

A determinantal identity

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Abstract

We prove an interesting identity for the sum of determinants, which is a generalization of the sum of a geometric progression. The proof is quite long and a number of other identities are proved along the way. Some of the more elementary ones are deferred to another section at the end.

1 The identity

We claim the following.

Theorem 1 *For any $n, N \in \mathbb{N}$ and $a_1, \dots, a_n \in \mathbb{C}$, the following identity holds.*

$$\begin{aligned} & \sum_{1 \leq x_1 < \dots < x_n \leq N} \begin{vmatrix} a_1^{x_1} & a_1^{x_2} & \cdots & a_1^{x_n} \\ a_2^{x_1} & a_2^{x_2} & \cdots & a_2^{x_n} \\ \vdots & \vdots & \cdots & \vdots \\ a_n^{x_1} & a_n^{x_2} & \cdots & a_n^{x_n} \end{vmatrix} \\ &= \prod_{k=1}^n \frac{a_k}{a_k - 1} \sum_{J \subset \{1, \dots, n\}} (-1)^{\nu(J^c)} \gamma(J) \gamma(J^c) \prod_{j \in J} a_j^N, \end{aligned} \quad (1)$$

where

$$\gamma(J) = \frac{1}{\prod_{\{i,j\} \subset J} (a_i a_j - 1)} \Delta(J), \quad (2)$$

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where

$$\Delta(J) = \begin{vmatrix} 1 & a_{j_1} & \cdots & a_{j_1}^{k-1} \\ 1 & a_{j_2} & \cdots & a_{j_2}^{k-1} \\ \vdots & \vdots & \cdots & \vdots \\ 1 & a_{j_k} & \cdots & a_{j_k}^{k-1} \end{vmatrix} \quad (3)$$

is a Van der Monde determinant if $J = \{j_1, \dots, j_k\}$ with $j_1 < \dots < j_k$ (and $\gamma(\emptyset) = \gamma(\{k\}) = 1$), and where

$$\nu(I) = \sum_{k \in I} k. \quad (4)$$

The proof uses a number of elementary results about determinants of this type, which are stated in Section 2.

Proof. We first sum over x_1 to write

$$\begin{aligned} & \prod_{k=1}^n \frac{a_k - 1}{a_k} \sum_{1 \leq x_1 < \dots < x_n \leq N} \begin{vmatrix} a_1^{x_1} & a_1^{x_2} & \cdots & a_1^{x_n} \\ a_2^{x_1} & a_2^{x_2} & \cdots & a_2^{x_n} \\ \vdots & \vdots & \cdots & \vdots \\ a_n^{x_1} & a_n^{x_2} & \cdots & a_n^{x_n} \end{vmatrix} \\ &= \sum_{1 \leq x_2 < \dots < x_n \leq N-1} \begin{vmatrix} a_1^{x_2} - 1 & a_1^{x_2+1} - 1 & a_1^{x_3}(a_1 - 1) & \cdots & a_1^{x_n}(a_1 - 1) \\ a_2^{x_2} - 1 & a_2^{x_2+1} - 1 & a_2^{x_3}(a_2 - 1) & \cdots & a_2^{x_n}(a_2 - 1) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_n^{x_2} - 1 & a_n^{x_2+1} - 1 & a_n^{x_3}(a_n - 1) & \cdots & a_n^{x_n}(a_n - 1) \end{vmatrix}. \end{aligned} \quad (5)$$

For $n = 2$ this becomes

$$\begin{aligned} & \sum_{x_2=1}^{N-1} \left\{ (a_1 a_2)^{x_2} \begin{vmatrix} 1 & a_1 \\ 1 & a_2 \end{vmatrix} - \begin{vmatrix} 1 & a_1^{x_2+1} - a_1^{x_2} \\ 1 & a_2^{x_2+1} - a_2^{x_2} \end{vmatrix} \right\} \\ &= \frac{(a_1 a_2)^N - a_1 a_2}{a_1 a_2 - 1} \begin{vmatrix} 1 & a_1 \\ 1 & a_2 \end{vmatrix} - \begin{vmatrix} 1 & a_1^N - a_1 \\ 1 & a_2^N - a_2 \end{vmatrix} \\ &= \frac{(a_1 a_2)^N - 1}{a_1 a_2 - 1} \begin{vmatrix} 1 & a_1 \\ 1 & a_2 \end{vmatrix} + a_1^N - a_2^N \\ &= \sum_{J \subset \{1, 2\}} (-1)^{\nu(J^c)} \gamma(J) \gamma(J^c) \prod_{j \in J} a_j^N. \end{aligned}$$

In general, we want to prove that

$$\begin{aligned} & \sum_{1 \leq x_2 < \dots < x_n \leq N-1} \left| \begin{array}{ccccc} a_1^{x_2} - 1 & a_1^{x_2+1} - 1 & a_1^{x_3}(a_1 - 1) & \cdots & a_1^{x_n}(a_1 - 1) \\ a_2^{x_2} - 1 & a_2^{x_2+1} - 1 & a_2^{x_3}(a_2 - 1) & \cdots & a_2^{x_n}(a_2 - 1) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_n^{x_2} - 1 & a_n^{x_2+1} - 1 & a_n^{x_3}(a_n - 1) & \cdots & a_n^{x_n}(a_n - 1) \end{array} \right| \\ &= \sum_{J \subset \{1, \dots, n\}} (-1)^{\nu(J^c)} \gamma(J) \gamma(J^c) \prod_{j \in J} a_j^N. \end{aligned} \quad (6)$$

We proceed by induction on n . First note that if $x_i = x_{i-1}$, the i -th column and the $i-1$ th columns are equal (except for $i = 3$, in which case the third column equals the difference of the second and first columns). We can therefore extend the sums to those cases and write the left-hand side of (6) as

$$\sum_{0 \leq x_2 \leq \dots \leq x_n \leq N-1} \left| \begin{array}{ccccc} a_1^{x_2} - 1 & a_1^{x_2+1} - 1 & a_1^{x_3}(a_1 - 1) & \cdots & a_1^{x_n}(a_1 - 1) \\ a_2^{x_2} - 1 & a_2^{x_2+1} - 1 & a_2^{x_3}(a_2 - 1) & \cdots & a_2^{x_n}(a_2 - 1) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_n^{x_2} - 1 & a_n^{x_2+1} - 1 & a_n^{x_3}(a_n - 1) & \cdots & a_n^{x_n}(a_n - 1) \end{array} \right|. \quad (7)$$

Expanding the left-hand side of (7) according to the last column it becomes

$$\begin{aligned} & \sum_{k=1}^n (-1)^{n-k} (a_k - 1) \sum_{x_n=0}^{N-1} a_k^{x_n} \sum_{0 \leq x_2 \leq \dots \leq x_{n-1} \leq x_n} \\ & \times \left| \begin{array}{ccccc} a_1^{x_2} - a_1 & a_1^{x_2+1} - a_1 & a_1^{x_3}(a_1 - 1) & \cdots & a_1^{x_{n-1}}(a_1 - 1) \\ a_2^{x_2} - a_2 & a_2^{x_2+1} - a_2 & a_2^{x_3}(a_2 - 1) & \cdots & a_2^{x_{n-1}}(a_2 - 1) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ [a_k^{x_2} - a_k & a_k^{x_2+1} - a_k & a_k^{x_3}(a_k - 1) & \cdots & a_k^{x_{n-1}}(a_k - 1)] \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_n^{x_2} - a_n & a_n^{x_2+1} - a_n & a_n^{x_3}(a_n - 1) & \cdots & a_n^{x_{n-1}}(a_n - 1) \end{array} \right|, \end{aligned} \quad (8)$$

where the square brackets around the k -th row indicate that this row is omitted. By the induction hypothesis, this equals

$$\begin{aligned} & \sum_{k=1}^n (-1)^{n-k} (a_k - 1) \sum_{x_n=0}^{N-1} a_k^{x_n} \\ & \times \sum_{J \subset \{1, \dots, n\} \setminus \{k\}} (-1)^{\nu_k(J^c)} \gamma(J) \gamma(J^c \setminus \{k\}) \prod_{j \in J} a_j^{x_n}, \end{aligned} \quad (9)$$

where $\nu_k(J^c)$ is given by

$$\nu_k(I) = \sum_{i \in I: i < k} i + \sum_{i \in I: i > k} (i - 1). \quad (10)$$

Now let us first consider the case that $|J| = n - 1$, that is $J = \{1, \dots, n\} \setminus \{k\}$. In that case $\nu_k(J^c) = 0$ for all k . The corresponding term is

$$\sum_{k=1}^n (-1)^{n-k} (a_k - 1) \sum_{x_n=0}^{N-1} a_k^{x_n} \gamma(\{1, \dots, n\} \setminus \{k\}) \prod_{j \neq k} a_j^{x_n}.$$

Multiplying by $\prod_{1 \leq i < j \leq n} (a_i a_j - 1)$ this becomes

$$\begin{aligned} & \sum_{k=1}^n (-1)^{n-k} (a_k - 1) \sum_{x_n=0}^{N-1} a_k^{x_n} \\ & \times \prod_{j \neq k} a_j^{x_n} \prod_{j \neq k} (a_j a_k - 1) \left| \begin{array}{cccc} 1 & a_1 & \cdots & a_1^{n-2} \\ \vdots & \vdots & \cdots & \vdots \\ [1 & a_k & \cdots & a_k^{n-1}] \\ \vdots & \vdots & \cdots & \vdots \\ 1 & a_n & \cdots & a_n^{n-2} \end{array} \right| \\ & = \sum_{x_n=0}^{N-1} (a_1 \dots a_n)^{x_n} \left| \begin{array}{ccccc} 1 & a_1 & \cdots & a_1^{n-2} & (a_1 - 1) \prod_{j=2}^n (a_1 a_j - 1) \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 1 & a_k & \cdots & a_k^{n-2} & (a_k - 1) \prod_{\substack{j=1 \\ j \neq k}}^n (a_j a_k - 1) \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 1 & a_n & \cdots & a_n^{n-2} & (a_n - 1) \prod_{j=1}^{n-1} (a_j a_n - 1) \end{array} \right|. \end{aligned} \quad (11)$$

Using Lemma 2.4 and summing over x_n this becomes

$$\left(\prod_{j=1}^n a_j^N - 1 \right) \left| \begin{array}{ccccc} 1 & a_1 & \cdots & a_1^{n-2} & a_1^{n-1} \\ 1 & a_2 & \cdots & a_2^{n-2} & a_2^{n-1} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 1 & a_n & \cdots & a_n^{n-2} & a_n^{n-1} \end{array} \right|.$$

The term with $\prod_{j=1}^n a_j^{N+1}$ is just the term $|J| = n$ of the right-hand side of (6) when divided again by $\prod_{1 \leq i < j \leq n} (a_i a_j - 1)$. The second term contributes to $J = \emptyset$.

Next consider the case $|J| = n - 2$ in the expression (9). This equals

$$\begin{aligned} & \sum_{k=1}^n (-1)^{n-k} (a_k - 1) \sum_{x_n=0}^{N-1} a_k^{x_n} \sum_{l \neq k} (-1)^{\nu_k(\{k,l\})} \gamma(\{k,l\}^c) \prod_{j \neq k,l} a_j^{x_n} \\ &= \sum_{l=1}^n \sum_{k \neq l} (-1)^{n-k} (a_k - 1) \sum_{x_n=0}^{N-1} \prod_{j \neq l} a_j^{x_n} (-1)^{\nu_k(\{k,l\})} \gamma(\{k,l\}). \end{aligned}$$

Here $\nu_k(J)$ is given by (10). Multiplying the l -th term by $\prod_{\substack{1 \leq i < j \leq n \\ i,j \neq l}} (a_i a_j - 1)$

it becomes

$$\begin{aligned} & \sum_{k \neq l} (-1)^{n-k} (a_k - 1) \sum_{x_n=0}^{N-1} \prod_{j \neq l} a_j^{x_n} (-1)^{\nu_k(\{k,l\})} \\ & \times \prod_{\substack{j=1 \\ j \neq k,l}}^n (a_j a_k - 1) \begin{vmatrix} 1 & a_1 & \cdots & a_1^{n-3} \\ \vdots & \vdots & \cdots & \vdots \\ [1 & a_k & \cdots & a_k^{n-3}] \\ \vdots & \vdots & \cdots & \vdots \\ [1 & a_l & \cdots & a_l^{n-3}] \\ \vdots & \vdots & \cdots & \vdots \\ 1 & a_n & \cdots & a_n^{n-3} \end{vmatrix}. \end{aligned}$$

Now,

$$\nu_k(\{k,l\}) = \begin{cases} l & \text{if } l < k; \\ l-1 & \text{if } l > k. \end{cases}$$

But, in the case $l > k$, the l -th row is below the k -th row so the number of rows below the k -th is only $n - 1 - k$. Performing the sum over k we therefore get

$$(-1)^l \frac{\prod_{j \neq l} a_j^N - 1}{\prod_{j \neq l} a_j - 1} \begin{vmatrix} 1 & a_1 & \cdots & a_1^{n-3} & (a_1 - 1) \prod_{j \neq 1,l} (a_1 a_j - 1) \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ [1 & a_l & \cdots & a_l^{n-3} & (a_l - 1) \prod_{j \neq l} (a_j a_l - 1)] \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 1 & a_n & \cdots & a_n^{n-3} & (a_n - 1) \prod_{j \neq l,n} (a_j a_n - 1) \end{vmatrix}.$$

We set $J' = J \cup \{k\} = \{l\}^c$ and note that $\nu(J'^c) = \nu(\{l\}) = l$. Using Lemma 2.4 again, we obtain

$$\left(\prod_{j \neq l} a_j^N - 1 \right) (-1)^{\nu(\{l\})} \begin{vmatrix} 1 & a_1 & \cdots & a_1^{n-3} & a_1^{n-2} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ [1 & a_l & \cdots & a_l^{n-3} & a_l^{n-2}] \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 1 & a_n & \cdots & a_n^{n-3} & a_n^{n-2} \end{vmatrix}.$$

Dividing again by $\prod_{\substack{1 \leq i < j \leq n \\ i,j \neq l}} (a_i a_j - 1)$ and summing over l this yields

$$\sum_{l=1}^n (-1)^{\nu(\{l\})} \gamma(\{l\}^c) \left(\prod_{j \neq l} a_j^N - 1 \right).$$

The term with $\prod_{j \neq l} a_j^N$ is just the term $J' = \{l\}^c$ in the right-hand side of (6). The other term contributes to the case $J' = \emptyset$.

We now consider the general case in (9). Again, we want to put $J' = J \cup \{k\}$. Then $(J')^c = J^c \setminus \{k\}$. Summing over x_n in (9) we get

$$\begin{aligned} & \sum_{\substack{J \subset \{1, \dots, n\} \\ J^c \neq \emptyset}} \sum_{k \in J^c} \left(\frac{a_k^N \prod_{j \in J} a_j^N - 1}{a_k \prod_{j \in J} a_j - 1} \right) \\ & \quad \times (-1)^{n-k} (a_k - 1) (-1)^{\nu_k(J^c)} \gamma(J) \gamma(J^c \setminus \{k\}). \end{aligned}$$

With $J' = J \cup \{k\}$ this is

$$\begin{aligned} & \sum_{\substack{J' \subset \{1, \dots, n\} \\ J' \neq \emptyset}} \left(\frac{\prod_{j \in J'} a_j^N - 1}{\prod_{j \in J'} a_j - 1} \right) \\ & \quad \times \sum_{k \in J'} (-1)^{n-k} (a_k - 1) (-1)^{\nu_k(J'^c \cup \{k\})} \gamma(J' \setminus \{k\}) \gamma((J')^c). \end{aligned}$$

As in the case $|J| = n - 2$, $\nu_k(J^c) = \nu(J'^c) - p$, where p is the number of $i \in J^c$ with $i > k$, which compensates for the number of rows below the k -th row omitted in the determinant for J . Applying Lemma 2.4, we therefore obtain

$$\sum_{\substack{J' \subset \{1, \dots, n\} \\ J' \neq \emptyset}} \left(\prod_{j \in J'} a_j^N - 1 \right) (-1)^{\nu(J'^c)} \gamma(J') \gamma(J'^c). \quad (12)$$

The terms corresponding to $\prod_{j \in J'} a_j^N$ agree with those in the right-hand side of equation (6) with $J \neq \emptyset$, so it remains to show that

$$-\sum_{\substack{J' \subset \{1, \dots, n\} \\ J' \neq \emptyset}} (-1)^{\nu(J'^c)} \gamma(J') \gamma(J'^c) = (-1)^{\nu(\emptyset)} \gamma(\{1, \dots, n\}). \quad (13)$$

Equivalently, with $I = J'^c$,

$$\sum_{I \subset \{1, \dots, n\}} (-1)^{\nu(I)} \prod_{i \in I} \prod_{j \in I^c} (a_i a_j - 1) \Delta(I) \Delta(I^c) = 0, \quad (14)$$

where

$$\Delta(I) = \begin{vmatrix} 1 & a_{i_1} & \dots & a_{i_1}^{p-1} \\ 1 & a_{i_2} & \dots & a_{i_2}^{p-1} \\ \vdots & \vdots & \dots & \vdots \\ 1 & a_{i_p} & \dots & a_{i_p}^{p-1} \end{vmatrix} \quad \text{for } I = \{i_1, \dots, i_p\}. \quad (15)$$

If $\nu(\{1, \dots, n\})$ is odd, the identity (14) is obvious by interchanging I and I^c . If $\nu(\{1, \dots, n\})$ is even, the terms I and I^c are equal to each other, so, by symmetry, we can assume that $|I| \leq |I^c|$. Then we can expand $\prod_{i \in I} \prod_{j \in I^c} (a_i a_j - 1)$ as before:

$$\prod_{i \in I} \prod_{j \in I^c} (a_i a_j - 1) = \sum_{p=0}^{|I||I^c|} \sum_{K \subset I \times I^c : |K|=p} (-1)^{|I||I^c|-|K|} \prod_{(i,j) \in K} a_i a_j. \quad (16)$$

Set $k = |I|$ so that $|I^c| = n - k$. We can reorder the points $i \in I$ such that the number n_i of points $(i, j) \in K$ for given $i \in I$, is non-decreasing. Given a non-decreasing sequence $(n_r)_{r=1}^k$, put $k_m = \#\{r : n_r = m\}$. Clearly, $n_r \leq n - k$ so $m \leq n - k$. Moreover, $\sum_{m=0}^{n-k} k_m = k$. We can then write

$$\begin{aligned} & \sum_{K \subset I \times I^c : |K|=p} \prod_{(i,j) \in K} a_i a_j \\ &= \sum_{\substack{0 \leq n_1 \leq n_2 \leq \dots \leq n_k \leq n - k \\ \sum_{r=1}^k n_r = p}} \sum_{\substack{(I_m)_{m=0}^k \in \Pi(I) : \\ |I_m|=k_m}} \prod_{m=1}^k \prod_{i \in I_m} a_i^m \prod_{r=1}^k \sum_{J_r \subset I^c} \prod_{j \in J} a_j, \end{aligned} \quad (17)$$

where $\Pi(I)$ is the set of partitions of I . We define

$$S_l = \sum_{J \subset I^c : |J|=l} \prod_{j \in J} a_j \quad (18)$$

and

$$A_{n_1, \dots, n_k} = \sum_{(I_m)_{m=0}^k \in \Pi(I) : |I_m| = k_m} \prod_{m=1}^k \prod_{i \in I_m} a_i^m, \quad (19)$$

so that

$$\sum_{K \subset I \times I^c : |K|=p} \prod_{(i,j) \in K} a_i a_j = \sum_{\substack{0 \leq n_1 \leq n_2 \leq \dots \leq n_k \leq n-k \\ \sum_{r=1}^k n_r = p}} A_{n_1, \dots, n_k} \prod_{r=1}^k S_{n_r}. \quad (20)$$

Denote $|\underline{n}| = \sum_{r=1}^k n_r$ and

$$\mathbb{N}_{\uparrow}^k = \{(n_1, \dots, n_k) \in \mathbb{Z}^k : 0 \leq n_1 \leq \dots \leq n_k \leq n - k\} \quad (21)$$

and introduce a lexicographic ordering according to

$$\underline{n} < \underline{m} \text{ if } n_r = m_r \text{ for } r > r_0, \quad n_{r_0} < m_{r_0}. \quad (22)$$

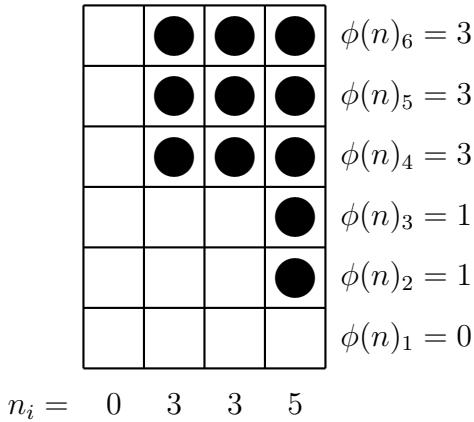
We define a map $\phi : \mathbb{N}_{\uparrow}^k \rightarrow \mathbb{N}_{\uparrow}^{n-k}$ by

$$\phi(\underline{n})_m = \begin{cases} 0 & \text{if } 1 \leq m \leq n - k - n_k, \\ 1 & \text{if } n - k - n_k + 1 \leq m \leq n - k - n_{k-1}, \\ \vdots & \\ k & \text{if } m \geq n - k - n_1 + 1. \end{cases} \quad (23)$$

That is,

$$\phi(\underline{n})_m = \min\{r \geq 0 : n_{k-r} \leq n - k - m\}. \quad (24)$$

For example, if $n = 10$ and $k = 4$, then $f(0, 3, 3, 5) = (0, 1, 1, 3, 3, 3)$. A pictorial representation of this map is obtained by filling squares of a $k \times (n - k)$ grid with beads; n_i on column i and $\phi(\underline{n})_j$ on row j . We order the rows from bottom to top:



Example 1. Consider the case where $k = 3$, $n = 7$ and $I = \{1, 2, 3\}$. Then (16) reads

$$\begin{aligned} & \prod_{i \in I} \prod_{j \in I^c} (a_i a_j - 1) \\ &= A_{4,4,4} S_4^3 - A_{3,4,4} S_3 S_4^2 \\ &\quad + A_{3,3,4} S_3^2 S_4 + A_{2,4,4} S_2 S_4^2 \\ &\quad - A_{3,3,3} S_3^3 - A_{2,3,4} S_2 S_3 S_4 - A_{1,4,4} S_1 S_4^2 \\ &\quad + A_{2,3,3} S_2 S_3^2 + A_{2,2,4} S_2^2 S_4 + A_{1,3,4} S_1 S_3 S_4 + A_{0,4,4} S_4^2 \\ &\quad - A_{2,2,3} S_2^2 S_3 - A_{1,3,3} S_1 S_3^2 - A_{1,2,4} S_1 S_2 S_4 - A_{0,3,4} S_3 S_4 \\ &\quad + A_{2,2,2} S_2^3 + A_{1,2,3} S_1 S_2 S_3 + A_{0,3,3} S_3^2 + A_{1,1,4} S_1^2 S_4 + A_{0,2,4} S_2 S_4 \\ &\quad - A_{1,2,2} S_1 S_2^2 - A_{1,1,3} S_1^2 S_3 - A_{0,2,3} S_2 S_3 - A_{0,1,4} S_1 S_4 \\ &\quad + A_{1,1,2} S_1^2 S_2 + A_{0,2,2} S_2^2 + A_{0,1,3} S_1 S_3 + A_{0,0,4} S_4 \\ &\quad - A_{1,1,1} S_1^3 - A_{0,1,2} S_1 S_2 - A_{0,0,3} S_3 \\ &\quad + A_{0,1,1} S_1^2 + A_{0,0,2} S_2 - A_{0,0,1} S_1 + 1. \end{aligned}$$

Here we have ordered the terms first according to p from largest ($p_{\max} = k(n-k) = 12$) to smallest ($p = 0$) and then according to the above lexicographic ordering. Here, for example, $A_{0,2,4} = a_1^4(a_2^2 + a_3^2) + a_2^4(a_1^2 + a_3^2) + a_3^4(a_1^2 + a_2^2)$, and $S_2 = \sum_{4 \leq j_1 < j_2 \leq 7} a_{j_1} a_{j_2}$.

Let us also define, for $n \in \mathbb{N}$ and $\underline{m} \in \mathbb{N}_0^n$,

$$\Delta_{\underline{m}}(J) = \begin{vmatrix} a_{j_1}^{m_1} & \cdots & a_{j_1}^{m_n} \\ \vdots & \cdots & \vdots \\ a_{j_n}^{m_1} & \cdots & a_{j_n}^{m_n} \end{vmatrix} \text{ if } J = \{j_1, \dots, j_n\}. \quad (25)$$

We then claim that there is an upper-triangular matrix $R_{\underline{n}, \underline{m}}$ such that

$$\prod_{r=1}^k S_{n_r} \Delta(I^c) = \sum_{\substack{\underline{m} \in \mathbb{N}_+^k: \\ |\underline{m}| = |\underline{n}|}} R_{\underline{n}, \underline{m}} \Delta_{\psi(\underline{m})}(I^c), \quad (26)$$

where $\psi(\underline{m})_r = \phi(\underline{m})_r + r - 1$. This follows easily by induction from Corollary 2.2 of Lemma 2.6, according to which

$$S_l \Delta_{\underline{m}}(I^c) = \sum_{\substack{\underline{m}' : m'_i - m_i = 0, 1 \\ \sum_{i=1}^k (m'_i - m_i) = l}} \Delta_{\underline{m}'}(I^c). \quad (27)$$

In terms of the pictorial representation, this means that multiplication by S_l corresponds to the addition of l additional beads on the right-most empty

sites of l different rows. This is equivalent to adding l beads on the upper most empty sites of a number of columns such that there are no new beads horizontally next to each other. Thus

$$S_l \Delta_{\psi(\underline{m})}(I^c) = \sum_{\substack{\underline{m}' : m_i \leq m'_i \leq m_{i+1} \\ \sum_{i=1}^k (m'_i - m_i) = l}} \Delta_{\psi(\underline{m}')}(I^c). \quad (28)$$

In particular, note that the minimal \underline{m}' (w.r.t. the above ordering) is obtained by adding beads to the upper-most incomplete rows. Note also that $|\underline{m}'| = |\underline{m}| + l$. Iterating, it follows that for $\underline{n} \in \mathbb{N}_+^k$,

$$\prod_{r=1}^k S_{n_r} \Delta(I^c) = \sum_{\substack{\underline{m} \in \mathbb{N}_+^k : \\ \underline{m} \geq \underline{n}; |\underline{m}| = |\underline{n}|}} R_{\underline{n}, \underline{m}} \Delta_{\psi(\underline{m})}(I^c), \quad (29)$$

where the matrix R is upper-triangular and has integer matrix elements given by the number of times a given configuration \underline{m} is obtained by iterating the above procedure. (Note that the number of non-zero n_r is the maximal length of a row of beads, i.e. $\phi(\underline{n})_{n-k}$. Also, $\underline{m} = \underline{n}$ only if the beads are placed in order from right to left starting with n_k , so $R_{\underline{n}, \underline{n}} = 1$.)

Example 2. In the case of Example 1, with $p = 6$, the matrix R is given by

$$R = \begin{pmatrix} 1 & 2 & 1 & 1 & 3 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where the rows are numbered from top to bottom and the columns from left to right in increasing lexicographic order, i.e. (222), (123), (033), (114), (024).

Next we consider the expressions $A_{\underline{n}}$. We claim that

$$A_{\underline{n}} \Delta(I) = \sum_{\substack{\underline{m} \in \mathbb{N}_+^k : \\ \underline{m} \leq \underline{n}; |\underline{m}| = |\underline{n}|}} (R^{-1})_{\underline{m}, \underline{n}} \Delta_{\tilde{\underline{m}}}(I), \quad (30)$$

where $\tilde{m}_i = m_i + i - 1$. Equivalently,

$$\sum_{\substack{\underline{n} \in \mathbb{N}_+^k : \\ \underline{n} \leq \underline{m}; |\underline{n}| = |\underline{m}|}} R_{\underline{n}, \underline{m}} A_{\underline{n}} \Delta(I) = \Delta_{\tilde{\underline{m}}}(I). \quad (31)$$

We prove this by induction on $p = |\underline{m}|$ and k . (Note that for $p = 1$, we have $\underline{n} = (0, \dots, 0, 1) = \underline{m}$ and $A_{0, \dots, 0, 1} = \sum_{i=1}^k a_i$ so that $A_{0, \dots, 0, 1}\Delta(I) = \Delta_{0, 1, \dots, n-k-2, n-k}$ follows from Corollary 2.1 of Lemma 2.5. If $k = 1$ then $I = \{i\}$ and $A_{n_1} = a_i^{n_1}$ so $A_{n_1}\Delta(\{i\}) = A_{n_1} = \Delta_{n_1}(\{i\})$.)

If $m_1 \geq 1$ then we define \underline{m}'' by $m_i'' = m_i - m_1$. Now it is easy to see that

$$R_{\underline{n}, \underline{m}} = 0 \text{ if } \sum_{r \leq r_0} n_r < \sum_{r \leq r_0} m_r \text{ for some } r_0 \geq 1. \quad (32)$$

In particular, if $m_1 \geq 1$ then $n_1 \geq m_1$, and in that case

$$R_{\underline{n}, \underline{m}} = R_{\underline{n}'', \underline{m}''}, \quad (33)$$

where $n_i'' = n_i - m_1$. Since $|\underline{n}''| = |\underline{n}| - km_1$, it follows from the induction hypothesis (w.r.t. p) that

$$\begin{aligned} \sum_{\substack{\underline{n} \in \mathbb{N}_+^k: \\ \underline{n} \leq \underline{m}; |\underline{n}| = |\underline{m}|}} R_{\underline{n}, \underline{m}} A_{\underline{n}} \Delta(I) &= A_{m_1, \dots, m_1} \sum_{\substack{\underline{n}'' \in \mathbb{N}_+^k: \\ \underline{n}'' \leq \underline{m}''; |\underline{n}''| = |\underline{m}| - km_1}} R_{\underline{n}'', \underline{m}''} A_{\underline{n}''} \Delta(I) \\ &= A_{m_1, \dots, m_1} \Delta_{\underline{m}''} = \Delta_{\underline{m}}. \end{aligned} \quad (34)$$

It remains to consider the case that $m_1 = 0$. If $m_1 = 0$ and also $n_1 = 0$, then we define $\underline{m}' = (m_2, \dots, m_k)$ and $\underline{n}' = (n_2, \dots, n_k)$ and we have

$$R_{\underline{n}, \underline{m}}^{(k)} = R_{\underline{n}', \underline{m}'}^{(k-1)}. \quad (35)$$

For ease of notation, we can assume that $I = \{1, \dots, k\}$. By induction w.r.t. k and expanding w.r.t. the first column, it follows that

$$\begin{aligned} \sum_{\substack{\underline{n} \in \mathbb{N}_+^k: \\ \underline{n} \leq \underline{m}; \sum_{r=1}^k n_r = p}} R_{\underline{n}, \underline{m}} A_{\underline{n}} \Delta(I) &= \Delta_{\psi(\underline{m})} \\ &+ \sum_{i=1}^k (-1)^{i-1} \sum_{s=1}^{m_k} a_i^s \sum_{\substack{\underline{n} \in \mathbb{N}_+^k: \\ \underline{n} \leq \underline{m}; |\underline{n}| = |\underline{m}|, s \in \{n_i\}}} R_{\underline{n}, \underline{m}} A_{\underline{n}^{(s)}} \Delta'(I \setminus \{i\}), \end{aligned} \quad (36)$$

where $\Delta'(I \setminus \{i\})$ denotes $\Delta_{1, 2, \dots, k-1}(I \setminus \{i\})$, and $\underline{n}^{(s)}$ is obtained from \underline{n} by omitting s , i.e. if $n_r = s$ then $n_i^{(s)} = n_i$ for $i < r$ and $n_i^{(s)} = n_{i+1}$ if $i \geq r$. (There may be more than one such r , namely, if $k_s > 1$, in which case we can simply choose one.) By the definition of R , and the formula (28) for S_l , we have that if $s \in \{n_i\}$ then

$$R_{\underline{n}, \underline{m}} = \sum_{\substack{q \in \mathbb{N}_+^{k-1}: \\ |q| = s, 0 \leq q_i \leq m_{i+1} - m_i}} R_{\underline{n}^{(s)}, \underline{m}' - \underline{q}}. \quad (37)$$

Inserting this, the remainder term becomes

$$\begin{aligned}
& \sum_{i=1}^k (-1)^{i-1} \sum_{s=1}^{m_k} a_i^s \sum_{\substack{\underline{n} \in \mathbb{N}_\uparrow^k : \\ \underline{n} \leq \underline{m}; |\underline{n}| = |\underline{m}|, s \in \{n_i\}}} R_{\underline{n}, \underline{m}} A_{\underline{n}^{(s)}} \Delta'(I \setminus \{i\}) \\
&= \sum_{s=1}^{m_k} \sum_{\substack{\underline{q} \in \mathbb{N}_\uparrow^{k-1} : \\ |\underline{q}| = s, 0 \leq q_i \leq m_{i+1} - m_i}} \sum_{i=1}^k (-1)^{i-1} a_i^s \\
&\quad \times \sum_{\substack{\underline{n}' \in \mathbb{N}_\uparrow^{k-1} : \\ \underline{n}' \leq \underline{m}' - \underline{q}; |\underline{n}'| = |\underline{m}'| - s}} R_{\underline{n}', \underline{m}' - \underline{q}} A_{\underline{n}'} \Delta'(I \setminus \{i\}) \\
&= \sum_{s=1}^{m_k} \sum_{\substack{\underline{q} \in \mathbb{N}_\uparrow^{k-1} : \\ |\underline{q}| = s, 0 \leq q_i \leq m_{i+1} - m_i}} \sum_{i=1}^k (-1)^{i-1} a_i^s \Delta'_{\widetilde{\underline{m}' - \underline{q}}} (I \setminus \{i\}) \\
&= \sum_{s=1}^{m_k} \sum_{\substack{\underline{q} \in \mathbb{N}_\uparrow^{k-1} : \\ |\underline{q}| = s, 0 \leq q_i \leq m_{i+1} - m_i}} \Delta_{(s, \widetilde{\underline{m}' - \underline{q}})}. \tag{38}
\end{aligned}$$

Example 3. To clarify this, consider Example 1 again and let $\underline{m} = (024)$. With the R -matrix of Example 2, we then have

$$\sum_{\substack{\underline{n} \in \mathbb{N}_\uparrow^3 : \\ \underline{n} \leq \underline{m}; |\underline{n}| = |\underline{m}|}} R_{\underline{n}, \underline{m}} A_{\underline{n}} \Delta(\{1, 2, 3\}) = \begin{vmatrix} \sum_{\substack{\underline{n} \leq (024); |\underline{n}| = 6}} R_{\underline{n}, 024} A_{\underline{n}} & a_1 & a_1^2 \\ \sum_{\substack{\underline{n} \leq (024); |\underline{n}| = 6}} R_{\underline{n}, 024} A_{\underline{n}} & a_2 & a_2^2 \\ \sum_{\substack{\underline{n} \leq (024); |\underline{n}| = 6}} R_{\underline{n}, 024} A_{\underline{n}} & a_3 & a_3^2 \end{vmatrix}.$$

In the i -th row, we separate out the terms where a_i has the power 0 in $A_{\underline{n}}$, in particular $n_1 = 0$. These are given by $A_{\underline{n}'}$ (as a function of a_j ($j \neq i$)). This yields

$$\begin{vmatrix} \sum_{\substack{\underline{n}' \leq (24); |\underline{n}'| = 6}} R_{\underline{n}', 24} A_{\underline{n}'} & a_1 & a_1^2 \\ \sum_{\substack{\underline{n}' \leq (24); |\underline{n}'| = 6}} R_{\underline{n}', 24} A_{\underline{n}'} & a_2 & a_2^2 \\ \sum_{\substack{\underline{n}' \leq (24); |\underline{n}'| = 6}} R_{\underline{n}', 24} A_{\underline{n}'} & a_3 & a_3^2 \end{vmatrix} = \begin{vmatrix} 1 & a_1^3 & a_1^6 \\ 1 & a_2^3 & a_2^6 \\ 1 & a_3^3 & a_3^6 \end{vmatrix} = \Delta_{\tilde{\underline{m}}}(\{1, 2, 3\}).$$

(For example,

$$\sum_{\substack{\underline{n}' \leq 24; |\underline{n}'| = 6}} R_{\underline{n}', 24} A_{\underline{n}'}(a_2, a_3) \begin{vmatrix} a_2 & a_2^2 \\ a_3 & a_3^2 \end{vmatrix} = \begin{vmatrix} a_2^3 & a_2^6 \\ a_3^3 & a_3^6 \end{vmatrix} \cdot .$$

The remaining terms have a_i^s for some $s \geq 1$ in the first column of the i -th row. They are

$$\sum_{s=1}^4 a_i^s \sum_{\substack{\underline{n} \in \mathbb{N}_+^3 : \\ \underline{n} \leq (024); |\underline{n}|=6, s \in \{n_i\}}} R_{\underline{n},024} A_{\underline{n}}.$$

Now, by equation (37),

$$\begin{aligned} R_{222,024} &= R_{22,04} + R_{22,13} + R_{22,22} = 3; \\ R_{123,024} &= R_{23,14} + R_{23,23} = R_{13,04} + R_{13,13} = R_{12,03} + R_{12,12} = 2; \\ R_{033,024} &= R_{33,24} = R_{03,03} = 1; \\ R_{114,024} &= R_{14,14} = R_{11,02} = 1 \text{ and } R_{024,024} = R_{24,24} = R_{04,04} = R_{02,02} = 1. \end{aligned}$$

(Note that $R_{03,12} = 0$ for example, and in the case of $R_{114,024}$, the term $R_{11,11}$ is not allowed because in that case $q_2 = 3$ whereas $m_3 - m_2 = 2$.)

For $s = 4$ we thus obtain $a_i^4(A_{02} + R_{114,024}A_{11}) = a_i^4(A_{02} + R_{11,02}A_{11})$, which yields the determinant

$$\begin{vmatrix} a_1^4(A_{02} + R_{11,02}A_{11}) & a_1 & a_1^2 \\ a_2^4(A_{02} + R_{11,02}A_{11}) & a_2 & a_2^2 \\ a_3^4(A_{02} + R_{11,02}A_{11}) & a_3 & a_3^2 \end{vmatrix} = \Delta_{4,1,4} = 0.$$

(Here we use induction w.r.t. k .) For $s = 3$ we obtain $a_i^3(A_{03} + R_{123,024}A_{12}) = a_i^3(A_{03} + R_{12,03}A_{12} + R_{12,12}A_{12})$. This yields the determinant

$$\begin{vmatrix} a_1^3(A_{03} + R_{12,03}A_{12} + R_{12,12}A_{12}) & a_1 & a_1^2 \\ a_2^3(A_{03} + R_{12,03}A_{12} + R_{12,12}A_{12}) & a_2 & a_2^2 \\ a_3^3(A_{03} + R_{12,03}A_{12} + R_{12,12}A_{12}) & a_3 & a_3^2 \end{vmatrix}$$

which equals $\Delta_{3,1,5} + \Delta_{3,2,4} = -\Delta_{1,3,5} - \Delta_{2,3,4}$. For $s = 2$ we get $a_i^2(A_{04} + R_{123,024}A_{13} + R_{222,024}A_{22}) = a_i^2(A_{04} + (R_{13,04} + R_{13,13})A_{13} + (R_{22,04} + R_{22,13} + R_{22,22})A_{22})$. This yields $\Delta_{2,1,6} + \Delta_{2,2,5} + \Delta_{2,3,4} = -\Delta_{1,2,6} + \Delta_{2,3,4}$. Finally, for $s = 1$ we have $a_i(R_{114,024}A_{14} + R_{123,024}A_{23}) = a_i(A_{14} + (R_{23,14} + R_{23,23})A_{23})$ and we obtain the determinants $\Delta_{1,2,6} + \Delta_{1,3,5}$. In total, we get $-\Delta_{1,3,5} - \Delta_{2,3,4} - \Delta_{1,2,6} + \Delta_{2,3,4} + \Delta_{1,2,6} + \Delta_{1,3,5} = 0$.

In general, we shall prove that the resulting determinants in (38) cancel in pairs. Consider a term $\Delta_{(s,\widetilde{m'-q})}$. It equals $\pm \Delta_{\widetilde{n}}$ for some $\underline{n} \in \mathbb{N}_+^k$. Conversely, now first suppose that $\underline{n}' = \underline{m}' - \underline{q}$ for some \underline{q} satisfying $0 \leq q_i \leq m_{i+1} - m_i$ and $|\underline{q}| = n_1 \geq 1$, i.e. there is a term with $s = n_1$. Then consider the

case $s = n_2 + 1$, where we need $\underline{m}' - \underline{q} = (n_1 - 1, n_3, \dots, n_k)$. Set $\tilde{q}_1 = q_1 + n_2 - n_1 + 1$ and $\tilde{q}_i = q_i$ for $i \geq 2$. Then $(n_1 - 1, n_3, \dots, n_k) = \underline{m}' - \tilde{\underline{q}}$ and $|\tilde{\underline{q}}| = n_2 + 1$. Moreover, since $1 \leq n_1 \leq n_2$ and $n_2 = m_2 - q_1 \leq m_2$, we have $\tilde{q}_1 = m_2 - n_1 + 1 \geq 0$ and $\tilde{q}_1 \leq m_2 = m_2 - m_1$. It follows that if $\Delta_{\tilde{n}}$ occurs in the sum (38) (with $s = n_1$) then $\Delta_{(n_2+1, \tilde{n}_1-1, \tilde{n}_3, \dots, \tilde{n}_k)}$ also occurs. But they cancel one another. Conversely, suppose that $\Delta_{(n_2+1, \tilde{n}_1-1, \tilde{n}_3, \dots, \tilde{n}_k)}$ occurs, so that $(n_1 - 1, n_3, \dots, n_k) = \underline{m}' - \tilde{\underline{q}}$ for some $\tilde{\underline{q}}$ such that $0 \leq \tilde{q}_i \leq m_{i+1} - m_i$ and $|\tilde{\underline{q}}| = n_2 + 1$. Define $q_1 = \tilde{q}_1 - n_2 + n_1 - 1$. Then we need that $0 \leq \tilde{q}_1 \leq m_2$, i.e. $0 \leq m_2 - n_2 \leq m_2$. Therefore, if $\Delta_{(n_2+1, \tilde{n}_1-1, \tilde{n}_3, \dots, \tilde{n}_k)}$ occurs then $\Delta_{\tilde{n}}$ also occurs provided $n_2 \leq m_2$. If this is not the case then $\Delta_{\tilde{n}}$ does not occur and we must start with $\Delta_{(n_2+1, \tilde{n}_1-1, \tilde{n}_3, \dots, \tilde{n}_k)}$. If this term does occur in (38) then there is \underline{q} such that $0 \leq q_i \leq m_{i+1} - m_i$, $|\underline{q}| = n_2 + 1$ and $(n_1 - 1, n_3, \dots, n_k) = \underline{m}' - \underline{q}$. Defining $\tilde{\underline{q}}$ by $\tilde{q}_2 = q_2 + n_3 - n_2 + 1$, $\tilde{q}_i = q_i$ for $i \neq 2$, we have $|\tilde{\underline{q}}| = n_3 + 2$ and $(n_1 - 1, n_2 - 1, n_4, \dots, n_k) = \underline{m}' - \tilde{\underline{q}}$. We need $0 \leq \tilde{q}_2 \leq m_3 - m_2$, i.e. $m_2 + 1 \leq n_3 \leq m_3 + 1$. But $n_3 \geq n_2 \geq m_2 + 1$ because we assumed that $\Delta_{\tilde{n}}$ does not occur. On the other hand $n_3 = m_3 - q_2 \leq m_3$. Therefore, $\Delta_{(n_2+1, \tilde{n}_1-1, \tilde{n}_3, \dots, \tilde{n}_k)}$ also occurs, and the two terms cancel each other.

More generally, suppose that $r \geq 2$ is an integer such that $\pm \Delta_n$ with $s = n_r + r - 1$ occurs in the sum (38). Then there exists $\underline{q} \in \mathbb{N}_0^{k-1}$ such that $0 \leq q_i \leq m_{i+1} - m_i$ and $|\underline{q}| = n_r + r - 1$ and $(n_1 - 1, \dots, n_{r-1} - 1, n_{r+1}, \dots, n_k) = (m_2 - q_1, \dots, m_k - q_{k-1})$. Therefore,

$$\begin{cases} m_i + 1 \leq n_i \leq m_{i+1} + 1 & \text{for } i < r; \\ m_i \leq n_{i+1} \leq m_{i+1} & \text{for } i \geq r. \end{cases} \quad (39)$$

Define $\underline{q}^{(r-1)}$ by $q_{r-1}^{(r-1)} = q_{r-1} - n_r + n_{r-1} - 1$ and $q_i^{(r-1)} = q_i$ for $i \neq r - 1$. Then $|\underline{q}^{(r-1)}| = |\underline{q}| - n_r + n_{r-1} + 1 = n_r + r - 1 - n_r + n_{r-1} - 1 = n_{r-1} + r - 2$ and $m_r - q_{r-1}^{(r-1)} = m_r - q_{r-1} + n_r - n_{r-1} + 1 = n_r$. Therefore the term with $s = n_{r-1} + r - 2$ also occurs provided $0 \leq q_{r-1}^{(r-1)} \leq m_r - m_{r-1}$. But, $m_r - q_{r-1} = n_r$, so this holds if $0 \leq m_r - n_r \leq m_r - m_{r-1}$. By (39), $n_{r-1} \geq m_{r-1} + 1$ and since $n_r \geq n_{r-1}$ the first inequality holds. Thus the term $s = n_{r-1} + r - 2$ also occurs if $n_r \leq m_r$.

Suppose now that this term does not occur. Then we conclude that $n_r \geq m_r + 1$. Now define $\underline{q}^{(r+1)}$ by $q_r^{(r+1)} = q_r + n_{r+1} - n_r + 1$ and $q_i^{(r+1)} = q_i$ for $i \neq r$. Then $(n_1 - 1, \dots, n_{r-1}, n_{r+2}, \dots, n_k) = \underline{m}' - \tilde{\underline{q}}$ since $m_{r+1} - q_r^{(r+1)} = m_{r+1} - (q_r + n_{r+1} - n_r + 1) = n_r - 1$. Also, $|\underline{q}^{(r+1)}| = n_{r+1} + r$. Moreover, $n_r \geq m_r + 1 \implies q_r^{(r+1)} \leq m_{r+1} - m_r$ and $n_{r+1} \leq m_{r+1} \implies n_r \leq m_{r+1} + 1 \implies q_r^{(r+1)} \geq 0$. Therefore the term with $s = n_{r+1} + r$ also occurs

and cancels the term $s = n_r + r - 1$.

We conclude that if the term $s = n_r + r - 1$ occurs then either $s = n_{r-1} + r - 2$ exists or $s = n_{r+1} + r$ exists, but not both. Note that $s \leq m_k$, so only terms $s = n_r + r - 1$ can exist where $n_r \leq m_k$. That means that if the term $s = n_k + k - 1$ occurs then the term $s = n_{k-1} + k - 2$ also occurs.

This proves that the sum (38) equals zero, and hence that (31) holds. Now, inserting (29) and (31) into (20) we have

$$\begin{aligned}
& \sum_{K \subset I \times I^c: |K|=p} \prod_{(i,j) \in K} a_i a_j \Delta(I) \Delta(I^c) \\
&= \sum_{\substack{\underline{n} \in \mathbb{N}_\uparrow^k: \\ |\underline{n}|=p}} A_{\underline{n}} \Delta(I) \sum_{\substack{\underline{m} \geq \underline{n}: \\ |\underline{m}|=p}} R_{\underline{n}, \underline{m}} \Delta_{\psi(\underline{m})}(I^c) \\
&= \sum_{\substack{\underline{m} \in \mathbb{N}_\uparrow^k: \\ |\underline{m}|=p}} \Delta_{\psi(\underline{m})}(I^c) \sum_{\substack{\underline{n} \leq \underline{m}: \\ |\underline{n}|=p}} R_{\underline{n}, \underline{m}} A_{\underline{n}} \Delta(I) \\
&= \sum_{\substack{\underline{m} \in \mathbb{N}_\uparrow^k: \\ |\underline{m}|=p}} \Delta_{\tilde{\underline{m}}}(I) \Delta_{\psi(\underline{m})}(I^c). \tag{40}
\end{aligned}$$

Inserting this into (16) and (14) we have

$$\begin{aligned}
& \sum_{I \subset \{1, \dots, n\}} (-1)^{\nu(I)} \prod_{i \in I} \prod_{j \in I^c} (a_i a_j - 1) \Delta(I) \Delta(I^c) \\
&= \sum_{I \subset \{1, \dots, n\}} (-1)^{\nu(I)} \sum_{p=0}^{|I| |I^c|} (-1)^{|I| |I^c| - p} \sum_{\substack{\underline{m} \in \mathbb{N}_\uparrow^k: \\ |\underline{m}|=p}} \Delta_{\tilde{\underline{m}}}(I) \Delta_{\psi(\underline{m})}(I^c) \\
&= \sum_{k=0}^n \sum_{p=0}^{k(n-k)} (-1)^{k(n-k)-p} \sum_{\substack{\underline{m} \in \mathbb{N}_\uparrow^k: \\ |\underline{m}|=p}} \\
&\quad \times \sum_{\substack{I \subset \{1, \dots, n\}: \\ |I|=k}} (-1)^{\nu(I)} \Delta_{\tilde{\underline{m}}}(I) \Delta_{\psi(\underline{m})}(I^c). \tag{41}
\end{aligned}$$

The last sum is an expansion of $\Delta_{\tilde{\underline{m}}, \psi(\underline{m})}(\{1, \dots, n\})$ w.r.t. the first k columns. In general,

$$\begin{aligned}
& \Delta_{m_1, \dots, m_n}(\{1, \dots, n\}) = \\
&= (-1)^{k(k+1)/2} \sum_{\substack{I \subset \{1, \dots, n\}: \\ |I|=k}} (-1)^{\nu(I)} \Delta_{m_1, \dots, m_k}(I) \Delta_{m_{k+1}, \dots, m_n}(I^c). \tag{42}
\end{aligned}$$

Indeed, for $k = 1$ we have

$$\Delta_{m_1, \dots, m_n}(\{1, \dots, n\}) = \sum_{i \in \{1, \dots, n\}} (-1)^{i-1} a_i^{m_1} \Delta_{m_2, \dots, m_n}(\{1, \dots, n\} \setminus \{i\}),$$

where $a_i^{m_1} = \Delta_{m_1}(\{i\})$. By induction we then have

$$\begin{aligned} \Delta_{m_1, \dots, m_n}(\{1, \dots, n\}) &= \sum_{i=1}^n (-1)^{i-1} a_i^{m_1} \Delta_{m_2, \dots, m_n}(\{1, \dots, n\} \setminus \{i\}) \\ &= (-1)^{k(k-1)/2} \sum_{i=1}^n (-1)^{i-1} a_i^{m_1} \\ &\quad \times \sum_{\substack{I \subset \{1, \dots, n\} \setminus \{i\}: \\ |I|=k-1}} (-1)^{\nu'_i(I)} \Delta_{m_2, \dots, m_k}(I) \Delta_{m_{k+1}, \dots, m_n}(I^c), \end{aligned}$$

where $\nu'_i(I) = \sum_{j \in I} j - \#\{j \in I : j > i\}$. Thus

$$\begin{aligned} \Delta_{m_1, \dots, m_n}(\{1, \dots, n\}) &= \\ &= (-1)^{k(k-1)/2} \sum_{\substack{I \subset \{1, \dots, n\}: \\ |I|=k}} \sum_{i \in I} (-1)^{\nu(I)-1-\#\{j \in I: j > i\}} \\ &\quad \times a_i^{m_1} \Delta_{m_2, \dots, m_k}(I) \Delta_{m_{k+1}, \dots, m_n}(I^c) \\ &= (-1)^{k(k-1)/2} \sum_{\substack{I \subset \{1, \dots, n\}: \\ |I|=k}} \sum_{i \in I} (-1)^{\nu(I)-k-\#\{j \in I: j < i\}} \\ &\quad \times a_i^{m_1} \Delta_{m_2, \dots, m_k}(I) \Delta_{m_{k+1}, \dots, m_n}(I^c) \\ &= (-1)^{k(k+1)/2} \sum_{\substack{I \subset \{1, \dots, n\}: \\ |I|=k}} (-1)^{\nu(I)} \Delta_{m_1, \dots, m_k}(I) \Delta_{m_{k+1}, \dots, m_n}(I^c). \end{aligned}$$

Thus, in order to prove (14), we want to show that

$$\sum_{k=0}^n \sum_{p=0}^{k(n-k)} (-1)^{k(n-k)-p+k(k+1)/2} \sum_{\substack{\underline{m} \in \mathbb{N}_+^k: \\ |\underline{m}|=p}} \Delta_{\tilde{m}, \psi(\underline{m})}(\{1, \dots, n\}) = 0. \quad (43)$$

First note that $\Delta_{\tilde{m}, \psi(\underline{m})}(\{1, \dots, n\}) = 0$ unless \tilde{m} and $\psi(\underline{m})$ have nothing in common and make up $\{0, 1, \dots, n-1\}$. In particular, $|\underline{m}| + |\phi(\underline{m})| + \frac{1}{2}k(k-1) + \frac{1}{2}(n-k)(n-k-1) = \frac{1}{2}n(n-1)$, i.e.

$$2p = \frac{1}{2}n(n-1) - \frac{1}{2}k(k-1) - \frac{1}{2}(n-k)(n-k-1) = k(n-k).$$

If $k(n - k)$ is odd, there is no nonzero term, so if n is even then k must also be even. We therefore need

$$\sum_{k=0}^n (-1)^{k(n-k)/2+k(k+1)/2} \sum_{\substack{\underline{m} \in \mathbb{N}_+^k : \\ |\underline{m}| = k(n-k)/2}} \Delta_{\tilde{\underline{m}}, \psi(\underline{m})}(\{1, \dots, n\}) = 0. \quad (44)$$

Next we argue that $\Delta_{\tilde{\underline{m}}, \psi(\underline{m})}(\{1, \dots, n\}) = 0$ unless $m_i + m_{k-i+1} = n - k$ for $i = 1, \dots, k$. Consider the case $i = 1$. In order that all the numbers below m_1 are present, we need $\phi(\underline{m})_j = 0$ for $j = 1, \dots, m_1$, while $\phi(\underline{m})_{m_1+1} \geq 1$. This means that the number of zeros in $\phi(\underline{m})$ equals m_1 , so $m_k = n - k - m_1$. The converse also holds. Similarly, for $i > 1$, we must have $\psi(\underline{m})_j = j+i-1$, i.e. $\phi(\underline{m})_j = i$, for $j = m_{i-1} + 1, \dots, m_i$ and $\phi(\underline{m})_{m_i+1} \geq i+1$. This implies that $m_{k-i+2} - m_{k-i+1} = m_i - m_{i-1}$. By induction, therefore $m_i + m_{k-i+1} = m_{i-1} + m_{k-i+2} = n - k$. In particular, if k is odd, then $n - k$ is even and $m_{(k+1)/2} = (n - k)/2$.

Consider first the case that n is even, and hence also k is even. Then we can count the number of possible solutions as follows. We put a number of markers between the numbers $i - 1$ and i equal to $m_i - m_{i-1}$, where $i = 1, \dots, k/2$. The total number of positions for the $(n-k)/2$ markers is then $n/2$ and the number of possible arrangements equals $\binom{n/2}{(n-k)/2} = \binom{n/2}{k/2}$. Note also, that if we move the $k/2$ last elements m_i ($i = k/2+1, \dots, k$) across all $\phi(\underline{m})_j$ ($j = 1, \dots, n-k$), then in order to put the \tilde{m}_i and $\psi(\underline{m})_j$ in increasing order, it remains to move each m_i with $i \leq k/2$ across equally many $\phi(\underline{m})_j$ to the right as we need to move m_{k-i+1} across $\phi(\underline{m})_j$ to the left. This means that in each case, the determinant $\Delta_{\tilde{\underline{m}}, \psi(\underline{m})} = (-1)^{k(n-k)/2} \Delta(\{1, \dots, n\})$. Inserting this into the left-hand side of (43) we obtain

$$\begin{aligned} & \sum_{k=0}^n (-1)^{k(n-k)/2+k(k+1)/2} \sum_{\substack{\underline{m} \in \mathbb{N}_+^k : \\ |\underline{m}| = k(n-k)/2}} \Delta_{\tilde{\underline{m}}, \psi(\underline{m})}(\{1, \dots, n\}) = \\ &= \sum_{\substack{k=0 \\ k \text{ even}}}^n (-1)^{k/2} \binom{n/2}{k/2} \Delta(\{1, \dots, n\}) = 0. \end{aligned} \quad (45)$$

(Note that if k is even, then $(-1)^{k(k+1)/2} = (-1)^{k/2}$.)

Analogously, if n is odd, then if k is even, the number of possibilities is $\binom{(n-1)/2}{k/2}$, and if k is odd then the number of possibilities is $\binom{(n-1)/2}{(k-1)/2}$.

The sign is again $(-1)^{k(n-k)/2}$ and we obtain

$$\begin{aligned} & \sum_{k=0}^{(n-1)/2} (-1)^{k(k+1)/2} \binom{(n-1)/2}{[k/2]} \Delta \\ &= \sum_{l=0}^{(n-1)/2} (-1)^l \binom{(n-1)/2}{l} \Delta + \sum_{l=0}^{(n-1)/2} (-1)^{l+1} \binom{(n-1)/2}{l} \Delta = 0. \quad (46) \end{aligned}$$

In both cases therefore (43) holds. The claim (14) is thus proved. This completes the proof of the theorem. \blacksquare

2 Lemmas

Lemma 2.1 *Let \mathcal{R} be a commutative ring. For $n \geq 3$ and $a_1, \dots, a_n \in \mathcal{R}$, and for $0 \leq k+l \leq n-2$,*

$$\left| \begin{array}{cccccc} 1 & a_1 & \cdots & a_1^{n-2} & a_1^l \sum_{2 \leq j_1 < \dots < j_k \leq n} a_{j_1} \dots a_{j_k} \\ 1 & a_2 & \cdots & a_2^{n-2} & a_2^l \sum_{\substack{1 \leq j_1 < \dots < j_k \leq n \\ j_r \neq 2}} a_{j_1} \dots a_{j_k} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 1 & a_n & \cdots & a_n^{n-2} & a_n^l \sum_{1 \leq j_1 < \dots < j_k \leq n-1} a_{j_1} \dots a_{j_k} \end{array} \right| = 0. \quad (47)$$

Proof. For $k = 0$ this is obvious.

We now proceed by induction on k :

$$\begin{aligned} & \left| \begin{array}{cccccc} 1 & a_1 & \cdots & a_1^{n-2} & a_1^l \sum_{2 \leq j_1 < \dots < j_k \leq n} a_{j_1} \dots a_{j_k} \\ 1 & a_2 & \cdots & a_2^{n-2} & a_2^l \sum_{\substack{1 \leq j_1 < \dots < j_k \leq n \\ j_r \neq 2}} a_{j_1} \dots a_{j_k} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 1 & a_n & \cdots & a_n^{n-2} & a_n^l \sum_{1 \leq j_1 < \dots < j_k \leq n-1} a_{j_1} \dots a_{j_k} \end{array} \right| \\ &= \left| \begin{array}{cccccc} 1 & a_1 & \cdots & a_1^{n-2} & a_1^l \sum_{1 \leq j_1 < \dots < j_k \leq n} a_{j_1} \dots a_{j_k} \\ 1 & a_2 & \cdots & a_2^{n-2} & a_2^l \sum_{1 \leq j_1 < \dots < j_k \leq n} a_{j_1} \dots a_{j_k} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 1 & a_n & \cdots & a_n^{n-2} & a_n^l \sum_{1 \leq j_1 < \dots < j_k \leq n-1} a_{j_1} \dots a_{j_k} \end{array} \right| \\ & - \left| \begin{array}{cccccc} 1 & a_1 & \cdots & a_1^{n-2} & a_1^{l+1} \sum_{2 \leq j_1 < \dots < j_{k-1} \leq n} a_{j_1} \dots a_{j_{k-1}} \\ 1 & a_2 & \cdots & a_2^{n-2} & a_2^{l+1} \sum_{\substack{1 \leq j_1 < \dots < j_{k-1} \leq n \\ j_r \neq 2}} a_{j_1} \dots a_{j_{k-1}} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 1 & a_n & \cdots & a_n^{n-2} & a_n^{l+1} \sum_{1 \leq j_1 < \dots < j_{k-1} \leq n-1} a_{j_1} \dots a_{j_{k-1}} \end{array} \right| = 0 \end{aligned}$$

provided $k + l \leq n - 2$. Indeed, the first term equals zero because the last column is a constant multiple of the $l + 1$ -th column, where $l \leq n - 2$. The second term equals zero by the induction hypothesis. \blacksquare

Similarly, we have also

Lemma 2.2 *Let \mathcal{R} be a commutative ring. For $n \geq 3$ and $a_1, \dots, a_n \in \mathcal{R}$, and for $1 \leq k, l \leq n - 1$, such that $k + l \geq n$,*

$$\left| \begin{array}{cccccc} 1 & a_1 & \cdots & a_1^{n-2} & a_1^l \sum_{\substack{2 \leq j_1 < \cdots < j_k \leq n \\ j_r \neq 2}} a_{j_1} \cdots a_{j_k} \\ 1 & a_2 & \cdots & a_2^{n-2} & a_2^l \sum_{\substack{1 \leq j_1 < \cdots < j_k \leq n \\ j_r \neq 2}} a_{j_1} \cdots a_{j_k} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 1 & a_n & \cdots & a_n^{n-2} & a_n^l \sum_{1 \leq j_1 < \cdots < j_k \leq n-1} a_{j_1} \cdots a_{j_k} \end{array} \right| = 0. \quad (48)$$

Proof. For $l \geq 1$ and $k = n - 1$ the final element in the i -th row equals $a_i^{l-1} a_1 \cdots a_n$ so the determinant is zero. For $k < n - 1$ we write

$$\begin{aligned} a_i^l \sum_{\substack{1 \leq j_1 < \cdots < j_k \leq n \\ j_r \neq i}} a_{j_1} \cdots a_{j_k} &= a_i^{l-1} \sum_{\substack{1 \leq j_1 < \cdots < j_k \leq n \\ j_r \neq i}} a_i a_{j_1} \cdots a_{j_k} \\ &= a_i^{l-1} \sum_{\substack{1 \leq j_1 < \cdots < j_{k+1} \leq n}} a_{j_1} \cdots a_{j_{k+1}} \\ &\quad - a_i^{l-1} \sum_{\substack{1 \leq j_1 < \cdots < j_{k+1} \leq n \\ j_r \neq i}} a_{j_1} \cdots a_{j_{k+1}}. \end{aligned}$$

Then first terms inserted into the determinant yield zero since $l - 1 \leq n - 2$, and the second terms yield zero by induction provided $l \geq 1$. \blacksquare

Lemma 2.3 *Let \mathcal{R} be a commutative ring. For $n \geq 3$ and $a_1, \dots, a_n \in \mathcal{R}$, and for $0 \leq k \leq n - 2$,*

$$\begin{aligned} &\left| \begin{array}{cccccc} 1 & a_1 & \cdots & a_1^{n-2} & a_1^{n-1-k} \sum_{2 \leq j_1 < \cdots < j_k \leq n} a_{j_1} \cdots a_{j_k} \\ 1 & a_2 & \cdots & a_2^{n-2} & a_2^{n-1-k} \sum_{\substack{1 \leq j_1 < \cdots < j_k \leq n \\ j_r \neq 2}} a_{j_1} \cdots a_{j_k} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 1 & a_n & \cdots & a_n^{n-2} & a_n^{n-1-k} \sum_{1 \leq j_1 < \cdots < j_k \leq n-1} a_{j_1} \cdots a_{j_k} \end{array} \right| \\ &= (-1)^k \left| \begin{array}{cccc} 1 & a_1 & \cdots & a_1^{n-2} & a_1^{n-1} \\ 1 & a_2 & \cdots & a_2^{n-2} & a_2^{n-1} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 1 & a_n & \cdots & a_n^{n-2} & a_n^{n-1} \end{array} \right|. \end{aligned} \quad (49)$$

Proof. For $k = 0$ the identity is tautological. For $k \geq 1$ we write again

$$\begin{aligned} a_i^{n-1-k} \sum_{\substack{1 \leq j_1 < \dots < j_k \leq n \\ j_r \neq i}} a_{j_1} \dots a_{j_k} &= a_i^{n-k-2} \sum_{\substack{1 \leq j_1 < \dots < j_{k+1} \leq n}} a_{j_1} \dots a_{j_{k+1}} \\ &\quad - a_i^{n-k-2} \sum_{\substack{1 \leq j_1 < \dots < j_{k+1} \leq n \\ j_r \neq i}} a_{j_1} \dots a_{j_{k+1}}. \end{aligned}$$

The first term yields zero and the result follows by induction. \blacksquare

As a corollary we have

Lemma 2.4 For $n \geq 3$ and $a_1, \dots, a_n \in \mathcal{R}$,

$$\begin{aligned} & \left| \begin{array}{ccccc} 1 & a_1 & \cdots & a_1^{n-2} & (a_1 - 1) \prod_{j=2}^n (a_1 a_j - 1) \\ 1 & a_2 & \cdots & a_2^{n-2} & (a_2 - 1) \prod_{\substack{j=1 \\ j \neq 2}}^n (a_2 a_j - 1) \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 1 & a_n & \cdots & a_n^{n-2} & (a_n - 1) \sum_{j=1}^{n-1} (a_j a_n - 1) \end{array} \right| \\ &= \left(\prod_{i=1}^n a_i - 1 \right) \left| \begin{array}{ccccc} 1 & a_1 & \cdots & a_1^{n-2} & a_1^{n-1} \\ 1 & a_2 & \cdots & a_2^{n-2} & a_2^{n-1} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 1 & a_n & \cdots & a_n^{n-2} & a_n^{n-1} \end{array} \right|. \end{aligned} \quad (50)$$

Proof. We expand

$$\prod_{\substack{j=1 \\ j \neq l}}^n (a_j a_l - 1) = \sum_{k=0}^{n-1} (-1)^{n-k-1} a_l^k \sum_{\substack{1 \leq j_1 < \dots < j_k \leq n \\ j_r \neq l}} a_{j_1} \dots a_{j_k}.$$

If n is even, $2k \neq n - 1$, so only a_i in the factor $a_i - 1$ contributes to the case $k = n - 1$. Then $l = k + 1$ so $k + l \geq n$ if $k \geq n/2$. By Lemma 2.2 these terms yield zero unless $l = n$ and $k = n - 1$. This is the highest-order term and yields

$$\prod_{i=1}^n a_i \left| \begin{array}{ccccc} 1 & a_1 & \cdots & a_1^{n-2} & a_1^{n-1} \\ 1 & a_2 & \cdots & a_2^{n-2} & a_2^{n-1} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 1 & a_n & \cdots & a_n^{n-2} & a_n^{n-1} \end{array} \right|.$$

If $k < n/2$, the only possible non-zero term is the case $k + l = n - 1$, i.e. $k = n/2 - 1$. By Lemma 2.3 this yields the contribution

$$(-1)^{n-1-k} (-1)^k \left| \begin{array}{ccccc} 1 & a_1 & \cdots & a_1^{n-2} & a_1^{n-1} \\ 1 & a_2 & \cdots & a_2^{n-2} & a_2^{n-1} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 1 & a_n & \cdots & a_n^{n-2} & a_n^{n-1} \end{array} \right| = - \left| \begin{array}{ccccc} 1 & a_1 & \cdots & a_1^{n-2} & a_1^{n-1} \\ 1 & a_2 & \cdots & a_2^{n-2} & a_2^{n-1} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 1 & a_n & \cdots & a_n^{n-2} & a_n^{n-1} \end{array} \right|. \blacksquare$$

Lemma 2.5 For $n \geq 2$, $1 \leq k \leq n - 1$, and $a_1, \dots, a_n \in \mathcal{R}$,

$$\begin{aligned}
& \left| \begin{array}{cccccc} \sum_{2 \leq j_1 < \dots < j_k \leq n} a_{j_1} \dots a_{j_k} & a_1 & \cdots & a_1^{n-1} \\ \sum_{\substack{1 \leq j_1 < \dots < j_k \leq n \\ j_r \neq 2}} a_{j_1} \dots a_{j_k} & a_2 & \cdots & a_2^{n-1} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{\substack{1 \leq j_1 < \dots < j_k \leq n \\ j_r \neq n}} a_{j_1} \dots a_{j_k} & a_n & \cdots & a_n^{n-1} \end{array} \right| \\
= & \left| \begin{array}{cccccc} 1 & a_1 & \cdots & a_1^{n-k-1} & a_1^{n-k+1} & \cdots & a_1^n \\ 1 & a_2 & \cdots & a_2^{n-k-1} & a_2^{n-k+1} & \cdots & a_2^n \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ 1 & a_n & \cdots & a_n^{n-k-1} & a_n^{n-k+1} & \cdots & a_n^n \end{array} \right|. \tag{51}
\end{aligned}$$

Proof. For $k = n - 1$, we have, expanding,

$$\begin{aligned}
& \left| \begin{array}{cccc} a_2 \dots a_n & a_1 & \cdots & a_1^{n-1} \\ a_1 a_3 \dots a_n & a_2 & \cdots & a_2^{n-1} \\ \vdots & \vdots & \cdots & \vdots \\ a_1 \dots a_{n-1} & a_n & \cdots & a_n^{n-1} \end{array} \right| \\
= & \sum_{j=1}^n (-1)^{j-1} a_1 \dots a_{j-1} a_{j+1} \dots a_n \left| \begin{array}{ccc} a_1 & \cdots & a_1^{n-1} \\ \vdots & \cdots & \vdots \\ [a_j & \cdots & a_j^{n-2}] \\ \vdots & \cdots & \vdots \\ a_n & \cdots & a_n^{n-1} \end{array} \right| \\
= & \sum_{j=1}^n (-1)^{j-1} \left| \begin{array}{ccc} a_1^2 & \cdots & a_1^n \\ \vdots & \cdots & \vdots \\ [a_j^2 & \cdots & a_j^n] \\ \vdots & \cdots & \vdots \\ a_n^2 & \cdots & a_n^n \end{array} \right| = \left| \begin{array}{cccc} 1 & a_1^2 & \cdots & a_1^n \\ \vdots & \vdots & \cdots & \vdots \\ 1 & a_n^2 & \cdots & a_n^n \end{array} \right|.
\end{aligned}$$

We proceed by induction and write similarly,

$$\begin{aligned}
& \left| \begin{array}{cccccc} \sum_{\substack{1 \leq j_1 < \dots < j_k \leq n \\ j_r \neq 1}} a_{j_1} \dots a_{j_k} & a_1 & \cdots & a_1^{n-1} \\ \sum_{\substack{1 \leq j_1 < \dots < j_k \leq n \\ j_r \neq 2}} a_{j_1} \dots a_{j_k} & a_2 & \cdots & a_2^{n-1} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{\substack{1 \leq j_1 < \dots < j_k \leq n \\ j_r \neq n}} a_{j_1} \dots a_{j_k} & a_n & \cdots & a_n^{n-1} \end{array} \right| \\
= & \sum_{j=1}^n (-1)^{j-1} \sum_{\substack{1 \leq j_1 < \dots < j_k \leq n \\ j_r \neq j}} a_{j_1} \dots a_{j_k} \left| \begin{array}{cccc} a_1 & \cdots & a_1^{n-1} \\ \vdots & \cdots & \vdots \\ [a_j & \cdots & a_j^{n-1}] \\ \vdots & \cdots & \vdots \\ a_n & \cdots & a_n^{n-1} \end{array} \right| \\
= & \sum_{j=1}^n (-1)^{j-1} \prod_{i \neq j} a_i \left| \begin{array}{cccccc} \sum_{\substack{1 \leq j_1 < \dots < j_k \leq n \\ j_r \neq j}} a_{j_1} \dots a_{j_k} & a_1 & \cdots & a_1^{n-2} \\ \vdots & \cdots & \vdots \\ ["] & a_j & \cdots & a_j^{n-2} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{\substack{1 \leq j_1 < \dots < j_k \leq n \\ j_r \neq j}} a_{j_1} \dots a_{j_k} & a_n & \cdots & a_n^{n-2} \end{array} \right| \\
& \left| \begin{array}{cccccc} \sum_{\substack{1 \leq j_1 < \dots < j_{k-1} \leq n \\ j_r \neq 1, j}} a_{j_1} \dots a_{j_{k-1}} & 1 & a_1 & \cdots & a_1^{n-3} \\ \vdots & \cdots & \vdots \\ ["] & a_j & \cdots & a_j^{n-3} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{\substack{1 \leq j_1 < \dots < j_{k-1} \leq n \\ j_r \neq j, n}} a_{j_1} \dots a_{j_{k-1}} & 1 & a_n & \cdots & a_n^{n-3} \end{array} \right| \\
= & \sum_{j=1}^n (-1)^{j-1} \prod_{i \neq j} a_i^2 \left| \begin{array}{cccccc} \sum_{\substack{2 \leq j_1 < \dots < j_k \leq n \\ j_r \neq j}} a_{j_1} \dots a_{j_k} & a_1 & \cdots & a_1^{n-2} \\ \vdots & \cdots & \vdots \\ ["] & a_j & \cdots & a_j^{n-2} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{\substack{1 \leq j_1 < \dots < j_k \leq n-1 \\ j_r \neq j}} a_{j_1} \dots a_{j_k} & a_n & \cdots & a_n^{n-2} \end{array} \right| \\
+ & \sum_{j=1}^n (-1)^{j-1} \prod_{i \neq j} a_i \left| \begin{array}{cccccc} \sum_{\substack{2 \leq j_1 < \dots < j_k \leq n \\ j_r \neq j}} a_{j_1} \dots a_{j_k} & a_1 & \cdots & a_1^{n-2} \\ \vdots & \cdots & \vdots \\ ["] & a_j & \cdots & a_j^{n-2} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{\substack{1 \leq j_1 < \dots < j_k \leq n-1 \\ j_r \neq j}} a_{j_1} \dots a_{j_k} & a_n & \cdots & a_n^{n-2} \end{array} \right|.
\end{aligned}$$

The first term equals zero by Lemma 2.1 since $k-1 \leq n-3$. By the induction hypothesis, the second term equals

$$\begin{aligned} & \sum_{j=1}^n (-1)^{j-1} \prod_{i \neq j} a_i \begin{vmatrix} 1 & a_1 & \cdots & a_1^{n-k-2} & a_1^{n-k} & \cdots & a_1^{n-1} \\ 1 & a_2 & \cdots & a_2^{n-k-2} & a_2^{n-k} & \cdots & a_2^{n-1} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ [1 & a_j & \cdots & a_j^{n-k-2} & a_n^{n-k} & \cdots & a_j^{n-1}] \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ 1 & a_n & \cdots & a_n^{n-k-2} & a_n^{n-k} & \cdots & a_n^{n-1} \end{vmatrix} \\ & = \begin{vmatrix} 1 & a_1 & \cdots & a_1^{n-k-1} & a_1^{n-k+1} & \cdots & a_1^n \\ 1 & a_2 & \cdots & a_2^{n-k-1} & a_2^{n-k+1} & \cdots & a_2^n \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ 1 & a_n & \cdots & a_n^{n-k-1} & a_n^{n-k+1} & \cdots & a_n^n \end{vmatrix}. \end{aligned}$$

■

Writing

$$\sum_{1 \leq j_1 < \cdots < j_k \leq n} a_{j_1} \cdots a_{j_k} = a_j \sum_{\substack{1 \leq j_1 < \cdots < j_{k-1} \leq n \\ j_r \neq j}} + \sum_{\substack{1 \leq j_1 < \cdots < j_k \leq n \\ j_r \neq j}}$$

we see that the first term vanishes if $k \leq n-1$ by Lemma 2.1 and we get

Corollary 2.1 For $n \geq 2$, $1 \leq k \leq n-1$, and $a_1, \dots, a_n \in \mathcal{R}$,

$$\begin{aligned} & \sum_{1 \leq j_1 < \cdots < j_k \leq n} a_{j_1} \cdots a_{j_k} \begin{vmatrix} 1 & a_1 & \cdots & a_1^{n-1} \\ 1 & a_2 & \cdots & a_2^{n-1} \\ \vdots & \vdots & \cdots & \vdots \\ 1 & a_n & \cdots & a_n^{n-1} \end{vmatrix} \\ & = \begin{vmatrix} 1 & a_1 & \cdots & a_1^{n-k-1} & a_1^{n-k+1} & \cdots & a_1^n \\ 1 & a_2 & \cdots & a_2^{n-k-1} & a_2^{n-k+1} & \cdots & a_2^n \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ 1 & a_n & \cdots & a_n^{n-k-1} & a_n^{n-k+1} & \cdots & a_n^n \end{vmatrix}. \end{aligned} \tag{52}$$

We generalise Lemma 2.5 further:

Lemma 2.6 Let \mathcal{R} be a commutative ring and $a_1, \dots, a_n \in \mathcal{R}$. Let $n \in \mathbb{N}$ and $m_1, \dots, m_{n-1} \in \mathbb{N}$ such that $1 \leq m_1 < \cdots < m_{n-1}$. Then, for any $k \in \mathbb{N}$

with $1 \leq k \leq n - 1$,

$$\begin{aligned}
& \left| \begin{array}{cccc} \sum_{2 \leq j_1 < \dots < j_k \leq n} a_{j_1} \dots a_{j_k} & a_1^{m_1} & \dots & a_1^{m_{n-1}} \\ \sum_{\substack{1 \leq j_1 < \dots < j_k \leq n \\ j_r \neq 2}} a_{j_1} \dots a_{j_k} & a_2^{m_1} & \dots & a_2^{m_{n-1}} \\ \vdots & \vdots & \dots & \vdots \\ \sum_{\substack{1 \leq j_1 < \dots < j_k \leq n \\ j_r \neq n}} a_{j_1} \dots a_{j_k} & a_n^{m_1} & \dots & a_n^{m_{n-1}} \end{array} \right| \\
= & \sum_{\substack{m_1 \leq m'_1 < \dots < m'_{n-1}: (\forall i) m'_i - m_i = 0, 1 \\ \#\{i: m'_i = m_i + 1\} = k}} \left| \begin{array}{cccc} 1 & a_1^{m'_1} & \dots & a_1^{m'_{n-1}} \\ \vdots & \vdots & \dots & \vdots \\ 1 & a_n^{m'_1} & \dots & a_n^{m'_{n-1}} \end{array} \right|. \quad (53)
\end{aligned}$$

Proof. We proceed as in the previous lemma and first note that

$$\begin{aligned}
& \left| \begin{array}{cccc} \sum_{2 \leq j_1 < \dots < j_{n-1} \leq n} a_{j_1} \dots a_{j_k} & a_1^{m_1} & \dots & a_1^{m_{n-1}} \\ \sum_{\substack{1 \leq j_1 < \dots < j_k \leq n \\ j_r \neq 2}} a_{j_1} \dots a_{j_k} & a_2^{m_1} & \dots & a_2^{m_{n-1}} \\ \vdots & \vdots & \dots & \vdots \\ \sum_{\substack{1 \leq j_1 < \dots < j_k \leq n \\ j_r \neq n}} a_{j_1} \dots a_{j_k} & a_n^{m_1} & \dots & a_n^{m_{n-1}} \end{array} \right| \\
= & \left| \begin{array}{cccc} a_2 \dots a_n & a_1^{m_1} & \dots & a_1^{m_{n-1}} \\ \prod_{i \neq 2} a_i & a_2^{m_1} & \dots & a_2^{m_{n-1}} \\ \vdots & \vdots & \dots & \vdots \\ a_1 \dots a_{n-1} & a_n^{m_1} & \dots & a_n^{m_{n-1}} \end{array} \right| \\
= & \sum_{j=1}^n (-1)^{j-1} \left| \begin{array}{c} a_1^{m_1+1} \dots a_1^{m_{n-1}+1} \\ \vdots \dots \vdots \\ [a_j^{m_1+1} \dots a_j^{m_{n-1}+1}] \\ \vdots \dots \vdots \\ a_n^{m-1+1} \dots a_n^{m_{n-1}+1} \end{array} \right| = \left| \begin{array}{cccc} 1 & a_1^{m_1+1} & \dots & a_1^{m_{n-1}+1} \\ \vdots & \vdots & \dots & \vdots \\ 1 & a_n^{m_1+1} & \dots & a_n^{m_{n-1}+1} \end{array} \right|.
\end{aligned}$$

Next we continue by induction as before:

$$\begin{aligned}
& \left| \begin{array}{cccc} \sum_{\substack{1 \leq j_1 < \dots < j_k \leq n \\ j_r \neq 1}} a_{j_1} \dots a_{j_k} & a_1^{m_1} & \dots & a_1^{m_{n-1}} \\ \sum_{\substack{1 \leq j_1 < \dots < j_k \leq n \\ j_r \neq 2}} a_{j_1} \dots a_{j_k} & a_2^{m_1} & \dots & a_2^{m_{n-1}} \\ \vdots & \vdots & \dots & \vdots \\ \sum_{\substack{1 \leq j_1 < \dots < j_k \leq n \\ j_r \neq n}} a_{j_1} \dots a_{j_k} & a_n^{m_1} & \dots & a_n^{m_{n-1}} \end{array} \right| \\
&= \sum_{j=1}^n (-1)^{j-1} \sum_{\substack{1 \leq j_1 < \dots < j_k \leq n \\ j_r \neq j}} a_{j_1} \dots a_{j_k} \left| \begin{array}{ccc} a_1^{m_1} & \dots & a_1^{m_{n-1}} \\ a_j^{m_1} & \dots & a_j^{m_{n-1}} \\ \vdots & \dots & \vdots \\ a_n^{m_1} & \dots & a_n^{m_{n-1}} \end{array} \right| \\
&= \sum_{j=1}^n (-1)^{j-1} \prod_{i \neq j} a_i^{m_1} \left| \begin{array}{ccccc} \sum_{\substack{1 \leq j_1 < \dots < j_k \leq n \\ j_r \neq j}} a_{j_1} \dots a_{j_k} & a_1^{m_2-m_1} & \dots & a_1^{m_{n-1}-m_1} \\ \vdots & \dots & \dots & \vdots \\ \sum_{\substack{1 \leq j_1 < \dots < j_k \leq n \\ j_r \neq j}} a_{j_1} \dots a_{j_k} & a_j^{m_2-m_1} & \dots & a_j^{m_{n-1}-m_1} \\ \vdots & \vdots & \dots & \vdots \\ \sum_{\substack{1 \leq j_1 < \dots < j_k \leq n \\ j_r \neq j}} a_{j_1} \dots a_{j_k} & a_n^{m_2-m_1} & \dots & a_n^{m_{n-1}-m_1} \end{array} \right| \\
&= \sum_{j=1}^n (-1)^{j-1} \prod_{i \neq j} a_i^{m_1+1} \left| \begin{array}{ccccc} \sum_{\substack{1 \leq j_1 < \dots < j_{k-1} \leq n \\ j_r \neq 1, j}} a_{j_1} \dots a_{j_{k-1}} & a_1^{m_2-m_1-1} & \dots & a_1^{m_{n-1}-m_1-1} \\ \vdots & \dots & \dots & \vdots \\ \sum_{\substack{1 \leq j_1 < \dots < j_{k-1} \leq n \\ j_r \neq j}} a_{j_1} \dots a_{j_{k-1}} & a_j^{m_2-m_1-1} & \dots & a_j^{m_{n-1}-m_1-1} \\ \vdots & \vdots & \dots & \vdots \\ \sum_{\substack{1 \leq j_1 < \dots < j_{k-1} \leq n \\ j_r \neq j, n}} a_{j_1} \dots a_{j_{k-1}} & a_n^{m_2-m_1-1} & \dots & a_n^{m_{n-1}-m_1-1} \end{array} \right| \\
&+ \sum_{j=1}^n (-1)^{j-1} \prod_{i \neq j} a_i^{m_1} \left| \begin{array}{ccccc} \sum_{\substack{2 \leq j_1 < \dots < j_k \leq n \\ j_r \neq j}} a_{j_1} \dots a_{j_k} & a_1^{m_2-m_1} & \dots & a_1^{m_{n-1}-m_1} \\ \vdots & \dots & \dots & \vdots \\ \sum_{\substack{2 \leq j_1 < \dots < j_k \leq n \\ j_r \neq j}} a_{j_1} \dots a_{j_k} & a_j^{m_2-m_1} & \dots & a_j^{m_{n-1}-m_1} \\ \vdots & \vdots & \dots & \vdots \\ \sum_{\substack{1 \leq j_1 < \dots < j_{k-1} \leq n-1 \\ j_r \neq j}} a_{j_1} \dots a_{j_k} & a_n^{m_2-m_1} & \dots & a_n^{m_{n-1}-m_1} \end{array} \right|.
\end{aligned}$$

Both determinants in the last expression are of the same form as the original, but of smaller size. By the induction hypothesis we therefore have that

$$\begin{aligned}
& \left| \begin{array}{cccc} \sum_{\substack{1 \leq j_1 < \dots < j_k \leq n \\ j_r \neq 1}} a_{j_1} \dots a_{j_k} & a_1^{m_1} & \dots & a_1^{m_{n-1}} \\ \sum_{\substack{1 \leq j_1 < \dots < j_k \leq n \\ j_r \neq 2}} a_{j_1} \dots a_{j_k} & a_2^{m_1} & \dots & a_2^{m_{n-1}} \\ \vdots & \vdots & \dots & \vdots \\ \sum_{\substack{1 \leq j_1 < \dots < j_k \leq n \\ j_r \neq n}} a_{j_1} \dots a_{j_k} & a_n^{m_1} & \dots & a_n^{m_{n-1}} \end{array} \right| \\
= & \sum_{j=1}^n (-1)^{j-1} \prod_{i \neq j} a_i^{m_1+1} \sum_{\substack{m_2-m_1-1 \leq m'_1 < \dots < m'_{n-2}: \\ (\forall i) m'_i - m_{i+1} - m_1 = 0, -1 \\ \#\{i: m'_i = m_{i+1} - m_1\} = k-1}} \left| \begin{array}{cccc} 1 & a_1^{m'_1} & \dots & a_1^{m'_{n-2}} \\ \vdots & \vdots & \dots & \vdots \\ [1 & a_j^{m'_1} & \dots & a_j^{m'_{n-2}}] \\ \vdots & \vdots & \dots & \vdots \\ 1 & a_n^{m'_1} & \dots & a_n^{m'_{n-2}} \end{array} \right| \\
& + \sum_{j=1}^n (-1)^{j-1} \prod_{i \neq j} a_i^{m_1} \sum_{\substack{m_2-m_1 \leq m'_1 < \dots < m'_{n-2}: \\ (\forall i) m'_i - m_{i+1} - m_1 = 0, 1 \\ \#\{i: m'_i = m_{i+1} - m_1 + 1\} = k}} \left| \begin{array}{cccc} 1 & a_1^{m'_1} & \dots & a_1^{m'_{n-2}} \\ \vdots & \vdots & \dots & \vdots \\ [1 & a_j^{m'_1} & \dots & a_j^{m'_{n-2}}] \\ \vdots & \vdots & \dots & \vdots \\ 1 & a_n^{m'_1} & \dots & a_n^{m'_{n-2}} \end{array} \right| \\
= & \sum_{j=1}^n (-1)^{j-1} \sum_{\substack{m_2 \leq m'_1 < \dots < m'_{n-2}: \\ (\forall i) m'_i - m_{i+1} = 0, 1 \\ \#\{i: m'_i = m_{i+1} + 1\} = k-1}} \left| \begin{array}{cccc} a_1^{m_1+1} & a_1^{m'_1} & \dots & a_1^{m'_{n-2}} \\ \vdots & \vdots & \dots & \vdots \\ [a_1^{m_1+1} & a_j^{m'_1} & \dots & a_j^{m'_{n-2}}] \\ \vdots & \vdots & \dots & \vdots \\ a_n^{m_1+1} & a_n^{m'_1} & \dots & a_n^{m'_{n-2}} \end{array} \right| \\
& + \sum_{j=1}^n (-1)^{j-1} \sum_{\substack{m_2-1 \leq m'_1 < \dots < m'_{n-2}: \\ (\forall i) m'_i - m_{i+1} - m_1 = 0, 1 \\ \#\{i: m'_i = m_{i+1} + 1\} = k}} \left| \begin{array}{cccc} a_1^{m_1} & a_1^{m'_1} & \dots & a_1^{m'_{n-2}} \\ \vdots & \vdots & \dots & \vdots \\ [a_j^{m_1} & a_j^{m'_1} & \dots & a_j^{m'_{n-2}}] \\ \vdots & \vdots & \dots & \vdots \\ a_n^{m_1} & a_n^{m'_1} & \dots & a_n^{m'_{n-2}} \end{array} \right|.
\end{aligned}$$

Summing over j we obtain

$$\begin{aligned}
& \left| \begin{array}{cccc} \sum_{\substack{1 \leq j_1 < \dots < j_k \leq n \\ j_r \neq 1}} a_{j_1} \dots a_{j_k} & a_1^{m_1} & \dots & a_1^{m_{n-1}} \\ \sum_{\substack{1 \leq j_1 < \dots < j_k \leq n \\ j_r \neq 2}} a_{j_1} \dots a_{j_k} & a_2^{m_1} & \dots & a_2^{m_{n-1}} \\ \vdots & \vdots & \dots & \vdots \\ \sum_{\substack{1 \leq j_1 < \dots < j_k \leq n \\ j_r \neq n}} a_{j_1} \dots a_{j_k} & a_n^{m_1} & \dots & a_n^{m_{n-1}} \end{array} \right| \\
&= \sum_{\substack{m_2 \leq m'_1 < \dots < m'_{n-2}: \\ (\forall i) m'_i - m_{i+1} = 0, 1 \\ \#\{i: m'_i = m_{i+1} + 1\} = k-1}} \left| \begin{array}{ccccc} 1 & a_1^{m_1+1} & a_1^{m'_1} & \dots & a_1^{m'_{n-2}} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & a_n^{m_1+1} & a_n^{m'_1} & \dots & a_n^{m'_{n-2}} \end{array} \right| \\
&+ \sum_{\substack{m_2 \leq m'_1 < \dots < m'_{n-2}: \\ (\forall i) m'_i - m_{i+1} = 0, 1 \\ \#\{i: m'_i = m_{i+1} + 1\} = k}} \left| \begin{array}{ccccc} 1 & a_1^{m_1} & a_1^{m'_1} & \dots & a_1^{m'_{n-2}} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & a_n^{m_1} & a_n^{m'_1} & \dots & a_n^{m'_{n-2}} \end{array} \right| \\
&= \sum_{\substack{m_1 \leq m'_1 < \dots < m'_{n-1}: \\ (\forall i) m'_i - m_i = 0, 1 \\ \#\{i: m'_i = m_i + 1\} = k}} \left| \begin{array}{ccccc} 1 & a_1^{m'_1} & \dots & a_1^{m'_{n-1}} \\ \vdots & \vdots & \dots & \vdots \\ 1 & a_n^{m'_1} & \dots & a_n^{m'_{n-1}} \end{array} \right|.
\end{aligned}$$

Iterating once more we get ■

Corollary 2.2 *Let \mathcal{R} be a commutative ring and $a_1, \dots, a_n \in \mathcal{R}$. Let $n \in \mathbb{N}$ and $m_1, \dots, m_{n-1} \in \mathbb{N}$ such that $1 \leq m_1 < \dots < m_{n-1}$. Then, for any $k \in \mathbb{N}$ with $1 \leq k \leq n$,*

$$\begin{aligned}
& \sum_{1 \leq j_1 < \dots < j_k \leq n} a_{j_1} \dots a_{j_k} \left| \begin{array}{ccccc} 1 & a_1^{m_1} & \dots & a_1^{m_{n-1}} \\ 1 & a_2^{m_1} & \dots & a_2^{m_{n-1}} \\ \vdots & \vdots & \dots & \vdots \\ 1 & a_n^{m_1} & \dots & a_n^{m_{n-1}} \end{array} \right| \\
&= \sum_{\substack{0 \leq m'_1 < \dots < m'_n: (\forall i) m'_i - m_{i-1} = 0, 1 \\ \#\{i: m'_i = m_{i-1} + 1\} = k}} \left| \begin{array}{ccccc} a_1^{m'_1} & \dots & a_1^{m'_n} \\ \vdots & \dots & \vdots \\ a_n^{m'_1} & \dots & a_n^{m'_n} \end{array} \right|,
\end{aligned} \tag{54}$$

where $m_0 = 0$.