le \prod_{k=\ln|\Lambda|+1}^{|\Lambda|^{1-\epsilon}} \frac{|\Lambda|}{n_k} e^{n_k} \\ &= \exp\left[-\sum_{k=\ln|\Lambda|+1}^{|\Lambda|^{1-\epsilon}} n_k(\ln(n_k/|\Lambda|) - 1)\right].\end{aligned} \tag{4.34}$$

(The exponent is essentially the entropy

$$\begin{aligned}s(\underline{\rho}) &= -\sum_{k=\ln|\Lambda|+1^{|\Lambda|^{1-\epsilon}}} \rho_k \ln\rho_k - |\Lambda|(1-\sum_k \rho_k)\ln(1-\sum_k\rho_k) \\ &\le -\sum_{k=\ln|\Lambda|+1^{|\Lambda|^{1-\epsilon}}} \rho_k(\ln\rho_k - 1).\end{aligned} \tag{4.35}$$

This is similar to the entropy in Example 6.1.)

Now note that if $k > \ln|\Lambda|$ then $n_k = 0$ or $n_k > |\Lambda|\, e^{-k}$ and hence

$$e^{n_k(-\ln(n_k/|\Lambda|)+1-(\tau-c_d)k)} < e^{-\tau' k\, n_k},$$

where $\tau' = \tau - c_d - 1 - 3^{-d}$.

Since Λ has regular boundary with constant K, there exists a contour Γ_n with $|\mathrm{Int}(\Gamma_n)| = n$, $|\Lambda| - |\Lambda|^{1-\epsilon} \le n < |\Lambda|$ such that $|\partial\Gamma_n| < K\,|\mathrm{Int}(\Gamma)|$. (Reduce the maximal contour $\Gamma_{\max}$ in the direction where Λ is widest.)

The possible $\Theta' \subset \Theta$ such that for $\Gamma \in \Theta'$, $|\Gamma| > \ln|\Lambda|$ and such that $\sum_{\Gamma\in\Theta'} |\mathrm{Int}(\Gamma)| > |\Lambda| - |\Lambda|^{1-\epsilon}$, can be enumerated by the choice of points and the numbers n_k with $\ln|\Lambda| < k \le |\Lambda|^{1-\epsilon}$ denoting the number of contours $\Gamma \in \Theta'$ with $|\Gamma| = k$. The corresponding Boltzmann factor is bounded by

$$\sum_{\{n_k\}} e^{-\tau' \sum_k k\, n_k}, \tag{4.36}$$

where $k = \ln|\Lambda| + 1, \dots, |\Lambda|^{1-\epsilon}$ and $\sum_k k\, n_k \le |\Lambda|^{1-\epsilon}$ and since $|\mathrm{Int}(\Gamma)| < (\kappa_d^{-1}|\Gamma|)^{d/(d-1)}$, $\sum_k k^{d/(d-1)} n_k \ge \kappa_d^{d/(d-1)} \sum_{\Gamma\in\Theta} |\mathrm{Int}(\Gamma)| > \kappa_d^{d/(d-1)}|\Lambda|$. Let $k_1 < \cdots < k_p$ be the values of k for which $n_k \ge 1$. Then

$$\sum_{\{n_k\}} e^{-\tau' \sum_k k\, n_k} \le \sum_{p=1}^{|\Lambda|^{1-\epsilon}} \sum_{\ln|\Lambda| < k_1 < \cdots < k_p < |\Lambda|^{1-\epsilon}} \prod_{i=1}^{p} \frac{e^{-\tau' k_i}}{1 - e^{-\tau' k_i}}.$$

For $p = 1$, obviously,

$$\sum_{k_1=\ln|\Lambda|}^{|\Lambda|^{1-\epsilon}} e^{-\tau' k_1} = \frac{e^{-\tau' \ln|\Lambda|} - e^{-\tau'(|\Lambda|^{1-\epsilon}+1)}}{1 - e^{\tau'}} < \frac{1}{1 - e^{-\tau'}} |\Lambda|^{-\tau'}.$$

To estimate this sum in case $p = 2$, we use the following inequality, valid for $x > 0$ and $\delta \in (0, 1)$,

$$x + (1 - x^{1/\delta})^{\delta} \ge 1 + \lambda x \text{ if } 0 \le x \le 2^{-\delta} \text{ where } \lambda = 2 - 2^{\delta}. \tag{4.37}$$

It follows from the concavity of the left-hand side.

Consider the case $p = 2$. We first note that $k_1 + k_2 \le |\Lambda|^{1-\epsilon}$ and $k_1^{d/(d-1)} + k_2^{d/(d-1)} = n^{1/\delta} \ge K^{-d/(d-1)}|\Lambda|$. Set $x = k_1/n$ in the inequality (4.37). Then

$x \le n/2^\delta$ and, since $k_2 = n(1-(k_1/n)^{1/\delta})^\delta$,

$$\sum_{\substack{\ln|\Lambda|<k_1<k_2\le|\Lambda|^{1-\epsilon}\\ k_1^{1/\delta}+k_2^{1/\delta}=n^{1/\delta}}} \frac{e^{-\tau'(k_1+k_2)}}{(1-e^{-\tau' k_1})(1-e^{-\tau' k_2})} \tag{4.38}$$

$$= \sum_{k_1=\ln|\Lambda|+1}^{n/2^\delta} \frac{e^{-\tau'(k_1+(n^{1/\delta}-k_1^{1/\delta})^\delta)}}{(1-e^{-\tau' k_1})(1-e^{-\tau' n(1-(k_1/n)^{1/\delta})^\delta})}$$

$$\le (1-|\Lambda|^{-\tau'})^2 \sum_{k_1=\ln|\Lambda|}^{n/2^\delta} e^{-\tau' n(1+\lambda k_1/n)}$$

$$\le 2e^{-\tau' n}\frac{|\Lambda|^{-\lambda\tau'}}{1-e^{-\tau'\lambda}}. \tag{4.39}$$

For $p>2$ we generalize the inequality (4.37):

Lemma 4.8 *Let $\delta \in (0,1)$. If $0 \le x_1 \le x_2 \le \cdots \le x_p$ and $\sum_{i=1}^p x_i^{1/\delta} = 1$ then*

$$\sum_{i=1}^p x_i \ge 1 + \sum_{i=1}^{p-1} \lambda_{p-i} x_i, \tag{4.40}$$

where $\lambda_i = 1 + i^\delta - (i+1)^\delta$ for $i = 1, \dots, p-1$.

Proof. Again the left-hand side is a concave function of $x_1, \dots, x_{p-1}$, writing $x_p = (1-\sum_{i=1}^{p-1} x_i^{1/\delta})^\delta$. It therefore suffices to check the inequality for the extremal points of the simplex bounded by the rays $x_1 = \cdots = x_j = 0$, $0 < x_{j+1} \le \cdots \le x_{p-1} \le (p-j)^{-\delta}$. The extremal points are 0 and $(0,\dots,0,(p-j)^{-\delta},\dots,(p-j)^{-\delta})$. In the latter points, the left-hand side equals $(p-j)^{1-\delta}$, and the right-hand side equals $1+\sum_{i=j+1}^{p-1}\lambda_{p-i}(p-j)^{-\delta} = 1+\sum_{i'=1}^{p-j-1}\lambda_i(p-j)^{-\delta} = 1+(p-j-(p-j)^\delta)(p-j)^{-\delta} = (p-j)^{1-\delta}$, which proves the lemma. (Here we used the identity $\lambda_i = (i+1-(i+1)^\delta)-(i-i^\delta)$ so that the sum telescopes.) ∎

Note also that it follows from the concavity of the function $x \mapsto x^\delta$ that $\lambda_{i+1} > \lambda_i \geq 2 - 2^\delta$. Repeating the above argument for $p = 2$, we now have

$$\begin{aligned}
&\sum_{\substack{\ln|\Lambda| < k_1 < \cdots < k_p \leq |\Lambda|^{1-\epsilon} \\ \sum_{i=1}^p k_i^{1/\delta} = n^{1/\delta}}} \prod_{i=1}^p \frac{e^{-\tau' k_i}}{1 - e^{-\tau' k_i}} && (4.41) \\
\leq\ & 2 \sum_{\substack{\ln|\Lambda| \leq k_1 \leq \cdots \leq k_{p-1} \leq n \\ \sum_{i=1}^{p-1} k_i^{1/\delta} \leq n^{1/\delta}/2}} e^{-\tau'(\sum_{i=1}^{p-1} k_i + (n^{1/\delta} - \sum_{i=1}^{p-1} k_i^{1/\delta})^\delta)} \\
\leq\ & 2 \sum_{k_0 \leq k_1 \leq \cdots \leq k_{p-1}} e^{-\tau' n(1 + \sum_{i=1}^{p-1} \lambda_{p-i} k_i / n))} \\
=\ & 2 e^{-\tau' n} e^{-\tau' k_0 \sum_{i=1}^{p-1} \lambda_i} \prod_{j=1}^{p-1} \frac{1}{1 - e^{-\tau' \sum_{i=1}^j \lambda_i}} \\
\leq\ & 2\, e^{-\tau' n} |\Lambda|^{-\tau'(p - p^\delta)} \prod_{j=1}^{p-1} \frac{1}{1 - e^{-\tau' \sum_{i=1}^j \lambda_i}}. && (4.42)
\end{aligned}$$

Note that $(1 - e^{-\tau' k_i})^{-1} < 1 + 2e^{-\tau' k_i} < e^{e^{-\tau' k_i}}$ and hence

$$\prod_{i=1}^p (1 - e^{-\tau' k_i})^{-1} < \exp[\sum_{i=1}^p e^{-\tau' k_i}] < 2.$$

It follows that the sum over p converges and moreover that the term $p = 1$ dominates as $|\Lambda| \to \infty$. We have seen above that there is a large contour Γ_n with $|\Gamma_n| \leq K|\Lambda|^\delta$. Since F satisfies the generalized Peierls inequality, this means that large contours Γ with (significantly) larger $|\Gamma|$ are exponentially less likely. ∎

The situation is illustrated in Figure 6.

Remark. In fact, Figure 6 is slightly deceptive, because it is advantageous to have a slightly smaller contour. The size $|\Gamma|$ is then smaller, whereas the volume $|\mathrm{Int}(\Gamma)|$ is smaller. This is advantageous because in general $\tau >> b$. The size reduction is only of order 1, however.

5 Spin models versus contour models

Pirogov-Sinai theory is a generalization of Peierls argument for the existence of a phase transition at low temperatures to the case of spin models without symmetry. Remember that Peierls' argument uses explicitly the spin-flip symmetry to derive Peierls' estimate (1.6). This symmetry ensures that the phase transition occurs at $h = 0$. In the absence of such symmetry, the value of the critical external fields is in general dependent on the temperature. This was illustrated in Example 3.1. To include arbitrary contours, the critical fields will be determined by a contraction mapping argument.

We consider the inhomogeneous Potts model given by the Hamiltonian of equation (3.11), i.e.

$$\Phi_{x-y}(s_x, s_y) = \sum_{1\le r<r'\le q} J_{r,r'}(\delta_{s_x,r}\delta_{s_y,r'} + \delta_{s_x,r'}\delta_{s_y,r}), \tag{5.1}$$

for $|x-y| = 1$, and we set for a boundary condition r,

$$\mathcal{H}_\Lambda(s_\Lambda \,|\, r) = \mathcal{H}_\Lambda(s_\Lambda) + \sum_{x\in\Lambda, y\in\Lambda^c:\, |x-y|=1} \Phi_{x-y}(s_x, r). \tag{5.2}$$

Given a boundary condition r we define the partition function by

Definition 5.1 *The finite-volume* ***partition function*** $\mathcal{Z}_\Lambda^{(r)}(\beta, \underline{h})$ *for the inhomogeneous Potts model with boundary condition r is given by*

$$\mathcal{Z}_\Lambda^{(r)}(\beta, \underline{h}) = \sum_{s_\Lambda\in\Omega^{(r)}(\Lambda)} e^{-\beta\widehat{\mathcal{H}}_\Lambda(s_\Lambda \,|\, r)}, \tag{5.3}$$

where

$$\widehat{\mathcal{H}}_\Lambda(s_\Lambda \,|\, r) = \mathcal{H}_\Lambda(s_\Lambda \,|\, r) - \sum_{x\in\Lambda}\langle \underline{h} - h_r, \delta_{s_x}\rangle = \mathcal{H}_\Lambda(s_\Lambda \,|\, r) - \sum_{x\in\Lambda}(h_{s_x} - h_r) \tag{5.4}$$

is the ***relative Hamiltonian****.*

Note that we have added a constant $h_r|\Lambda|$ to the Hamiltonian, which of course does not affect the equilibrium state. We want to relate the distribution of external contours to that of a contour model and therefore define a corresponding crystal partition function.

Definition 5.2 *For a given (external) contour* (Γ, s_Γ) *with external boundary condition* r*, we define the* ***crystal partition function*** *by*

$$\mathcal{Z}_r((\Gamma, s_\Gamma), \beta, \underline{h}) =$$
$$e^{-\beta \widehat{\mathcal{H}}_\Gamma(s_\Gamma \,|\, r)} \prod_{m=1}^{q} \sum_{\underline{s} \in \Omega^{(m)}(\mathrm{Int}_m(\Gamma))} e^{-\beta \widehat{\mathcal{H}}_{\mathrm{Int}_m(\Gamma)}(\underline{s} \,|\, m) + \beta(h_m - h_r)|\mathrm{Int}_m(\Gamma)|}. \tag{5.5}$$

Crucially, the following identity then holds.

$$\mathcal{Z}_\Lambda^{(r)}(\beta, \underline{h}) = \sum_{\Theta \in \tilde{\mathcal{D}}_r(\Lambda)} \prod_{(\Gamma, s_\Gamma) \in \Theta} \mathcal{Z}_r((\Gamma, s_\Gamma), \beta, \underline{h}), \tag{5.6}$$

where $\tilde{\mathcal{D}}_r(\Lambda)$ denotes the set of exterior boundaries with boundary condition r. We also denote by $\mathcal{C}_r(\Lambda)$ the collection of contours in Λ with exterior boundary condition r.

We have the following correspondence.

Theorem 5.1 *Assume that* $3\,||\underline{h}|| < \rho := d2^{-d} J_{\min}$ *and* $\tau = \frac{1}{3}\beta\rho - 1 > 3c_d$*. Then there exist contour functionals* $F_r \in \mathcal{F}_w$ $(r = 1, \ldots, q)$ *on contours* $(\Gamma, s_\Gamma) \in \mathcal{C}_r$ *with boundary condition* r *respectively, satisfying the Peierls condition (4.1) and such that*

$$\mathcal{Z}_r((\Gamma, s_\Gamma), \beta, \underline{h}) = e^{b_r\,|\mathrm{Int}(\Gamma)|} \Xi((\Gamma, s_\Gamma) \,|\, F_r) \tag{5.7}$$

where

$$b_r = M - \beta h_r - P(F_r), \tag{5.8}$$

with $M = \max_{r=1}^{q}(P(F_r) + \beta h_r)$*. Moreover,* $F_1, \ldots, F_q$ *are continuous functions of* $J_{r,r'}$ $(1 \le r < r' \le q)$*.*

Proof. First observe that the identities (5.6) and (5.7) imply that

$$\begin{aligned}
\mathcal{Z}_\Lambda^{(r)}(\beta,\underline{h}) &= \sum_{\Theta\in\tilde{\mathcal{D}}_r(\Lambda)} \prod_{(\Gamma,s_\Gamma)\in\Theta} \mathcal{Z}_r((\Gamma,s_\Gamma),\beta,\underline{h}) \\
&= \sum_{\Theta\in\tilde{\mathcal{D}}_r(\Lambda)} \prod_{(\Gamma,s_\Gamma)\in\Theta} e^{b_r|\mathrm{Int}(\Gamma)|}\Xi((\Gamma,s_\Gamma)\,|\,F_r) \\
&= \sum_{\Theta\in\tilde{\mathcal{D}}_r(\Lambda)} \prod_{(\Gamma,s_\Gamma)\in\Theta} e^{b_r|\mathrm{Int}(\Gamma)|}e^{-F_r(\Gamma,s_\Gamma)} \sum_{\partial\in\mathcal{D}_r(\mathrm{Int}(\Gamma))} e^{-F_r(\partial)} \\
&= \sum_{\partial\in\mathcal{D}_r(\Lambda)} e^{-F_r(\partial)} \prod_{(\Gamma,s_\Gamma)\in\Theta(\partial)} e^{b_r|\mathrm{Int}(\Gamma)|} \\
&= \Xi_\Lambda(F_r,b_r).
\end{aligned} \tag{5.9}$$

Conversely, if $\mathcal{Z}_\Lambda^{(r)}(\beta,\underline{h}) = \Xi_\Lambda(F_r,b_r)$ then

$$\begin{aligned}
&\sum_{\Theta\in\tilde{\mathcal{D}}_r(\Lambda)} \prod_{(\Gamma,s_\Gamma)\in\Theta} \mathcal{Z}_r((\Gamma,s_\Gamma),\beta,\underline{h}) \\
&\quad= \sum_{\Theta\in\tilde{\mathcal{D}}_r(\Lambda)} \prod_{(\Gamma,s_\Gamma)\in\Theta} e^{b_r|\mathrm{Int}(\Gamma)|}\Xi((\Gamma,s_\Gamma)\,|\,F_r)
\end{aligned}$$

from which it follows that $\mathcal{Z}_r((\Gamma,s_\Gamma),\beta,\underline{h}) = e^{b_r|\mathrm{Int}(\Gamma)|}\Xi((\Gamma,s_\Gamma)\,|\,F_r)$.

Inserting the definition (5.5), we have that for $(\Gamma,s_\Gamma)\in\mathcal{C}_r$,

$$\begin{aligned}
&e^{b_r|\mathrm{Int}(\Gamma)|}\Xi((\Gamma,s_\Gamma)\,|\,F_r) \\
&\quad= \mathcal{Z}_r((\Gamma,s_\Gamma),\beta,\underline{h}) \\
&\quad= e^{-\beta\widehat{\mathcal{H}}_\Gamma(s_\Gamma\,|\,r)} \prod_{m=1}^{q} e^{\beta(h_m-h_r)|\mathrm{Int}_m(\Gamma)|}\mathcal{Z}_{\mathrm{Int}_m(\Gamma)}^{(m)}(\beta,\underline{h}) \\
&\quad= e^{-\beta\widehat{\mathcal{H}}_\Gamma(s_\Gamma\,|\,r)} \prod_{m=1}^{q} e^{\beta(h_m-h_r)|\mathrm{Int}_m(\Gamma)|}\Xi_{\mathrm{Int}_m(\Gamma)}(F_m,b_m).
\end{aligned} \tag{5.10}$$

Since by definition 4.1, $\Xi((\Gamma,s_\Gamma)\,|\,F_r) = e^{-F_r(\Gamma,s_\Gamma)}\Xi_{\mathrm{Int}(\Gamma)}(F_r)$, it follows from (4.21) that

$$\ln\Xi((\Gamma,s_\Gamma)\,|\,F_r) = P(F_r)\,|\mathrm{Int}(\Gamma)| - F_r(\Gamma,s_\Gamma) + \sum_{m=1}^{q}\Delta_{\mathrm{Int}_m(\Gamma)}(F_r). \tag{5.11}$$

On the other hand

$$\ln\Xi_{\mathrm{Int}_m(\Gamma)}(F_m,b_m) = (P(F_m)+b_m)\,|\mathrm{Int}_m(\Gamma)| + \Delta_{\mathrm{Int}_m(\Gamma)}(F_m,b_m). \tag{5.12}$$

Hence

$$
\begin{aligned}
&(P(F_r)+b_r)\,|\mathrm{Int}(\Gamma)| - F_r(\Gamma,s_\Gamma) + \sum_{m=1}^{q} \Delta_{\mathrm{Int}_m(\Gamma)}(F_r) \\
&= \quad -\beta\widehat{\mathcal{H}}_\Gamma(s_\Gamma\,|\,r) - \beta h_r|\mathrm{Int}(\Gamma)| \\
&\qquad + \sum_{m=1}^{q}\left[(P(F_m)+b_m+\beta h_m)\,|\mathrm{Int}_m(\Gamma)| + \Delta_{\mathrm{Int}_m(\Gamma)}(F_m,b_m)\right].
\end{aligned}
\tag{5.13}
$$

Inserting the identity (5.8), we have for $(\Gamma, s_\Gamma) \in \mathcal{C}_r$,

$$
F_r(\Gamma,s_\Gamma) = \beta\widehat{\mathcal{H}}_\Gamma(s_\Gamma\,|\,r) + T_r(\underline{F},\beta,\underline{h}), \tag{5.14}
$$

where the map $T_r(\underline{F},\beta,\underline{h})$ is defined by

$$
T_r(\underline{F},\beta,\underline{h}) = \widehat{T}_r(\underline{F},\underline{b}),
$$

in which b_m is given by equation (5.8) in terms of h_m and $P(F_m)$,

$$
b_m = -\beta h_m - P(F_m) + M,
$$

and

$$
\widehat{T}_r(\underline{F},\underline{b}) = \sum_{m=1}^{q}\left[\Delta_{\mathrm{Int}_m(\Gamma)}(F_r) - \Delta_{\mathrm{Int}_m(\Gamma)}(F_m,b_m)\right]. \tag{5.15}
$$

Note that it follows from the assumption $3||\underline{h}|| < d2^{-d}J_{\min} = \rho$ that

$$
\widehat{\mathcal{H}}_\Gamma(s_\Gamma\,|\,r) = \mathcal{H}_\Gamma(s_\Gamma) - \sum_{x\in\Gamma}\langle \underline{h} - h_r, \delta_{s_x}\rangle > \frac{1}{3}\rho\,|\Gamma|
$$

because every cube of side 2 in Γ contains at least one site with a spin different from the others, giving a contribution $dJ_{r,r'}$ so $\mathcal{H}_\Gamma(s_\Gamma) > dJ_{\min}2^{-d}|\Gamma|$, and by assumption, $\max(h_{r'} - h_r) < \frac{2}{3}\rho$. Then, by the inequalities (4.27),

$$
\tilde{T}_r(\underline{F},\underline{b})(\Gamma) \geq -2e^{-\tau}\sum_{m=1}^{q}|\partial(\mathrm{Int}_m(\Gamma))| \geq -2e^{-\tau}|\Gamma|,
$$

and therefore

$$
\beta\widehat{\mathcal{H}}_\Gamma(s_\Gamma\,|\,r) + T_r(\underline{F},\beta,\underline{h}) \geq (\frac{1}{3}\beta\rho - 2e^{-\tau})|\Gamma| > \tau|\Gamma|.
$$

We next prove that the operator $\underline{T} = (T_r)_{r=1}^q$ has small norm, which implies the existence of a unique solution $\underline{F}$. In fact, by Theorem 4.5,

$$
\begin{aligned}
&|\widehat{T}_r(\underline{F},\underline{b})(\Gamma) - \widehat{T}_r(\underline{F}',\underline{b}')(\Gamma)| \\
&\le \sum_{m=1}^{q} \big(|\Delta_{\mathrm{Int}_m(\Gamma)}(F_r) - \Delta_{\mathrm{Int}_m(\Gamma)}(F_r')| \\
&\qquad\qquad + |\Delta_{\mathrm{Int}_m(\Gamma)}(F_m, b_m) - \Delta_{\mathrm{Int}_m(\Gamma)}(F_m', b_m')|\big) \\
&\le 2\sum_{m=1}^{q} |b_r' - b_r|\,|\mathrm{Int}_m(\Gamma)| \\
&\qquad + 2\sum_{m=1}^{q} (\frac{1}{6} e^{a\,\mathrm{diam}(\mathrm{Int}_m(\Gamma))} + e^{-\tau})|\mathrm{Int}_m(\Gamma)|\,||F_m - F_m'||_w
\end{aligned} \tag{5.16}
$$

and hence

$$
\begin{aligned}
&|T_r(\underline{F},\beta,\underline{h})(\Gamma) - T_r(\underline{F}',\beta,\underline{h}')(\Gamma)| \\
&\le 2(||\beta\underline{h} - \beta'\underline{h}'|| + e^{-\tau}||\underline{F} - \underline{F}'||_w)\,|\mathrm{Int}(\Gamma)| \\
&\quad + (\frac{1}{3} e^{a\delta(\Gamma)} + 2e^{-\tau})||\underline{F} - \underline{F}'||_w\,|\mathrm{Int}(\Gamma)|,
\end{aligned} \tag{5.17}
$$

where we have defined

$$
||\underline{F} - \underline{F}'||_w = \max_{r=1}^{q} ||F_r - F_r'||_w,
$$

and where we also used the continuity of the pressure, Theorem 4.4. Recalling the definition of the norm (4.26), we have

$$
||\underline{T}(\underline{F},\beta,\underline{h}) - \underline{T}(\underline{F}',\beta',\underline{h}')||_w \le 2\,||\beta\underline{h} - \beta'\underline{h}'|| + (\frac{1}{3} + 4e^{-\tau})||\underline{F} - \underline{F}'||_w. \tag{5.18}
$$

In particular, $\underline{T}$ is a contraction for fixed $\underline{h}$ and there exists a unique solution to the set of equations (5.14). Moreover, the inequality (5.18) also implies the continuity of the solution in $\mathcal{H}_\Gamma$ and $\underline{h}$ and hence $J_{r,r'}$ subject to the conditions $2||\underline{h}|| < d3^{-d}J_{\min} = \rho$ and $\tau = \beta\rho - 1 > 3c_d$. ∎

In the following we need to improve the inequality (4.27). In fact the lower bound is also of order $|\partial\Lambda|$:

Corollary 5.1 *Assume that $||F_r|| < +\infty$. Then $-\beta f(\beta, \underline{h}) = b_r + P(F_r)$. Moreover, The surface tension of the parametric contour model with $b_r > 0$ satisfies*

$$-(b_r + P(F_r) + ||\Phi|| + 2||\underline{h}|| + e^{-\tau})|\partial\Lambda| \leq \Delta_\Lambda(F, b) \leq e^{-\tau}|\partial\Lambda|. \qquad (5.19)$$

Therefore the solutions F_r in fact belong to $\mathcal{F}$, i.e. $||F_r|| < +\infty$.

Proof. Since by (5.9),

$$\begin{aligned}
\mathcal{Z}_\Lambda^{(r)}(\beta, \underline{h}) &= \Xi_\Lambda(F_r, b_r) \\
&= \sum_{\partial \in \mathcal{D}_r(\Lambda)} e^{-F_r(\partial)} \prod_{\Gamma \in \Theta(\partial)} e^{b_r |\mathrm{Int}(\Gamma)|} \\
&\leq e^{b_r|\Lambda|} \sum_{\partial \in \mathcal{D}_r(\Lambda)} e^{-F_r(\partial)},
\end{aligned}$$

we have the upper bound

$$-\beta f(\beta, \underline{h}) = \lim_{\Lambda \to \mathbb{Z}^d} \frac{1}{|\Lambda|} \ln \mathcal{Z}_\Lambda^{(r)}(\beta, \underline{h}) \leq b_r + \lim_{\Lambda \to \infty} \frac{1}{|\Lambda|} \ln \Xi_\Lambda(F_r) = b_r + P(F_r).$$

On the other hand, if $||F_r|| < +\infty$, we can choose the maximal contour consisting of $\Gamma_{\max} = \{x \in \Lambda : d(x, \Lambda^c) = 1\}$ and obtain, writing $\Lambda_1 = \Lambda \setminus \Gamma_{\max}$,

$$\begin{aligned}
\mathcal{Z}_\Lambda^{(r)}(\beta, \underline{h}) &\geq \sum_{\partial \in \mathcal{D}_r(\Lambda_1)} e^{-F_r(\partial) - F_r(\Gamma_{\max})} e^{b_r|\Lambda_1|} \\
&\geq e^{-||F_r||\,|\Gamma_{\max}|} e^{(b_r + P(F_r))|\Lambda_1| + \Delta_{\Lambda_1}(F_r)} \\
&\geq e^{-(||F_r|| + b_r + P(F_r))\,|\Gamma_{\max}|} e^{(b_r + P(F_r))|\Lambda| + \Delta_{\Lambda_1}(F_r)}.
\end{aligned}$$

Since $|\Delta_{\Lambda_1}(F_r)| \leq e^{-\tau}|\partial\Lambda_1|$, it follows that

$$-\beta f(\beta, \underline{h}) \geq b_r + P(F_r).$$

The upper bound of (5.19) is (4.27). Similarly also the lower bound if $b_r = 0$. To prove the lower bound if $b_r > 0$, let $\Gamma_{\max}$ be a maximal contour given by the inner boundary of Λ with inner boundary condition $\tilde{r}$ where

$b_{\tilde{r}} = 0$ (this means $s_{\Gamma_{\max}} = \tilde{r}$). Then we have

$$\begin{aligned}
\mathcal{Z}_{\Lambda}^{(r)}(\beta, \underline{h}) &\geq e^{-\beta \widehat{\mathcal{H}}_{\Gamma_{\max}}(\tilde{r}\,|\,r)} \mathcal{Z}_{\Lambda_1}^{(\tilde{r})}(\beta, \underline{h}) \\
&= e^{-\beta \widehat{\mathcal{H}}_{\Gamma_{\max}}(\tilde{r}\,|\,r)} \Xi_{\Lambda_1}(F_{\tilde{r}}) \\
&= e^{-\beta \widehat{\mathcal{H}}_{\Gamma_{\max}}(\tilde{r}\,|\,r)} e^{|\Lambda_1|\, P(F_{\tilde{r}}) + \Delta_{\Lambda_1}(F_{\tilde{r}})} \\
&= e^{-\beta \widehat{\mathcal{H}}_{\Gamma_{\max}}(\tilde{r}\,|\,r)} e^{(P_r + b_r)|\partial\Lambda| + \Delta_{\Lambda_1}(F_{\tilde{r}})} e^{|\Lambda|\,(P(F_r) + b_r)}.
\end{aligned}$$

Since $\widehat{\mathcal{H}}_{\Gamma}(s_\Gamma\,|\,r) \leq (||\Phi|| + 2\,||\underline{h}||)|\Gamma|$ the lower bound follows. Inserting into the formula (5.15), we conclude that $||F_r|| < +\infty$. ∎

It follows that in case of a boundary condition r with $b_r > 0$, there is a large contour close to the boundary. This is illustrated in Figure 6 where the black colour corresponds to a spin value r with $b_r > 0$.

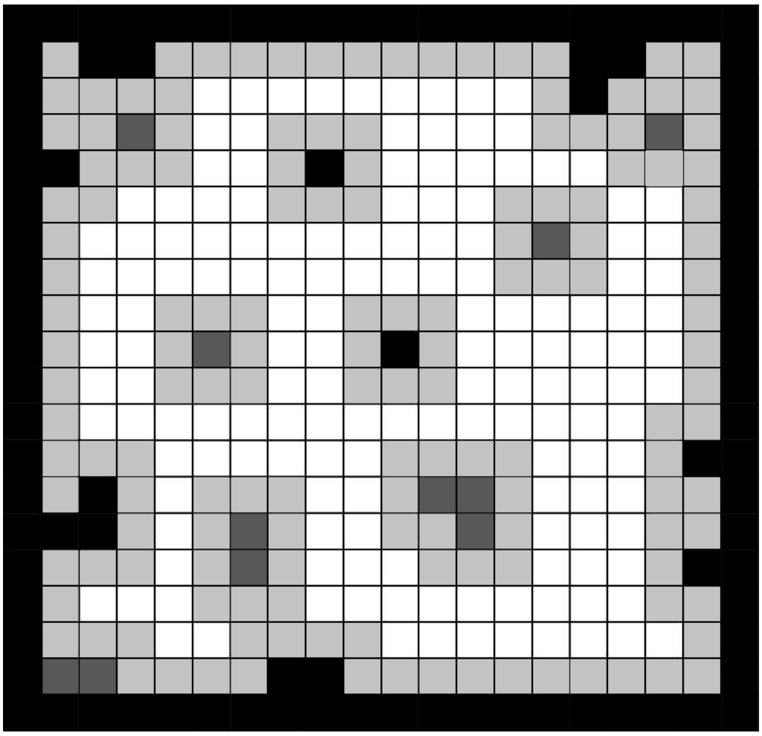

Figure 6. A large contour in the Potts model with $q = 3$.

6 The low-temperature phase diagram

We are now ready to describe the low-temperature phase diagram in general. First, however, let us consider again the example of the 3-state Potts model, Example 3.1.

EXAMPLE 6.1. *Phase diagram of the 3-state Potts model.*

In order to obtain the complete phase diagram of Example 3.1, we need to consider also the case where $h_2 \neq 0$ (we can assume $h_1 = 0$ because adding a constant to the Hamiltonian does not change its thermodynamics). In that case, it is also possible that the phases 1 and 3 or 2 and 3 coexist. As in Example 3.1 we have

$$f_1(\rho_2, \rho_3) = (2dJ - h_2)\rho_2 + (2d\tilde{J} - h_3)\rho_3 - \frac{1}{\beta} s(\rho_2, \rho_3),$$

and minimising,

$$f_{1,\min} = -\frac{1}{\beta}(\rho_2 + \rho_3),$$

where

$$\rho_2 = e^{-\beta(2dJ - h_2)} \text{ and } \rho_3 = e^{-\beta(2d\tilde{J} - h_3)}.$$

Similarly,

$$f_2(\rho_1, \rho_3) = 2dJ\rho_1 - h_2(1 - \rho_1 - \rho_3) + (2d\tilde{J} - h_3)\rho_3 - \frac{1}{\beta} s(\rho_1, \rho_3),$$

and minimising,

$$f_{2,\min} = -h_2 - \frac{1}{\beta}(\rho_1 + \rho_3),$$

where

$$\rho_1 = e^{-\beta(2dJ + h_2)} \text{ and } \rho_3 = e^{-\beta(2d\tilde{J} + h_2 - h_3)},$$

and

$$f_3(\rho_1, \rho_2) = 2d\tilde{J}(\rho_1 + \rho_2) - h_2\rho_2 - h_3(1 - \rho_1 - \rho_2) - \frac{1}{\beta} s(\rho_1, \rho_2),$$

and minimising,

$$f_{3,\min} = -h_3 - \frac{1}{\beta}(\rho_1 + \rho_2),$$

where

$$\rho_1 = e^{-\beta(2d\tilde{J}+h_3)} \text{ and } \rho_2 = e^{-\beta(2d\tilde{J}-h_2+h_3)}.$$

We have already seen that the states 1 and 2 coexist if $h_2 = 0$ and

$$h_3 \le h_{3,c} \approx \frac{1}{\beta}(e^{-2\beta dJ} - e^{-2\beta d\tilde{J}}). \tag{6.1}$$

Equating $f_{1,\min} = f_{3,\min}$, we see that 1 and 3 coexist if

$$h_3 = \frac{1}{\beta}\left(e^{-\beta(2dJ-h_2)} + e^{-\beta(2d\tilde{J}-h_3)} - e^{-\beta(2d\tilde{J}-h_2+h_3)} - e^{-\beta(2d\tilde{J}+h_3)}\right). \tag{6.2}$$

Clearly, for large β, $|h_2| \ll 1$ and $|h_3| \ll 1$, so we can approximate this by

$$h_3 \approx h_{3,c} + h_2(e^{-2\beta dJ} - e^{-2\beta d\tilde{J}}). \tag{6.3}$$

Note that this free energy is minimal only if $h_2 < 0$. It is a straight line with small slope approximating the negative h_2-axis as $\beta \to \infty$.

Similarly, the states 2 and 3 coexist if $f_{2,\min} = f_{3,\min}$, i.e.

$$h_3 = h_2 + \frac{1}{\beta}\left(e^{-\beta(2dJ+h_2)} + e^{-\beta(2d\tilde{J}+h_2-h_3)} - e^{-\beta(2d\tilde{J}+h_3)} - e^{-\beta(2d\tilde{J}-h_2+h_3)}\right). \tag{6.4}$$

This is approximately

$$h_3 \approx h_{3,c} + h_2 - h_2(e^{-2\beta dJ} + 2e^{-2\beta d\tilde{J}}), \tag{6.5}$$

which is a straight line to the right of the **triple point** $(0, h_{3,c})$ tending to the line $h_3 = h_2$ as $\beta \to \infty$.

In general, the low-temperature phase diagram can be described as follows.

Theorem 6.1 (Pirogov-Sinai) *For β large enough, there is a neighbourhood V_0 of $0 \in \mathbb{R}^{q-1}$ and a homeomorphism $I_\beta : V_0 \to U_0$, a neighbourhood of 0 in $O_q = \{(b_1, \dots, b_q) \in \mathbb{R}^q : \min_{m=1}^q b_m = 0\}$ such that if $m_1 < \dots < m_N$ are the spin values for which $I_\beta(h_2 - h_1, \dots, h_q - h_1)_{m_i} = 0$, then there exist exactly N translation-invariant limit-Gibbs measures $\mu_{\beta,\underline{h}}^{(m_i)}$ representing distinct co-existing phases.*

Proof. We may assume that $h_1 = 0$ by replacing h_r by $h_r - h_1$ since adding a constant to the Hamiltonian does not change the equilibrium states. By the previous theorem, given $J_{r,r'}$ $(1 \le r < r' \le q)$ and assuming $\tau = \frac{1}{3}\beta\rho - 1 \ge 3c_d$, where $\rho = d2^{-d}J_{\min}$, there is a continuous map from a neighbourhood of 0 given by $3||\underline{h}|| < \rho$, mapping $\underline{h}$ to a q-tuple of contour functionals $(F_1, \dots, F_q)$, such that (5.7) holds with b_r given by (5.8). We therefore set $I_\beta(\underline{h})_r = -\beta h_r - P(F_r) + M$. It follows immediately from Theorem 4.4 that the map I_β is continuous.

Conversely, given $\underline{b}$ and $J_{r,r'}$ with $1 \le r < r' \le q$, we can determine $F_1, \dots, F_q$ from a rewriting of equation (5.14) as

$$F_r(\Gamma, s_\Gamma) = \beta\mathcal{H}_\Gamma(s_\Gamma) - \sum_{x\in\Gamma}\sum_{m=1}^{q}(b_m + P(F_m) - b_r - P(F_r))\,\delta_{s_x,m} + \widehat{T}_r(\underline{F}, \underline{b}), \tag{6.6}$$

with

$$\widehat{T}_r(\underline{F}, \underline{b})(\Gamma) = \sum_{m=1}^{q}\left[\Delta_{\mathrm{Int}_m(\Gamma)}(F_r) - \Delta_{\mathrm{Int}_m(\Gamma)}(F_m, b_m)\right]. \tag{6.7}$$

By equation (5.16),

$$||\underline{\widehat{T}}(\underline{F}, \underline{b}) - \underline{\widehat{T}}(\underline{F}', \underline{b}')||_w \le 2||\underline{b} - \underline{b}'|| + \frac{1}{2}||\underline{F} - \underline{F}'||_w$$

and in particular, for given $\underline{b}$, $\tilde{T}$ is a contraction. We then put

$$\pi_\beta(\underline{b})_r = \frac{1}{\beta}(-b_r - P(F_r) + b_1 + P(F_1))$$

for $r = 1, \dots, q$. Then it follows from Theorem 4.4 that

$$||\pi_\beta(\underline{b})|| \le \frac{1}{\beta}(||\underline{b} - b_m|| + e^{-\tau}).$$

We claim that $\pi_\beta(I_\beta(\underline{h})) = \underline{h}$ and $I_\beta(\pi_\beta(\underline{b})) = \underline{b}$. For, if $I_\beta(\underline{h}) = \underline{b}$, then $\underline{b} = -\beta\underline{h} - P(\underline{F}) + M$, where $F_1, \dots, F_q$ satisfy

$$\begin{aligned} F_r(\Gamma, s_\Gamma) &= \beta\Big(\mathcal{H}_\Gamma(s_\Gamma) - \sum_{x\in\Gamma}(h_{s_x} - h_r)\Big) + T_r(\underline{F}, \beta, \underline{h}) \\ &= \beta\mathcal{H}_\Gamma(s_\Gamma) - \sum_{x\in\Gamma}\sum_{m=1}^{q}(b_m + P(F_m) - b_r - P(F_r))\delta_{s_x,m} \\ &\quad +\tilde{T}_r(\underline{F}, \underline{b}), \end{aligned}$$

and hence $\pi_\beta(\underline{b}) = \underline{h}$. The converse is analogous. If $||\underline{b}|| < \frac{1}{3}\beta\rho - \beta e^{-\tau}$ then $||\underline{h}|| < \frac{1}{3}\rho$ and the inverse map is well-defined.

Now suppose that for a given boundary condition r, $b_r = 0$. Then, according to the corollary of Theorem 4.2, the correlation of external contours decays exponentially. As in the case of the Ising model, we show that this implies that the corresponding equilibrium state $\mu^{(r)}_{\beta,\underline{h}}$ is mixing. Denote $\mu^{(r)}_\Lambda$ the Gibbs state on Λ given by

$$\mu^{(r)}_\Lambda(s_\Lambda) = \frac{1}{\mathcal{Z}^{(r)}_\Lambda(\beta,\underline{h})} \exp\left[-\beta\mathcal{H}_\Lambda(s_\Lambda\,|\,r) + \beta\sum_{x\in\Lambda}(h_{s_x} - h_r)\right].$$

The corresponding distribution of external contours is given by

$$\mu^{(r)}_\Lambda(\Theta) = \frac{1}{\mathcal{Z}^{(r)}_\Lambda(\beta,\underline{h})} \prod_{(\Gamma,s_\Gamma)\in\Theta} \mathcal{Z}_r((\Gamma,s_\Gamma),\beta,\underline{h})$$

for external boundaries $\Theta \in \mathcal{D}_r(\Lambda)$. By the identification with contour models in Theorem 5.1, we see that this equals

$$\frac{1}{\Xi_\Lambda(F_r)} \prod_{(\Gamma,s_\Gamma)\in\Theta} \Xi((\Gamma,s_\Gamma)\,|\,F_r) = \tilde{\mu}_\Lambda(\Theta),$$

which is the probability of an external boundary in the contour model. It follows that the thermodynamic limit of the correlation functions

$$\sum_{\tilde{\Theta}\supset\Theta} \mu^{(r)}_\Lambda(\tilde{\Theta}) = \tilde{\rho}_\Lambda(\Theta)$$

exists.

Consider a finite subset $A \subset \mathbb{Z}^d$. We need to relate the probability of a given configuration s_A to that of exterior contours. If for some $x \in A$, $s_x = r$, then it is possible that x is not surrounded by a contour. Given an exterior boundary Θ, let A_1 be the subset of A surrounded by a contour and A_2 the subset not surrounded by a contour. Define

$$\tilde{\mathcal{D}}_x(\Theta) = \{\Theta' \supset \Theta : \, (\exists\Gamma \in \Theta')\, x \in \Gamma \cup \mathrm{Int}(\Gamma)\}.$$

The complement of the set of $\Theta' \supset \Theta$ such that for all $\Gamma' \in \Theta'$, $A_2 \cap (\Gamma' \cup \mathrm{Int}(\Gamma')) = \emptyset$, is the set $\Theta' \supset \Theta$ such that there exists $\Gamma' \in \Theta'$ such that

$A_2 \cap (\Gamma' \cup \mathrm{Int}(\Gamma')) \neq \emptyset$, i.e. $\Gamma' \in \bigcup_{x \in A_2} \tilde{\mathcal{D}}_x(\Theta)$. Hence, be the inclusion-exclusion principle (4.7), the probability of the event that $A_2 \cap (\Gamma \cup \mathrm{Int}(\Gamma)) \neq \emptyset$ for any contour Γ, given that $A_1 = A \cap \bigcup_{\Gamma \in \Theta} (\Gamma \cup \mathrm{Int}(\Gamma))$, is given by

$$\tilde{\rho}_\Lambda(\Theta) - \sum_{I \subset A_2;\, I \neq \emptyset} (-1)^{|I|-1} \tilde{\mu}_\Lambda\big(\cap_{x \in I} \tilde{\mathcal{D}}_x(\Theta)\big).$$

In terms of correlation functions this can be written as

$$\tilde{\rho}_\Lambda(\Theta) - \sum_{I \subset A_2;\, I \neq \emptyset} (-1)^{|I|-1} \sum_{\Theta' \supset \Theta;\, \Theta' \in \tilde{\mathcal{D}}_I} \tilde{\rho}_\Lambda(\Theta'),$$

where

$$\tilde{\mathcal{D}}_I = \Big\{ \Theta' : \forall \Gamma' \in \Theta', I \cap (\Gamma' \cup \mathrm{Int}(\Gamma')) \neq \emptyset \\ \text{and } I \subset \bigcup_{\Gamma' \in \Theta'} (\Gamma' \cup \mathrm{Int}(\Gamma')) \Big\} \tag{6.8}$$

This therefore converges as $\Lambda \to \mathbb{Z}^d$. (Note that the sum converges because the number of Γ' containing a point of A_2 grows like $|\Gamma'|^{c_d}$ whereas $\tilde{\rho}(\Theta') \leq e^{-\tau ||\Theta'||}$.) We will denote the limiting probability by $\tilde{\rho}(\Theta; A_1, A_2)$, i.e.

$$\tilde{\rho}(\Theta; A_1, A_2) = \tilde{\rho}(\Theta) + \sum_{I \subset A_2;\, I \neq \emptyset} (-1)^{|I|} \sum_{\Theta' \supset \Theta;\, \Theta' \in \tilde{\mathcal{D}}_I} \tilde{\rho}(\Theta'). \tag{6.9}$$

The distribution of s_A is then

$$\begin{aligned} \mu^{(r)}(s_A) &= \sum_{A_1 \subset A} \sum_{\Theta \in \tilde{\mathcal{D}}_{A_1} \cap \bigcap_{x \in A \setminus A_1} \tilde{\mathcal{D}}_x^c} \tilde{\rho}(\Theta; A_1, A \setminus A_1) \\ &\quad \times \prod_{x \in A \setminus A_1} \delta_{s_x, r} \prod_{\Gamma \in \Theta} \mu^{(r)}_{\Gamma \cup \mathrm{Int}(\Gamma)}(s_{A_1}), \end{aligned} \tag{6.10}$$

where

$$\tilde{\mathcal{D}}_x = \tilde{\mathcal{D}}_x(\emptyset) = \{\Theta : (\exists \Gamma \in \Theta)\, x \in \Gamma \cup \mathrm{Int}(\Gamma)\}.$$

Note that in case $A_1 = \emptyset$, $\tilde{\mathcal{D}}_{A_1} = \emptyset$, so this contribution to the sum in (6.10) is

$$\tilde{\rho}(\emptyset; \emptyset, A) \prod_{x \in A} \delta_{s_x, r}$$

The correlation function

$$\tilde{\rho}(\emptyset;\emptyset,A) = 1 + \sum_{\emptyset\neq I\subset A} (-1)^{|I|} \sum_{\Theta'\in\tilde{\mathcal{D}}_I} \tilde{\rho}(\Theta')$$

is the probability that no contour of Θ surrounds a point of A.

Note also that the expression (6.10) agrees with (3.5) in the case of the Ising model with $A = \{0, z\}$.

We next prove that the distribution (6.10) satisfies the mixing property (2.1). We can take f and g to be of the form $f = \prod_{x\in A} \delta_{s_x,\overline{s}_x}$ and $g = \prod_{x\in B} \delta_{s_x,\overline{s}_x}$ for fixed $\overline{s}_x$. Then

$$\begin{aligned}
\mathbb{E}^{\mu^{(r)}}&(\tau_z(f)\, g) = \mu^{(r)}(\overline{s}_{\tau_z(A)}\overline{s}_B) \\
&= \sum_{A_1\subset A,\, B_1\subset B} \; \sum_{\Theta\in\tilde{\mathcal{D}}_{\tau_z(A_1)\cup B_1}\cap\bigcap_{x\in\tau_z(A_2)\cup B_2}\tilde{\mathcal{D}}_x^c} \tilde{\rho}(\Theta;\tau_z(A_1)\cup B_1, \tau_z(A_2)\cup B_2) \\
&\quad\times \prod_{x\in A_2\cup B_2} \delta_{\overline{s}_x,r} \prod_{\Gamma\in\Theta} \mu^{(r)}_{\Gamma\cup\mathrm{Int}(\Gamma)}(s_{\tau_z(A_1)\cup B_1})
\end{aligned}$$

In this expression, the terms where there exists $\Gamma\in\Theta$ such that $\tau_z(A)\cap(\Gamma\cup\mathrm{Int}(\Gamma))\neq\emptyset$ and $B\cap(\Gamma\cup\mathrm{Int}(\Gamma))\neq\emptyset$ tend to zero because $\tilde{\rho}(\Gamma;\tau_z(A_1)\cup B_1,\tau_z(A_2)\cup B_2)\leq e^{-\tau|\Gamma|}\leq e^{-\tau 2^{d-1}(|z|-\mathrm{diam}(A\cup B))}$. In the remaining terms, we can write $\Theta = \Theta_1\cup\Theta_2$, where $\Theta_1\in\tilde{\mathcal{D}}_{\tau_z(A_1)}$ and $\Theta_2\in\tilde{\mathcal{D}}_{B_1}$. Moreover, if $\Theta_1\in\tilde{\mathcal{D}}_{\tau_z(A_1)}$ and $\Theta_1\in\tilde{\mathcal{D}}_x$ for some $x\in B_2$ then there is a $\Gamma_1\in\Theta_1$ such that $\tau_z(A_1)\cap(\Gamma_1\cup\mathrm{Int}(\Gamma_1))\neq\emptyset$ and $x\in\Gamma_1\cup\mathrm{Int}(\Gamma_1)$ and hence $\tilde{\rho}(\Theta')\leq e^{-\tau|\Gamma_1|}\leq e^{-\tau 2^{d-1}(|z|-\mathrm{diam}(A\cup B))}$ for all $\Theta'\supset\Theta$. We can therefore ignore the condition $\Theta_1\in\bigcap_{x\in B_2}\tilde{\mathcal{D}}_x^c$, and similarly also the condition $\Theta_2\in\bigcap_{x\in\tau_z(A_2)}\tilde{\mathcal{D}}_x^c$.

The resulting expression is

$$\begin{aligned}
&\sum_{A_1\subset A,\, B_1\subset B} \; \sum_{\substack{\Theta=\Theta_1\cup\Theta_2: \\ \Theta_1\in\tilde{\mathcal{D}}_{\tau_z(A_1)}\cap\bigcap_{x\in\tau_z(A_2)}\tilde{\mathcal{D}}_x^c,\, \Theta_2\in\tilde{\mathcal{D}}_{B_1}\cap\bigcap_{x\in B_2}\tilde{\mathcal{D}}_x^c}} \\
&\quad\times \tilde{\rho}(\Theta;\tau_z(A_1)\cup B_1,\tau_z(A_2)\cup B_2) \prod_{x\in A_2\cup B_2}\delta_{\overline{s}_x,r}\prod_{\Gamma\in\Theta}\mu^{(r)}_{\Gamma\cup\mathrm{Int}(\Gamma)}(\overline{s}_{\tau_z(A_1)\cup B_1}).
\end{aligned}$$

This tends to

$$
\begin{aligned}
\mathbb{E}^{\mu^{(r)}}(f)\mathbb{E}^{\mu^{(r)}}(g) &= \mu^{(r)}(\overline{s}_A)\mu^{(r)}(\overline{s}_B) \\
&= \sum_{A_1\subset A,\, B_1\subset B} \; \sum_{\Theta_1\in\tilde{\mathcal{D}}_{A_1}\cap\bigcap_{x\in A_2}\tilde{\mathcal{D}}_x^c} \; \sum_{\Theta_2\in\tilde{\mathcal{D}}_{B_1}\cap\bigcup_{x\in B_2}\tilde{\mathcal{D}}_x^c} \\
&\quad \times \tilde{\rho}(\Theta_1; A_1, A_2)\tilde{\rho}(\Theta_2; B_1, B_2) \\
&\quad \times \prod_{x\in A_2\cup B_2} \delta_{\overline{s}_x, r} \prod_{\Gamma\in\Theta_1} \mu^{(r)}_{\Gamma\cup\mathrm{Int}(\Gamma)}(\overline{s}_{A_1}) \prod_{\Gamma\in\Theta_2} \mu^{(r)}_{\Gamma\cup\mathrm{Int}(\Gamma)}(\overline{s}_{B_1})
\end{aligned}
$$

since by the corollary of Theorem 4.2,

$$
\begin{aligned}
&|\tilde{\rho}(\Theta; \tau_z(A_1)\cup B_1, \tau_z(A_2)\cup B_2) - \tilde{\rho}(\Theta_1; \tau_z(A_1), \tau_z(A_2))\tilde{\rho}(\Theta_2; B_1, B_2)| \\
&\quad \leq 2^{|A_2|+|B_2|}\, e^{(c_d-\tau)(||\Theta_1||+||\Theta_2||+d(\Theta_1,\Theta_2))}. \qquad (6.11)
\end{aligned}
$$

(Note that $\tilde{\rho}(\Theta_1; \tau_z(A_1), \tau_z(A_2)) = \tilde{\rho}(\tau_{-z}(\Theta_1); A_1, A_2)$ and if $\Gamma \in \Theta_1$ then $B_1 \cap (\Gamma \cup \mathrm{Int}(\Gamma)) = \emptyset$ while $\mu^{(r)}_{\Gamma\cup\mathrm{Int}(\Gamma)}(s_{\tau_z(A_1)}) = \mu^{(r)}_{\tau_{-z}(\Gamma\cup\mathrm{Int}(\Gamma))}(s_{A_1})$.)

It is clear that the measures $\mu^{(r)}$ with $b_r = 0$ are not identical because $\mu^{(r)}(\delta_{s_0,r}) \sim 1$ by Peierls' argument. Since they are extremal measures, they must be singular w.r.t. each other.

Now consider values of r such that $b_r > 0$. We want to show that such values do not correspond to different phases. By Lemma 4.7, there is a large contour with total area/volume almost equal to Λ. (Here we take Λ to be a large square/cube and assume $\epsilon < 1/d$ so that $|\partial\Lambda|^{1+\epsilon} \ll |\Lambda|$.)

We still need to show that the internal regions $\mathrm{Int}_m(\Gamma)$ with $\Gamma \in \Theta(\partial)$ and $b_m > 0$ are insignificant compared to $\mathrm{Int}_r(\Gamma)$ with $b_r = 0$. By Lemma 4.7 we can consider $\mathrm{Int}(\Gamma)$ as a region Λ' which also has a regular boundary. Consider the relation (5.10). We want to show that with high probability,

$$
e^{\beta h_m|\Lambda'|}\Xi_{\Lambda'}(F_m, b_m) \ll e^{\beta h_r|\Lambda'|}\Xi_{\Lambda'}(F_r)
$$

if $b_m > 0$ and $b_r = 0$. By Lemma 4.7, we have that with high probability

$$\begin{aligned}
\Xi_{\Lambda'}(F_m, b_m) &\le (1-\eta)^{-1} \sum_{\partial \in \mathcal{D}(\Lambda') \colon \Theta(\partial) \in \mathcal{B}_{\Lambda'}(\epsilon)} e^{-F_m(\partial)} \prod_{\Gamma \in \Theta(\partial)} e^{b_m |\mathrm{Int}(\Gamma)|} \\
&\le 2 \sum_{\partial \in \mathcal{D}(\Lambda') \colon \Theta(\partial) \in \mathcal{B}_{\Lambda'}(\epsilon)} e^{-F_m(\partial)} e^{b_m |\Lambda'|} \\
&\le 2 e^{(b_m + \beta h_m)|\Lambda'|} \sum_{\partial \colon \Theta(\partial) \in \mathcal{B}_{\Lambda'}(\epsilon)} e^{-F_m(\partial)} \\
&\le 2 e^{(b_m + \beta h_m)|\Lambda'|} \\
&\quad \times \sum_{\Theta \in \mathcal{B}_{\Lambda'}(\epsilon)} e^{-F_m(\Theta)} \sum_{\partial \subset \bigcup_{\Gamma \in \Theta} \mathrm{Int}(\Gamma)} e^{-F_m(\partial)} \\
&\le 2 e^{(M - P(F_m))|\Lambda'|} \\
&\quad \times \sum_{\Theta \in \mathcal{B}_{\Lambda'}(\epsilon)} e^{-F_m(\Theta)} \prod_{\Gamma \in \Theta} e^{P(F_m)|\mathrm{Int}(\Gamma)| + \Delta_{\mathrm{Int}(\Gamma)}(F_m)}. \\
&\le 2 e^{(M - P(F_m))|\Lambda'|} \\
&\quad \times \sum_{\Theta \in \mathcal{B}_{\Lambda'}(\epsilon)} e^{-\tau \sum_{\Gamma \in \Theta} |\Gamma|} e^{P(F_m)|\Lambda'| + \sum_{\Gamma \in \Theta} \Delta_{\mathrm{Int}(\Gamma)}(F_m)} \\
&\le 2 e^{(P(F_r) + \beta h_r)|\Lambda'|} \sum_{\Theta \in \mathcal{B}_{\Lambda'}(\epsilon)} e^{-(\tau - e^{-\tau}) \sum_{\Gamma \in \Theta} |\Gamma|} \\
&= 2 e^{\beta h_r |\Lambda'|} \Xi_{\Lambda'}(F_r) e^{-\Delta_{\Lambda'}(F_r)} e^{-(\tau - e^{-\tau}) \kappa_d |\Lambda'|^{\delta}}, \qquad (6.12)
\end{aligned}$$

The resulting bound tends to 0 as $|\Lambda'| \to \infty$.

This implies that the boundary condition r is much more likely than m once the area is large.

We now follow Gallavotti and Miracle-Solé[10] and use the following lemma.

Lemma 6.1 *Let μ be a translation invariant probability measure on $\Omega_q^{\mathbb{Z}^d}$. Suppose that there is a subset $S_0 \subset \{1, \ldots, q\}$ such that for any function $f : \Omega_q^{\mathbb{Z}^d} \to \mathbb{R}$ with bounded support $A \subset \mathbb{Z}^d$, there exists a family of numbers $\alpha_{\Lambda,r} \in [0,1]$ for all finite $\Lambda \subset \mathbb{Z}^d$ and $r \in S_0$ such that*

$$\lim_{\Lambda \to \mathbb{Z}^d} \left| \mathrm{Av}\mathbb{E}^{\mu}_{\Lambda}(f) - \sum_{r \in S_0} \alpha_{\Lambda,r} \mathbb{E}^{\mu^{(r)}}(f) \right| = 0,$$

[10]G. Gallavotti and S. Miracle-Solé, Equilibrium states of the Ising model in the two-phase region, *Phys. Rev.* B**5**, 2555–9 (1972)

where Av *is the averaging operator defined by*

$$\mathrm{Av}\mathbb{E}^{\mu}_{\Lambda}(f) = \frac{1}{|\Lambda|}\sum_{z\in\Lambda}\mathbb{E}^{\mu}_{\Lambda}(\tau_z(f)).$$

Then μ belongs to the convex hull of the measures $\mu^{(r)}$ with $r \in S_0$.

Given f, let $a = \mathrm{diam}(A)$ be the diameter of the support of f. Then $\tau_z(f)$ has support $\tau_z(A)$ which, with high probability, is contained in $\mathrm{Int}(\Gamma)$ for some $\Gamma \in \Theta(\partial)$ unless z belongs to an A-neighbourhood of $\bigcup_{\Gamma\in\Theta(\partial)}\Gamma$, having total volume at most $a^d c\,|\partial\Lambda|$ by the above lemma. Analogous to equation (6.10) the expectation of f is given by

$$\begin{aligned}\mathbb{E}^{\mu}(f) \;=\; & \sum_{A_1\subset A}\;\sum_{\Theta\in\tilde{\mathcal{D}}_{A_1}\cap\bigcap_{x\in A_2}\tilde{\mathcal{D}}^c_x}\tilde{\rho}(\Theta; A_1, A_2) \\ & \times\prod_{x\in A_2}\delta_{s_x,r}\prod_{\Gamma\in\Theta}\mathbb{E}^{\mu^{(r)}}_{\Gamma\cup\mathrm{Int}(\Gamma)}(f|_{\Gamma\cup\mathrm{Int}(\Gamma)}),\end{aligned} \tag{6.13}$$

If $\tau_z(A) \subset \mathrm{Int}(\Gamma)$ for a given $\Gamma \in \Theta(\partial)$ then $A_1 = A$ and $\Theta(\partial) = \{\Gamma\}$ and $\mathbb{E}^{\mu^{(r)}}_{\Gamma\cup\mathrm{Int}(\Gamma)}(f) = \mathbb{E}^{\mu^{(m)}}_{\mathrm{Int}(\Gamma)}(f)$. Since, moreover, $\tilde{\rho}(\Theta; A, \emptyset) = \tilde{\rho}(\Theta)$, it follows that

$$\lim_{\Lambda\to\mathbb{Z}^d}\mathrm{Av}\mathbb{E}^{\mu}_{\Lambda}(f) = \sum_{m\colon b_m=0}\;\sum_{\Gamma\in\bigcap_{x\in A}\mathcal{C}^{(m)}_x}\tilde{\rho}(\{\Gamma\}; A, \emptyset)\mathbb{E}^{\mu^{(m)}}_{\mathrm{Int}(\Gamma)}(f), \tag{6.14}$$

where $\mathcal{C}^{(m)}_x$ is the set of contours Γ with inner boundary condition m containing x, i.e. such that $x \in \mathrm{Int}_m(\Gamma)$. This completes the proof of Theorem 6.1. ∎

Remark. In fact, the proof is incomplete since we have only considered uniform boundary conditions. The proof in case of mixed boundary conditions is analogous, however. In that case one needs to consider also contours which are not closed but connect to the boundary of Λ. This was first done for the Ising model by Gallavotti and Miracle-Solé[11]. Indeed, the above proof in case $b_r > 0$ is based on their article. The extension to the general case was done by Martirosyan[12]. However, his short note contains only a very brief outline of the proof, with many essential details omitted.

[11] G. Gallavotti and S. Miracle-Solé, *loc. cit.*

[12] D. G. Martirosyan: On the question of an upper bound on the number of periodic Gibbs states for models of a lattice gas. *Usp. Mat. Nauk* **30**, 181–2 (1975) (In Russian).