

On the Wasserstein distance and Dobrushin's uniqueness theorem.

May 26, 2023

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Abstract

In this paper, we revisit Dobrushin's uniqueness theorem for Gibbs measures of lattice systems of interacting particles at thermal equilibrium. In a nutshell, Dobrushin's uniqueness theorem provides a practical way to derive sufficient conditions on the inverse-temperature and model parameters assuring uniqueness of Gibbs measures by reducing the uniqueness problem to a suitable estimate of the Wasserstein distance between pairs of 1-point Gibbs measures with different boundary conditions. After proving a general result of completeness for the Wasserstein distance, we reformulate Dobrushin's uniqueness theorem in a convenient form for lattice systems of interacting particles described by Hamiltonians that are not necessarily translation-invariant and with a general complete metric space as single-spin space. Subsequently, we give a series of applications. After proving a uniqueness result at high-temperature that we extend to the Ising and Potts models, we focus on classical lattice systems for which the local Gibbs measures are convex perturbations of Gaussian measures. We show that uniqueness holds at all temperatures by constructing suitable couplings with the 1-point Gibbs measures as marginals. The decay of correlation functions is also discussed.

MSC-2010: 82B05, 82B10, 82B20, 82B26 (Primary); 28C20, 46G10, 60B05, 60B10 (Secondary).

Keywords: (limit) Gibbs distributions, Gibbs measures, DLR approach, Dobrushin's uniqueness theorem, Gaussian measures, Decay of correlations.

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1 Introduction.

intro

A core problem of classical and quantum equilibrium statistical mechanics is the description of equilibrium thermodynamic properties of large systems of interacting particles, see, e.g., [50]. Literature on mathematical characterisations of the equilibrium thermodynamics of interacting particle systems is extensive: for an account of rigorous methods, see, e.g., [36, 41, 28, 11, 47, 44].

A standard way to describe the equilibrium thermodynamics of a large system of interacting particles consists in constructing the equilibrium Gibbs states of the associated infinite system at a given 'temperature' $\beta^{-1} > 0$ and given values of the system parameters. For systems undergoing structural phase transitions, it is expected that the set of equilibrium Gibbs states consists of more than one element corresponding to the different phases. Below, we give a brief account of some rigorous constructions of equilibrium Gibbs states in classical and quantum lattice systems, with an emphasis on lattice systems with unbounded single-spin spaces (also called state spaces).

Equilibrium Gibbs states of classical lattice systems of interacting particles may be represented by *Gibbs measures*, also called *limit Gibbs distributions* or *DLR* (Dobrushin-Lanford-Ruelle) *measures*. A Gibbs measure is the distribution of a random field on the infinite lattice admitting a prescribed family of conditional distributions, see [13, 34, 35, 14, 15, 16, 17, 18, 32] for pioneering works. Local Gibbs measures may describe local equilibrium Gibbs states of systems of interacting particles in finite domains (the particle interactions are not limited to the interior of these domains) at thermal equilibrium with their exteriors where the configurations of particles are held fixed. The latter play the role of boundary conditions and, as such, determine conditions for the distributions. Gibbs measures are then defined by means of the family of local Gibbs measures as solutions of the equilibrium DLR equation. This approach, called *DLR approach*, is standard, see, e.g., [23] and references therein. Note that this approach allows one to define Gibbs states of infinite lattice systems without resorting to any limiting procedures.

Local equilibrium Gibbs states of finite-volume quantum systems are traditionally defined as positive normalised linear functionals on the C^* -algebra of bounded operators on a Hilbert space, satisfying the KMS (Kubo-Martin-Schwinger) boundary conditions relative to the time-evolution automorphisms describing the dynamics of the systems, see, e.g., [27] and [11, Sec. 5.3]. Accordingly, equilibrium Gibbs states are constructed as limiting states on the C^* -algebra of quasi-local observables satisfying the KMS conditions. They are the so-called *KMS states*. Within this limiting procedure, the formulation of the KMS conditions relative to the time-evolution automorphisms is problematic when considering some quantum lattice systems described by unbounded operators, see, e.g., [39, 38]. To construct equilibrium Gibbs states of quantum lattice systems

of interacting particles, an alternative to the above algebraic approach is the so-called *Euclidean approach*, see [26, 1, 5]. The Euclidean approach relies on a one-to-one correspondence between local Gibbs states, as functionals on C^* -algebras of observables, and local Gibbs measures, as Feynman-Kac measures on β -periodic path spaces. To connect both, the Matsubara functions play a key role as they uniquely determine the local Gibbs states. For further details, see, e.g., [6, Sec. 6] and [30, Sec. 2.5]. As such, equilibrium Gibbs states of quantum lattice systems may be represented, analogously to classical lattice systems, by Gibbs measures defined through the DLR approach but with local Gibbs measures living on infinite-dimensional path spaces. They are often called *Euclidean Gibbs measures*, see, e.g., [5].

In the study of Gibbs measures in lattice systems with unbounded single-spin spaces, two main mathematical problems arise: existence and uniqueness. For the case of compact single-spin spaces the existence problem is simpler, see e.g., [14, 23, 47] and [22, Chap. 6]. At this stage, it should be noted that, as shown in [25], when the set of Gibbs measures consists of several elements, some may have no physical relevance. As suggested by Euclidean quantum field theories, see, e.g., [46, 25], the Gibbs measures of interest are those for which the sequence of their moments satisfies some *a priori* growth limitations at infinity. They are the so-called *tempered Gibbs measures*. Uniqueness of Gibbs measures characterizes the absence of (first-order) phase transitions.

The existence problem may be solved by constructing Gibbs measures as thermodynamic limits. This may be achieved by proving that the family of local Gibbs measures, indexed by an increasing sequence of bounded regions filling the whole lattice, has at least one limit point in the weak topology, and that this limit point is a Gibbs measure. Sufficient conditions were derived by Dobrushin [14, 15, 17] and [18, Thm. 1]. See also [48, Thm. 1.3]. Dobrushin's existence criteria have been applied to some classical lattice models of Euclidean lattice field theories with single-spin space \mathbb{R} in [12, 7]. Therein, some restrictions on the boundary conditions are necessary. One of the key methods is Ruelle's technique of superstability estimates in [42, 43, 33] which requires the interactions to be translation invariant, superstable and with pair-potentials growing at most quadratically. Applied to some models of Euclidean lattice field theories with single-spin space \mathbb{R} , it is proved in [12] that the family of local Gibbs measures associated with a wide class of boundary conditions has at least one accumulation point in the set of '*superstable*' Gibbs measures, a subset of the set of tempered Gibbs measures, see [12, Thm. 1.2]. This result was extended to some quantum anharmonic lattice systems with superstable interactions in [39, Thm 2.6] by extending Ruelle's technique to quantum statistical mechanics, see [37, 38]. See also [30, Thm. 3.1] covering a wide class of quantum anharmonic lattice systems. In general, for quantum lattice systems where the single-spin spaces are infinite-dimensional, verifying Dobrushin's existence criteria turns out to be challenging, see, e.g., [48, Sec. I.5]. A brief review of methods can be found in [6, Sec. 2].

Turning to the problem of uniqueness of Gibbs measures, sufficient conditions were derived by Dobrushin in [14, 16] (case of compact single-spin spaces), [18, Thm. 4] and [19, Thm. 1] (an extension). Dobrushin's uniqueness theorem [18, Thm. 4] reduces the uniqueness problem to a suitable estimate of the Wasserstein distance between pairs of 1-point (i.e., lattice site) Gibbs measures subject to different boundary conditions. In particular, uniqueness holds provided that the Dobrushin matrix coefficients satisfy the condition of *weak dependence* [18, Eq. (5.2)]. In practice, this allows one to derive sufficient conditions on β and model parameters assuring uniqueness. Dobrushin's criterion has been applied to some classical lattice models of Euclidean lattice field theories with single-spin space \mathbb{R} in [12, Sec. 2]. The key ingredient is an expression of the Wasserstein distance for probability measures on \mathbb{R} in terms of their distribution functions. The same criterion has also been applied to some quantum anharmonic lattice models of quantum crystals in [2, 4] (see also [30, Thm 3.4]). Therein, the key ingredient is the representation of the Wasserstein distance by means of the Kantorovich-Rubinstein duality theorem, see, e.g., [21]. Finally, we mention that cluster expansion techniques are used in [40] to prove uniqueness of Gibbs measures in the high temperature regime (in 1-dimensional lattices, uniqueness holds at all temperatures) for a class of quantum anharmonic lattice systems with superstable interactions. In [3] a quantum crystal with double-well potential is shown to have a phase transition using a Peierls-type argument.

As such, Dobrushin's uniqueness theorem [18, Thm. 4] provides a way to determine sufficient conditions on β and model parameters assuring uniqueness of Gibbs measures by deriving a suitable estimate of the Wasserstein distance between pairs of 1-point Gibbs measures with different boundary conditions. However, this criterion remains difficult to verify in general.

In this paper, we revisit Dobrushin's uniqueness theorem for Gibbs measures of lattice systems of interacting particles at thermal equilibrium in the case where the single-spin space is a general complete metric space. This covers both classical and quantum lattice systems. The main ingredient in the proof of existence of limit Gibbs distribution is the completeness of the Wasserstein space of first order. We show that this is guaranteed whenever the underlying single-spin space is a complete metric space. Our proof does not require separability, and relies on the construction of a well-chosen Prokhorov compact to show the uniform tightness of the probability measures. Further, the method we use to prove the equivalence of weak convergence and convergence in Wasserstein metric would allow the result to be extended to uniform spaces, replacing the Wasserstein metric by a Wasserstein uniformity. In addition to revisiting the proof of Dobrushin's uniqueness theorem, another motivation of this paper is to provide some alternative techniques to prove uniqueness of the limit Gibbs distribution for some classical lattice systems with possibly infinite-range pair potentials. First, we prove a general result of uniqueness at high temperatures with application to the classical Heisenberg model. An extension of the uniqueness result to the Ising and Potts models is given. Second, we revisit the Gaussian free field model with n -dimensional spins. When the local Gibbs measures are convex perturbations of Gaussian measures, we derive a suitable estimate of the Wasserstein distance between pairs of 1-point Gibbs measures by constructing appropriate couplings with the 1-point Gibbs measures as marginals.

Our paper is organised as follows. In Section 2, we recall some properties of the Wasserstein distance and give the completeness result in Theorem 2.1. After recalling some definitions of local and limit Gibbs distributions, Theorem 3.5 in Section 3 is our reformulation of Dobrushin's uniqueness theorem for classical or quantum lattice systems of interacting particles when the single-spin space is a general metric space. Two corollaries related to the decay of correlations in the case of nearest-neighbour interactions are also given. The proof of Theorem 3.5 and the proof of the corollaries are given in Section 4. In Section 5, we focus on some applications of Theorem 3.5 at high-temperature. The main result is Proposition 5.1 with application to the classical Heisenberg model. The Ising and Potts models are treated in Section 5.2. In Section 6, we focus on some applications of Theorem 3.5 to classical lattice systems. The Gaussian free field model with n -dimensional spins is revisited in Section 6.1, see Proposition 6.1. The perturbation by a convex self-interaction potential is treated in Section 6.2, see Propositions 6.5 and 6.9. Section 7 contains the proof of Theorem 2.1. The key result for the proof of Theorem 2.1 (ii) is Proposition 7.5. Section 8 is the Appendix. The quantum harmonic crystal model is revisited in Section 8.2, see Proposition 8.4. The decay of Gaussian correlations in classical and quantum lattice systems is discussed in Section 8.3.

2 Wasserstein distance and completeness.

Wapro

Let (\mathcal{X}, ρ) be a metric space. Denote by $\mathcal{B}(\mathcal{X})$ the Borel σ -algebra of subsets of \mathcal{X} . Let $\mathcal{M}^+(\mathcal{X})$ denote the set of all finite non-negative Radon measures on \mathcal{X} and let

$$\mathcal{P}(\mathcal{X}) := \{\mu \in \mathcal{M}^+(\mathcal{X}) : \mu(\mathcal{X}) = 1\},$$

be the set of all Radon probability measures on \mathcal{X} . For any $\mu, \nu \in \mathcal{P}(\mathcal{X})$, define the set of Radon couplings of μ and ν as

$$\Xi_{\mathcal{X}}(\mu, \nu) := \{\sigma \in \mathcal{P}(\mathcal{X} \times \mathcal{X}) : \sigma(A \times \mathcal{X}) = \mu(A), \sigma(\mathcal{X} \times A) = \nu(A) \text{ for all } A \in \mathcal{B}(\mathcal{X})\}.$$

Let $\rho_W : \mathcal{P}(\mathcal{X}) \times \mathcal{P}(\mathcal{X}) \rightarrow \mathbb{R}_+ \cup \{\infty\}$ be the Wasserstein distance defined as (see, e.g., [49])

$$\rho_W(\mu, \nu) := \inf_{\sigma \in \Xi_{\mathcal{X}}(\mu, \nu)} \int_{\mathcal{X} \times \mathcal{X}} \rho(x, y) \sigma(dx, dy). \quad (2.1)$$

rhoW

Introduce the set

$$\mathcal{P}_1(\mathcal{X}) := \left\{ \mu \in \mathcal{P}(\mathcal{X}) : \int_{\mathcal{X}} \rho(x, x_0) \mu(dx) < \infty, \text{ for some (and hence all) } x_0 \in \mathcal{X} \right\}, \quad (2.2) \quad \boxed{\text{P1def}}$$

which is independent of x_0 . Here is the main result of this section

complete

Theorem 2.1 (i). *The Wasserstein distance ρ_W is a metric on $\mathcal{P}_1(\mathcal{X})$.*
(ii). *If (\mathcal{X}, ρ) is a complete metric space, then so is $(\mathcal{P}_1(\mathcal{X}), \rho_W)$.*

The proof of Theorem 2.1 is postponed to Sec. 7 for reader's convenience. We point out that the proof of Theorem 2.1 (ii) relies on Prokhorov's theorem, see, e.g., [10, Sec. IX.5.5]. For a proof with the separability assumption, see, e.g., [9]. We note that Theorem 2.1 is used in the proof of existence in Dobrushin's theorem in Sec. 4.2. Hereafter, \mathcal{X} as in Theorem 2.1 will play the role of state space (i.e., single-spin space).

3 Dobrushin's uniqueness theorem revisited.

Dobs

3.1 Gibbs distributions.

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In this section, we recall the definition of local Gibbs distributions and limit Gibbs distributions. Our definitions below are taken from [48]. See also, e.g., [34, 14, 15, 16, 17, 32, 18, 33, 23].

Notations. For dimension $d \in \mathbb{N}$, let $\mathcal{S} := \{\Gamma \subset \mathbb{Z}^d : 0 < |\Gamma| < \infty\}$ be the (countably infinite) set of all non-empty finite subsets of \mathbb{Z}^d . Here and hereafter, $|\Lambda|$ denotes the cardinality of $\Lambda \in \mathcal{S}$, $\Lambda^c := \mathbb{Z}^d \setminus \Lambda$ its complement and $\partial\Lambda := \{j' \in \Lambda^c : \exists j \in \Lambda, |j - j'| = 1\}$ its boundary. In the following, the state space \mathcal{X} is assumed to be a metric space (note that all the definitions below hold for \mathcal{X} a topological space). The space of finite configurations in $\Lambda \in \mathcal{S}$ and the space of all possible configurations are respectively \mathcal{X}^Λ and $\mathcal{X}^{\mathbb{Z}^d}$ endowed with the product topology and equipped with the Borel σ -algebra $\mathcal{B}(\mathcal{X}^\Lambda)$ and $\mathcal{B}(\mathcal{X}^{\mathbb{Z}^d})$ respectively. Note that the latter coincides with the σ -algebra generated by the cylinder sets. Configurations in \mathcal{X}^Λ , $\Lambda \in \mathcal{S}$ will be often denoted by $\underline{\xi}_\Lambda$ and a configuration $\underline{\xi}_\Lambda$ will be often decomposed as the concatenation $\underline{\xi}_\Lambda = \underline{\xi}_\Delta \underline{\xi}_{\Lambda \setminus \Delta}$ for a given $\Delta \subset \Lambda$. Let $\mathcal{P}(\mathcal{X}^\Lambda)$, $\Lambda \in \mathcal{S}$ or $\Lambda = \mathbb{Z}^d$ denote the set of all Radon probability measures on \mathcal{X}^Λ . Let $\mathcal{P}_1(\mathcal{X}^\Lambda)$, $\Lambda \in \mathcal{S}$ denote the subset similar to (2.2) but with a metric $\rho^{(\Lambda)}$ on \mathcal{X}^Λ . Let $\{\Pi_\Lambda\}_{\Lambda \in \mathcal{S}}$ denote the family of projection maps $\Pi_\Lambda : \mathcal{X}^{\mathbb{Z}^d} \rightarrow \mathcal{X}^\Lambda$.

Given $\Lambda \in \mathcal{S}$, the energy of a configuration $\underline{\xi}_\Lambda \in \mathcal{X}^\Lambda$ is defined by

$$H(\underline{\xi}_\Lambda) := \sum_{X \in \mathcal{S}: X \subset \Lambda} \mathcal{V}_X(\underline{\xi}_X), \quad (3.1) \quad \boxed{\text{HLam}}$$

where for every $X \in \mathcal{S}$, the function $\mathcal{V}_X : \mathcal{X}^X \rightarrow \mathbb{R}$ stands for the joint interaction energy of the $\underline{\xi}$'s inside X . The family $\{\mathcal{V}_X\}_{X \in \mathcal{S}}$ is commonly called a *potential*, and typically, the \mathcal{V}_X 's are assumed to be $\mathcal{B}(\mathcal{X}^X)$ -measurable and such that the series in (3.1), and in (3.2) below, exist.

The interaction energy between a configuration $\underline{\xi}_\Lambda \in \mathcal{X}^\Lambda$ and $\underline{\eta}_{\Lambda^c} \in \mathcal{X}^{\Lambda^c}$ is defined as

$$H(\underline{\xi}_\Lambda, \underline{\eta}_{\Lambda^c}) := \sum_{\substack{X \in \mathcal{S}: X \cap \Lambda \neq \emptyset, \\ X \cap \Lambda^c \neq \emptyset}} \mathcal{V}_X(\underline{\xi}_{X \cap \Lambda} \underline{\eta}_{X \cap \Lambda^c}). \quad (3.2) \quad \boxed{\text{interact}}$$

In the following, given $\Lambda \in \mathcal{S}$, configurations in \mathcal{X}^{Λ^c} will play the role of boundary conditions.

The total energy of a configuration $\underline{\xi}_\Lambda$ under the boundary condition $\underline{\eta}_{\Lambda^c}$ is then defined as

$$H_\Lambda(\underline{\xi}_\Lambda | \underline{\eta}_{\Lambda^c}) := H(\underline{\xi}_\Lambda) + H(\underline{\xi}_\Lambda, \underline{\eta}_{\Lambda^c}). \quad (3.3) \quad \boxed{\text{totHam}}$$

Definition 3.1 *The local Gibbs distribution at inverse temperature $\beta > 0$ for the domain $\Lambda \in \mathcal{S}$ under the boundary condition $\underline{\eta}_{\Lambda^c} \in \mathcal{X}^{\Lambda^c}$ is a probability measure on $(\mathcal{X}^\Lambda, \mathcal{B}(\mathcal{X}^\Lambda))$ defined as*

$$\mu_\Lambda^\beta(A | \underline{\eta}_{\Lambda^c}) := \frac{1}{Z_\Lambda^\beta(\underline{\eta}_{\Lambda^c})} \int_A \exp\left(-\beta H_\Lambda(\underline{\xi}_\Lambda | \underline{\eta}_{\Lambda^c})\right) \prod_{j \in \Lambda} \mu_0(d\xi_j), \quad A \in \mathcal{B}(\mathcal{X}^\Lambda), \quad (3.4) \quad \boxed{\text{condexp}}$$

where $H_\Lambda(\cdot|\underline{\eta}_{\Lambda^c})$ is defined in (3.3), $Z_\Lambda^\beta(\underline{\eta}_{\Lambda^c})$ is a normalisation constant (called partition function)

$$Z_\Lambda^\beta(\underline{\eta}_{\Lambda^c}) := \int_{\mathcal{X}^\Lambda} \exp\left(-\beta H_\Lambda(\underline{\xi}_\Lambda|\underline{\eta}_{\Lambda^c})\right) \prod_{j \in \Lambda} \mu_0(d\xi_j), \quad (3.5)$$

partfunc

and the (single-spin) measure μ_0 is a given a priori measure on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$, not necessarily bounded.

We now turn to the definition of limit Gibbs distributions. We refer the reader to the disintegration theorem in Sec. 8.1 for the existence of conditional probability measures.

defG

Definition 3.2 A limit Gibbs distribution at inverse temperature $\beta > 0$ corresponding to the formal Hamiltonian $H : \mathcal{X}^{\mathbb{Z}^d} \rightarrow \mathbb{R}$ given by a potential as follows

$$H(\underline{\xi}) := \sum_{X \subset \mathbb{Z}^d} \mathcal{V}_X(\underline{\xi}_X),$$

is a probability measure μ^β on $(\mathcal{X}^{\mathbb{Z}^d}, \mathcal{B}(\mathcal{X}^{\mathbb{Z}^d}))$ such that, for any domain $\Lambda \in \mathcal{S}$,

- (i). For μ^β -a.e. $\underline{\xi} = (\underline{\xi}_\Lambda, \underline{\xi}_{\Lambda^c})$, $H_\Lambda(\underline{\xi}_\Lambda|\underline{\xi}_{\Lambda^c})$ in (3.3) and $Z_\Lambda^\beta(\underline{\xi}_{\Lambda^c})$ in (3.5) are finite;
- (ii). The conditional probability measure induced by μ^β on $(\mathcal{X}^\Lambda, \mathcal{B}(\mathcal{X}^\Lambda))$ under the boundary condition $\underline{\eta}_{\Lambda^c} \in \mathcal{X}^{\Lambda^c}$ coincides $\mu_{\Lambda^c}^\beta$ -a.e. with the local Gibbs distribution in (3.4).

frek1

Remark 3.3 The conditional probability measure in (ii) above is relative to the projection map Π_{Λ^c} , see Sec. 8.1. Denoting it by $\mu^\beta(\cdot|\underline{\eta}_{\Lambda^c})$ for a fixed configuration $\underline{\eta}_{\Lambda^c} \in \mathcal{X}^{\Lambda^c}$, (ii) reads

$$\mu^\beta(\cdot|\underline{\eta}_{\Lambda^c}) \circ \Pi_\Lambda^{-1} = \mu_\Lambda^\beta(\cdot|\underline{\eta}_{\Lambda^c}).$$

Note that the measure $\mu^\beta(\cdot|\underline{\eta}_{\Lambda^c})$ is concentrated on the set $\{\underline{\xi} \in \mathcal{X}^{\mathbb{Z}^d} : \Pi_{\Lambda^c} \underline{\xi} = \underline{\eta}_{\Lambda^c}\}$.

equiDLR

Remark 3.4 Connection with Gibbsian specification and equilibrium DLR equation. From the local Gibbs distribution in (3.4), we can associate on $(\mathcal{X}^{\mathbb{Z}^d}, \mathcal{B}(\mathcal{X}^{\mathbb{Z}^d}))$ the probability measure

$$\pi_\Lambda^\beta(A|\underline{\eta}) := \int_{\mathcal{X}^\Lambda} \mathbb{I}_A(\underline{\xi}_\Lambda \Pi_{\Lambda^c} \underline{\eta}) \mu_\Lambda^\beta(d\underline{\xi}_\Lambda | \Pi_{\Lambda^c} \underline{\eta}), \quad A \in \mathcal{B}(\mathcal{X}^{\mathbb{Z}^d}), \underline{\eta} \in \mathcal{X}^{\mathbb{Z}^d}.$$

$\{\pi_\Lambda^\beta\}_{\Lambda \in \mathcal{S}}$ forms a family of proper probability kernels from $\mathcal{B}(\mathcal{X}^{\Lambda^c})$ to $\mathcal{B}(\mathcal{X}^{\mathbb{Z}^d})$, see, e.g., [23]. By virtue of the additive structure of the Hamiltonian, the family satisfies the consistency relation

$$\pi_{\Lambda'}^\beta, \pi_\Lambda^\beta(A|\underline{\eta}) := \int_{\mathcal{X}^{\mathbb{Z}^d}} \pi_\Lambda^\beta(A|\underline{\xi}) \pi_{\Lambda'}^\beta(d\underline{\xi}|\underline{\eta}) = \pi_{\Lambda'}^\beta(A|\underline{\eta}), \quad \Lambda \subset \Lambda', \quad \Lambda, \Lambda' \in \mathcal{S}.$$

Due to this feature, $\{\pi_\Lambda^\beta\}_{\Lambda \in \mathcal{S}}$ is called a Gibbsian specification in Georgii's terminology. In the DLR formalism, a limit Gibbs distribution at $\beta > 0$ is defined as a measure $\mu^\beta \in \mathcal{P}(\mathcal{X}^{\mathbb{Z}^d})$ satisfying

$$\mu^\beta \pi_\Lambda^\beta(A) := \int_{\mathcal{X}^{\mathbb{Z}^d}} \pi_\Lambda^\beta(A|\underline{\xi}) \mu^\beta(d\underline{\xi}) = \mu^\beta(A), \quad (3.6)$$

DLReq

for all $\Lambda \in \mathcal{S}$ and all $A \in \mathcal{B}(\mathcal{X}^{\mathbb{Z}^d})$. (3.6) is the equilibrium DLR equation. μ^β is said to be specified by $\{\pi_\Lambda^\beta\}_{\Lambda \in \mathcal{S}}$. This definition is equivalent to Definition 3.2 (ii), see, e.g., [23, Rem. 1.24].

3.2 Dobrushin's uniqueness theorem.

Secthm

Theorem 3.5 below is our reformulation of Dobrushin's uniqueness theorem [18, Thm. 4] for classical lattice systems of interacting particles at thermal equilibrium.

Dobrushin

Theorem 3.5 Let (\mathcal{X}, ρ) be a complete metric space and μ_0 a given a priori measure on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$. Let $H : \mathcal{X}^{\mathbb{Z}^d} \rightarrow \mathbb{R}$ be a formal Hamiltonian of the form

$$H(\underline{\xi}) = \sum_{X \subset \mathbb{Z}^d} \mathcal{V}_X(\underline{\xi}_X). \quad (3.7) \quad \text{Hamm}$$

Let $\beta > 0$ be fixed. Assume the following,

(C1). For all $X \subset \mathbb{Z}^d$, the functions $\mathcal{V}_X : \mathcal{X}^X \rightarrow \mathbb{R}$ are continuous;

(C2). Given $j \in \mathbb{Z}^d$ and $\underline{\eta} \in \mathcal{X}^{\mathbb{Z}^d \setminus \{j\}}$, the local partition functions defined as

$$Z_j^\beta(\underline{\eta}) := \int_{\mathcal{X}} \exp \left(-\beta \sum_{X \subset \mathbb{Z}^d: j \in X} \mathcal{V}_X(\xi_j \underline{\eta}_{X \setminus \{j\}}) \right) \mu_0(d\xi_j), \quad (3.8) \quad \text{locpartf}$$

are finite and bounded uniformly in $j \in \mathbb{Z}^d$;

(C3). There exists $\xi^* \in \mathcal{X}$ such that

$$c_0 := \sup_{j \in \mathbb{Z}^d} \int_{\mathcal{X}} \rho(\xi, \xi^*) \mu_j^\beta(d\xi | \underline{\xi}_{\mathbb{Z}^d \setminus \{j\}}^*) < \infty, \quad (3.9) \quad \text{unifi}$$

where $\xi_k^* = \xi^*$ for all $k \in \mathbb{Z}^d$.

Given $j \in \mathbb{Z}^d$ and $\underline{\eta} \in \mathcal{X}^{\mathbb{Z}^d \setminus \{j\}}$, the 1-point Gibbs distribution reads

$$\mu_j^\beta(A | \underline{\eta}) := \frac{1}{Z_j^\beta(\underline{\eta})} \int_A \exp \left(-\beta \sum_{X \subset \mathbb{Z}^d: j \in X} \mathcal{V}_X(\xi_j \underline{\eta}_{X \setminus \{j\}}) \right) \mu_0(d\xi_j), \quad A \in \mathcal{B}(\mathcal{X}). \quad (3.10) \quad \text{condma}$$

Then there exists a unique limit Gibbs distribution $\mu^\beta \in \mathcal{P}(\mathcal{X}^{\mathbb{Z}^d})$, associated with the Hamiltonian (3.7), with marginal distributions satisfying

$$\sup_{j \in \mathbb{Z}^d} \int_{\mathcal{X}} \rho(\xi, \xi^*) \mu_j^\beta(d\xi) < \infty,$$

provided that, for all $j \in \mathbb{Z}^d$ and for all $(\underline{\eta}, \underline{\eta}') \in \mathcal{X}^{\mathbb{Z}^d \setminus \{j\}} \times \mathcal{X}^{\mathbb{Z}^d \setminus \{j\}}$,

$$\rho_W \left(\mu_j^\beta(\cdot | \underline{\eta}), \mu_j^\beta(\cdot | \underline{\eta}') \right) \leq \sum_{\substack{l \in \mathbb{Z}^d \\ l \neq j}} r(l, j) \rho(\eta_l, \eta'_l), \quad (3.11) \quad \text{Dcond}$$

holds for some constants $r(l, j) \geq 0$ satisfying, for all $j \in \mathbb{Z}^d$,

$$\sum_{\substack{l \in \mathbb{Z}^d \\ l \neq j}} r(l, j) \leq \lambda < 1. \quad (3.12) \quad \text{cond}$$

Remark 3.6 For lattice systems described by formal Hamiltonians of the form (3.7), uniqueness of the limit Gibbs distribution usually means absence of phase transition. If uniqueness holds for all $\beta > 0$, the system is said to be stable.

extqutm

Remark 3.7 In Theorem 3.5, (\mathcal{X}, ρ) is a general complete metric space. The latter could be the infinite-dimensional space of all continuous and periodic functions on $[0, \beta]$ equipped with the supremum norm, so that Theorem 3.5 also applies to quantum lattice systems. See Sec. 8.2.

DLReqaa

Remark 3.8 Under the conditions of Theorem 3.5, the equilibrium DLR equation (see Remark 3.4) is satisfied in the following sense. For any $\beta > 0$, for all $\Lambda_0, \Lambda \in \mathcal{S}$ such that $\Lambda_0 \subset \Lambda$, for all $A \in \mathcal{B}(\mathcal{X}^{\Lambda_0})$ and all $B \in \mathcal{B}(\mathcal{X}^{\Lambda \setminus \Lambda_0})$ bounded,

$$\int_B \mu_{\Lambda_0}^\beta(A | \underline{\xi}_{\Lambda \setminus \Lambda_0}) \bar{\mu}_{\Lambda \setminus \Lambda_0}^\beta(d\xi_{\Lambda \setminus \Lambda_0}) = \bar{\mu}_\Lambda^\beta(A \times B), \quad (3.13) \quad \text{DLReq}$$

where $\bar{\mu}^\beta$ denotes the limit Gibbs distribution from Theorem 3.5 (to distinguish in the notation the marginals of the limit Gibbs distribution from the local Gibbs distributions).

nearestn

Remark 3.9 In the case of nearest-neighbour interactions only, the conditions (3.11)-(3.12) can be simply replaced with

$$\rho_W \left(\mu_j^\beta(\cdot | \underline{\eta}), \mu_j^\beta(\cdot | \underline{\eta}') \right) \leq \frac{\lambda}{2d} \sum_{l \in N_1(j)} \rho(\eta_l, \eta'_l), \quad (3.14)$$

nniupbd

where $0 < \lambda < 1$ and $N_1(j) := \{j' \in \mathbb{Z}^d : |j' - j| = 1\}$ is the set of nearest-neighbours of $j \in \mathbb{Z}^d$.

Below, we give a remark on the assumptions of Theorem 3.5. By a bounded boundary condition, we mean any configuration $\underline{\vartheta} \in \mathcal{X}^{\mathbb{Z}^d}$ such that $\sup_{j \in \mathbb{Z}^d} \rho(\xi^*, \vartheta_j) < \infty$. Given $\beta > 0$ and $\Delta, \Gamma \in \mathcal{S}$ such that $\Delta \subset \Gamma$ and $\underline{\vartheta} \in \mathcal{X}^{\mathbb{Z}^d}$, introduce on $(\mathcal{X}^\Delta, \mathcal{B}(\mathcal{X}^\Delta))$ the probability measure

$$\mu_\Delta^\Gamma(\cdot | \underline{\vartheta}_{\Gamma^c}) := \mu_\Gamma^\beta(\cdot | \underline{\vartheta}_{\Gamma^c}) \circ (\Pi_\Delta^\Gamma)^{-1}, \quad (3.15)$$

tildemud

where $\mu_\Gamma^\beta(\cdot | \underline{\vartheta}_{\Gamma^c})$ is the local Gibbs distribution for the domain Γ and $\Pi_\Delta^\Gamma : \mathcal{X}^\Gamma \rightarrow \mathcal{X}^\Delta$ the projection map. Given $\Lambda \in \mathcal{S}$, define the metric $\rho^{(\Lambda)}$ on \mathcal{X}^Λ as

$$\rho^{(\Lambda)}(\underline{\xi}_\Lambda, \underline{\xi}'_\Lambda) := \sum_{j \in \Lambda} \rho(\xi_j, \xi'_j),$$

and let $\rho_W^{(\Lambda)}$ denote the corresponding Wasserstein distance on $\mathcal{P}(\mathcal{X}^\Lambda)$ defined similarly to (2.1) but with the metric $\rho^{(\Lambda)}$.

recondt

Remark 3.10 Assumptions (C1)-(C3) together guarantee that, for any domain $\Lambda \in \mathcal{S}$, there exists a constant $c_\infty > 0$ such that,

$$\sup_{j \in \Lambda} \int_{\mathcal{X}} \rho(\xi, \xi^*) \mu_j^\Lambda(d\xi | \xi_{\Lambda^c}^*) \leq c_\infty.$$

See Sec. 4.2 for further details. In particular, the above guarantees that $\mu_\Lambda^\beta(\cdot | \xi_{\Lambda^c}^*) \in \mathcal{P}_1(\mathcal{X}^\Lambda)$.

3.3 Some corollaries.

corollaries

The two corollaries of Theorem 3.5 below follow from the proof of existence of Dobrushin's uniqueness theorem in Sec. 4.2. The proofs are deferred to Sec. 4.3.

coroll1

Corollary 3.11 Consider the special case of nearest-neighbour interactions, i.e., where $\mathcal{V}_X = 0$ unless $|X| = 1$, or $|X| = 2$ and $X = \{k, l\}$ with $|k - l| = 1$. Let $\beta > 0$ be fixed and assume that the conditions of Theorem 3.5 hold. Then, given a finite domain $\Delta \in \mathcal{S}$, there exists a constant $C > 0$ such that, for any $\Lambda, \Lambda' \in \mathcal{S}$ with $\Delta \subset \Lambda \subset \Lambda'$ and for any bounded boundary condition $\underline{\vartheta} \in \mathcal{X}^{\mathbb{Z}^d}$,

$$\rho_W^{(\Delta)} \left(\mu_\Delta^\Lambda(\cdot | \underline{\vartheta}_{\Lambda^c}), \mu_{\Delta'}^{\Lambda'}(\cdot | \underline{\vartheta}_{\Lambda'^c}) \right) \leq C \lambda^{\text{dist}(\Delta, \partial \Lambda)},$$

where $\text{dist}(\Delta, \partial \Lambda) := \inf_{j \in \Delta, j' \in \Lambda^c} |j - j'| > 0$, and the two probability measures $\mu_\Delta^\Lambda(\cdot | \underline{\vartheta}_{\Lambda^c}), \mu_{\Delta'}^{\Lambda'}(\cdot | \underline{\vartheta}_{\Lambda'^c})$ on $(\mathcal{X}^\Delta, \mathcal{B}(\mathcal{X}^\Delta))$ are defined similarly to (3.15). In particular, given a Lipschitz-continuous function $f : \mathcal{X}^\Delta \rightarrow \mathbb{R}$, i.e. such that,

$$\left| f(\underline{\xi}_\Delta) - f(\underline{\xi}'_\Delta) \right| \leq c_f \sum_{j \in \Delta} \rho(\xi_j, \xi'_j),$$

for a constant $c_f > 0$, we have,

$$\left| \int_{\mathcal{X}^\Delta} f(\underline{\xi}_\Delta) \mu_\Delta^\beta(d\xi_\Delta | \underline{\vartheta}_{\Lambda^c}) - \int_{\mathcal{X}^{\Lambda'}} f(\underline{\xi}'_\Delta) \mu_{\Delta'}^\beta(d\xi'_{\Delta'} | \underline{\vartheta}_{\Lambda'^c}) \right| \leq c_f C \lambda^{\text{dist}(\Delta, \partial \Lambda)}.$$

A similar statement can be made concerning the decay of correlations. For further related results, see, e.g., [24, 29, 20], [31, Sec. 7], and references therein.

corol2

Corollary 3.12 *Consider the special case of nearest-neighbour interactions, i.e., where $\mathcal{V}_{\mathcal{X}} = 0$ unless $|X| = 1$, or $|X| = 2$ and $X = \{k, l\}$ with $|k - l| = 1$. Let $\beta > 0$ be fixed and assume that the conditions of Theorem 3.5 hold. Let $\Delta, \Delta' \in \mathcal{S}$ such that $\Delta \cap \Delta' = \emptyset$. Let $f : \mathcal{X}^\Delta \rightarrow \mathbb{R}$ and $g : \mathcal{X}^{\Delta'} \rightarrow \mathbb{R}$ be bounded continuous functions. Then, there exists a constant $C > 0$ such that, if μ^β is the limit-Gibbs measure at inverse temperature β , then*

$$\left| \int_{\mathcal{X}^{\Delta \cup \Delta'}} f(\underline{\xi}_\Delta) g(\underline{\xi}_{\Delta'}) \mu_{\Delta \cup \Delta'}^\beta(d\underline{\xi}_{\Delta \cup \Delta'}) - \int_{\mathcal{X}^\Delta} f(\underline{\xi}_\Delta) \mu_\Delta^\beta(d\underline{\xi}_\Delta) \int_{\mathcal{X}^{\Delta'}} g(\underline{\xi}_{\Delta'}) \mu_{\Delta'}^\beta(d\underline{\xi}_{\Delta'}) \right| \leq C \lambda^{\text{dist}(\Delta, \Delta')}.$$

4 Proof of Dobrushin's uniqueness theorem.

DobSecc

We recall some notations from Sec. 3.1. Given $\Delta, \Lambda \in \mathcal{S}$ such that $\Delta \subset \Lambda$, let $\Pi_\Lambda : \mathcal{X}^{\mathbb{Z}^d} \rightarrow \mathcal{X}^\Lambda$ and $\Pi_\Delta^\Lambda : \mathcal{X}^\Lambda \rightarrow \mathcal{X}^\Delta$ be the projection maps. Given a probability measure $\nu \in \mathcal{P}(\mathcal{X}^{\mathbb{Z}^d})$, let $\{\nu_\Lambda\}_{\Lambda \in \mathcal{S}}$ be the family of marginal distributions of ν defined as $\nu_\Lambda := \nu \circ \Pi_\Lambda^{-1} \in \mathcal{P}(\mathcal{X}^\Lambda)$. This means that $\nu_\Lambda(A) = \nu(A \times \mathcal{X}^{\Lambda^c})$ for any $A \in \mathcal{B}(\mathcal{X}^\Lambda)$. Let $\{\nu(\cdot | \underline{\eta}_{\Lambda^c}), \underline{\eta}_{\Lambda^c} \in \mathcal{X}^{\Lambda^c}\}$ be the conditional probability measures in $\mathcal{P}(\mathcal{X}^{\mathbb{Z}^d})$ relative to Π_{Λ^c} . Note that $\nu(\{\underline{\xi} \in \mathcal{X}^{\mathbb{Z}^d} : \Pi_{\Lambda^c} \underline{\xi} = \underline{\eta}_{\Lambda^c}\} | \underline{\eta}_{\Lambda^c}) = 1$.

4.1 Proof of uniqueness.

The uniqueness follows from the following result

propDobp

Proposition 4.1 *Let $\mu, \nu \in \mathcal{P}(\mathcal{X}^{\mathbb{Z}^d})$ be probability measures with the same family of 1-point conditional distributions $\mu_k(\cdot | \underline{\xi})$, $k \in \mathbb{Z}^d$ and $\underline{\xi} \in \mathcal{X}^{\mathbb{Z}^d \setminus \{k\}}$. Given $\xi^* \in \mathcal{X}$, assume that there exists a constant $C > 0$ such that*

$$\max \left\{ \int_{\mathcal{X}^{\mathbb{Z}^d}} \rho(\xi_j, \xi^*) \mu(d\underline{\xi}), \int_{\mathcal{X}^{\mathbb{Z}^d}} \rho(\xi_j, \xi^*) \nu(d\underline{\xi}) \right\} \leq C, \quad (4.1) \quad \text{bd2C}$$

holds uniformly in $j \in \mathbb{Z}^d$. Further, assume that for all $j \in \mathbb{Z}^d$ and all $(\underline{\xi}, \underline{\xi}') \in \mathcal{X}^{\mathbb{Z}^d \setminus \{j\}} \times \mathcal{X}^{\mathbb{Z}^d \setminus \{j\}}$,

$$\rho_W(\mu_j(\cdot | \underline{\xi}), \nu_j(\cdot | \underline{\xi}')) \leq \sum_{\substack{l \in \mathbb{Z}^d \\ l \neq j}} r(l, j) \rho(\xi_l, \xi'_l), \quad (4.2) \quad \text{iily}$$

holds for some constants $r(l, j) \geq 0$ satisfying, for all $j \in \mathbb{Z}^d$,

$$\sum_{\substack{l \in \mathbb{Z}^d \\ l \neq j}} r(l, j) \leq \lambda < 1. \quad (4.3) \quad \text{consumf}$$

Then $\mu = \nu$.

4.1.1 Proof of Proposition 4.1.

The proof is based on the following Lemma, the proof of which is deferred to Sec. 4.1.2.

Dobpr

Lemma 4.2 *Let $\mu, \nu \in \mathcal{P}(\mathcal{X}^{\mathbb{Z}^d})$ be probability measures. Assume that for all $\epsilon > 0$ there exists a coupling $\sigma \in \Xi_{\mathcal{X}^{\mathbb{Z}^d}}(\mu, \nu)$ such that, for every $j \in \mathbb{Z}^d$,*

$$\int_{\mathcal{X}^{\mathbb{Z}^d} \times \mathcal{X}^{\mathbb{Z}^d}} \rho(\xi_j, \xi'_j) \sigma(d\underline{\xi}, d\underline{\xi}') < \gamma_j + \epsilon,$$

with $\gamma_j \geq 0$. Further, assume that for all $j \in \mathbb{Z}^d$ and all $(\underline{\xi}, \underline{\xi}') \in \mathcal{X}^{\mathbb{Z}^d \setminus \{j\}} \times \mathcal{X}^{\mathbb{Z}^d \setminus \{j\}}$, (4.2) holds for some constants $r(l, j) \geq 0$ satisfying (4.3).

Then, given a finite domain $\Lambda \in \mathcal{S}$, for all $\epsilon > 0$, there exists another $\tilde{\sigma} \in \Xi_{\mathcal{X}^{\mathbb{Z}^d}}(\mu, \nu)$ such that, for every $j \in \mathbb{Z}^d$,

$$\int_{\mathcal{X}^{\mathbb{Z}^d} \times \mathcal{X}^{\mathbb{Z}^d}} \rho(\xi_j, \xi'_j) \tilde{\sigma}(d\underline{\xi}, d\underline{\xi}') < \chi(j) + \epsilon, \quad (4.4) \quad \boxed{\text{lem2es1}}$$

where $\chi(j) = \gamma_j$ for $j \notin \Lambda$, and for $j \in \Lambda$,

$$\chi(j) = \sum_{l \in \Lambda^c} \gamma_l \sum_{k \in \Lambda} \sum_{n=0}^{\infty} r(l, k) (R_{\Lambda}^n)_{k,j},$$

where the matrix R_{Λ} is defined as

$$(R_{\Lambda})_{k,j} := \begin{cases} r(k, j), & \text{if } j, k \in \Lambda \text{ and } j \neq k, \\ 0, & \text{if } j, k \in \Lambda \text{ and } j = k \end{cases}. \quad (4.5) \quad \boxed{\text{mattRlbd}}$$

We now turn to

Proof of Proposition 4.1. We first note that, given $\sigma \in \Xi_{\mathcal{X}^{\mathbb{Z}^d}}(\mu, \nu)$, (4.1) ensures that,

$$\begin{aligned} \int_{\mathcal{X}^{\mathbb{Z}^d} \times \mathcal{X}^{\mathbb{Z}^d}} \rho(\xi_j, \xi'_j) \sigma(d\underline{\xi}, d\underline{\xi}') &\leq \int_{\mathcal{X}^{\mathbb{Z}^d} \times \mathcal{X}^{\mathbb{Z}^d}} (\rho(\xi_j, \xi^*) + \rho(\xi^*, \xi'_j)) \sigma(d\underline{\xi}, d\underline{\xi}') \\ &\leq \int_{\mathcal{X}^{\mathbb{Z}^d}} \rho(\xi_j, \xi^*) \mu(d\underline{\xi}) + \int_{\mathcal{X}^{\mathbb{Z}^d}} \rho(\xi_j, \xi^*) \nu(d\underline{\xi}) \leq 2C, \end{aligned}$$

uniformly in $j \in \mathbb{Z}^d$. Let $\Lambda_0 \in \mathcal{S}$ be fixed. For any $D > 0$ define the set

$$\Lambda_D := \{k \in \mathbb{Z}^d : \exists j \in \Lambda_0, |k - j| \leq D\}.$$

Fix $0 < \epsilon < 1$ and let $N \in \mathbb{N}$ be large enough so that

$$2C \frac{\lambda^{N+2}}{1 - \lambda} < \frac{\epsilon}{2}. \quad (4.6) \quad \boxed{\text{usef1}}$$

By assumption, there exists $D_1 > 0$ such that

$$\sup_{j \in \Lambda_0} \sum_{l \in \Lambda_{D_1}^c} r(l, j) < \frac{\epsilon}{2C}. \quad (4.7) \quad \boxed{\text{sumeps1}}$$

Continuing by recursion, there exist $D_1 < D_2 < \dots < D_{N+1}$ such that

$$\sup_{j \in \Lambda_{D_m}} \sum_{l \in \Lambda_{D_{m+1}}^c} r(l, j) < \frac{\epsilon}{2C}, \quad m = 1, \dots, N. \quad (4.8) \quad \boxed{\text{sumeps2}}$$

Set $\Lambda = \Lambda_{D_{N+1}}$. Applying Lemma 4.2, there exists $\tilde{\sigma} \in \Xi_{\mathcal{X}^{\mathbb{Z}^d}}(\mu, \nu)$ such that,

$$\int_{\mathcal{X}^{\mathbb{Z}^d} \times \mathcal{X}^{\mathbb{Z}^d}} \rho(\xi_j, \xi'_j) \tilde{\sigma}(d\underline{\xi}, d\underline{\xi}') < \chi(j) + \frac{\epsilon}{2}, \quad j \in \mathbb{Z}^d,$$

where $\chi(j) = 2C$ for $j \notin \Lambda$, and,

$$\chi(j) = 2C \sum_{l \in \Lambda^c} \sum_{k \in \Lambda} r(l, k) \sum_{n=0}^{\infty} (R_{\Lambda}^n)_{k,j}, \quad j \in \Lambda. \quad (4.9) \quad \boxed{\text{chide}}$$

Define the matrix norm of R_{Λ} by

$$\|R_{\Lambda}\| := \sup_{k \in \Lambda} \sum_{j \in \Lambda} |(R_{\Lambda})_{j,k}|. \quad (4.10) \quad \boxed{\text{nommatx}}$$

Note that $\|R_\Lambda\| \leq \lambda < 1$ by assumption. To estimate (4.9) with $j \in \Lambda_0$, we split the sum over n into the terms with $n \leq N$ and the rest, and note first that

$$2C \sum_{l \in \Lambda^c} \sum_{k \in \Lambda} r(l, k) \sum_{n=N+1}^{\infty} (R_\Lambda^n)_{k,j} \leq 2C \sup_{k \in \Lambda} \sum_{\substack{l \in \mathbb{Z}^d \\ l \neq k}} r(l, k) \sum_{n=N+1}^{\infty} \|R_\Lambda\|^n \leq 2C \frac{\lambda^{N+2}}{1-\lambda} < \frac{\epsilon}{2},$$

uniformly in $j \in \Lambda_0$. Here, we used (4.6) in the right-hand side of the last inequality. The remaining terms, we write as follows

$$2C \sum_{l \in \Lambda^c} \left(r(l, j) + \sum_{n=1}^N \sum_{k_1, \dots, k_n \in \Lambda} r(l, k_n) \left(\prod_{i=1}^{n-1} r(k_{n-i+1}, k_{n-i}) \right) r(k_1, j) \right), \quad j \in \Lambda_0.$$

Now,

$$2C \sup_{j \in \Lambda_0} \sum_{l \in \Lambda^c} r(l, j) < \epsilon,$$

by the definition of Λ and (4.7) since $D_{N+1} > D_1$. Similarly, for $n \geq 1$, if $k_n \in \Lambda_{D_n}$, we obtain

$$2C \sum_{l \in \Lambda^c} \sum_{n=1}^N \sum_{k_n \in \Lambda_{D_n}} r(l, k_n) (R_\Lambda^n)_{k_n, j} \leq 2C \sup_{k \in \Lambda} \sum_{\substack{l \in \mathbb{Z}^d \\ l \neq k}} r(l, k) \sum_{n=1}^N \|R_\Lambda\|^n \leq \frac{\lambda}{1-\lambda} \epsilon,$$

uniformly in $j \in \Lambda_0$. The term where $k_n \in \Lambda \setminus \Lambda_{D_n}$ can be written as

$$2C \sum_{l \in \Lambda^c} \sum_{n=1}^N \sum_{p=1}^{n-1} \sum_{k_n \in \Lambda \setminus \Lambda_{D_n}} \cdots \sum_{k_{n-p+1} \in \Lambda \setminus \Lambda_{D_{n-p+1}}} \sum_{k_{n-p} \in \Lambda_{D_{n-p}}} r(l, k_n) \times \left(\prod_{s=1}^p r(k_{n-s+1}, k_{n-s}) \right) (R_\Lambda^{n-p})_{k_{n-p}, j}.$$

This is bounded by

$$2C \sum_{n=1}^N \sum_{p=1}^{n-1} \left(\sup_{k \in \Lambda} \sum_{\substack{l \in \mathbb{Z}^d \\ l \neq k}} r(l, k) \right) \|R_\Lambda^{p-1}\| \sup_{k' \in \Lambda_{D_{n-p}}} \left(\sum_{k'' \in \Lambda_{D_{n-p+1}}^c} r(k'', k') \right) \|R_\Lambda^{n-p}\| \leq \sum_{n=1}^N \sum_{p=1}^{n-1} \lambda^n \epsilon < \frac{\lambda}{(1-\lambda)^2} \epsilon,$$

where we used (4.8) in the right-hand side of the first inequality. In total, we obtain the following upper bound

$$\begin{aligned} \rho^{(\Lambda_0)}(\mu_{\Lambda_0}, \nu_{\Lambda_0}) &\leq \sum_{j \in \Lambda_0} \int_{\mathcal{X}^{\mathbb{Z}^d} \times \mathcal{X}^{\mathbb{Z}^d}} \rho(\xi_j, \xi'_j) \tilde{\sigma}(d\xi, d\xi') \\ &< \left(1 + \frac{\lambda}{1-\lambda} + \frac{\lambda}{(1-\lambda)^2} \right) |\Lambda_0| \epsilon = \frac{1}{(1-\lambda)^2} |\Lambda_0| \epsilon. \end{aligned}$$

Letting $\epsilon \rightarrow 0$, we conclude that $\mu_{\Lambda_0} = \nu_{\Lambda_0}$. Since $\Lambda_0 \subset \mathbb{Z}^d$ is an arbitrary finite subset and the measures are entirely determined by their marginals, $\mu = \nu$. \square

4.1.2 Proof of Lemma 4.2.

We start by proving a local version of the Lemma. Given $k \in \mathbb{Z}^d$, for all $\epsilon > 0$, there exists $\tilde{\sigma} \in \Xi_{\mathcal{X}^{\mathbb{Z}^d}}(\mu, \nu)$ such that, for every $j \in \mathbb{Z}^d$,

$$\int_{\mathcal{X}^{\mathbb{Z}^d} \times \mathcal{X}^{\mathbb{Z}^d}} \rho(\xi_j, \xi'_j) \tilde{\sigma}(d\underline{\xi}, d\underline{\xi}') < \gamma'_j + \epsilon, \quad (4.11) \quad \text{need11}$$

where $\gamma'_j = \gamma_j$ for $j \neq k$, and

$$\gamma'_k = \sum_{\substack{j \in \mathbb{Z}^d \\ j \neq k}} r(j, k) \gamma_j. \quad (4.12) \quad \text{need12}$$

Let $\epsilon > 0$ be fixed. Let $\sigma \in \Xi_{\mathcal{X}^{\mathbb{Z}^d}}(\mu, \nu)$ be such that, for every $j \neq k$,

$$\int_{\mathcal{X}^{\mathbb{Z}^d \setminus \{k\}} \times \mathcal{X}^{\mathbb{Z}^d \setminus \{k\}}} \rho(\xi_j, \xi'_j) \sigma_{\mathcal{X}^{\mathbb{Z}^d \setminus \{k\}}}(d\underline{\xi}, d\underline{\xi}') < \gamma_j + \frac{\epsilon}{2}. \quad (4.13) \quad \text{lem1es1}$$

By (4.2), for all $(\underline{\xi}, \underline{\xi}') \in \mathcal{X}^{\mathbb{Z}^d \setminus \{k\}} \times \mathcal{X}^{\mathbb{Z}^d \setminus \{k\}}$, there exists $\sigma_{\underline{\xi}, \underline{\xi}'} \in \Xi_{\mathcal{X}}(\mu_k(\cdot | \underline{\xi}), \nu_k(\cdot | \underline{\xi}'))$ such that

$$\int_{\mathcal{X} \times \mathcal{X}} \rho(\xi, \xi') \sigma_{\underline{\xi}, \underline{\xi}'}(d\xi, d\xi') \leq \rho_W(\mu_k(\cdot | \underline{\xi}), \nu_k(\cdot | \underline{\xi}')) + \frac{\epsilon}{2} \leq \sum_{\substack{j \in \mathbb{Z}^d \\ j \neq k}} r(j, k) \rho(\xi_j, \xi'_j) + \frac{\epsilon}{2}. \quad (4.14) \quad \text{lem1es2}$$

For $A, B \in \mathcal{B}(\mathcal{X}^{\mathbb{Z}^d})$, define $\tilde{\sigma} \in \mathcal{P}(\mathcal{X}^{\mathbb{Z}^d} \times \mathcal{X}^{\mathbb{Z}^d})$ as follows

$$\tilde{\sigma}(A \times B) := \int_{\mathcal{X}^{\mathbb{Z}^d} \times \mathcal{X}^{\mathbb{Z}^d}} \mathbb{I}_{A \times B}((\underline{\xi}, \xi), (\underline{\xi}', \xi')) \sigma_{\mathcal{X}^{\mathbb{Z}^d \setminus \{k\}}}(d\underline{\xi}, d\underline{\xi}') \sigma_{\underline{\xi}, \underline{\xi}'}(d\xi, d\xi').$$

Then taking $B = \mathcal{X}^{\mathbb{Z}^d}$,

$$\begin{aligned} \tilde{\sigma}(A \times \mathcal{X}^{\mathbb{Z}^d}) &= \int_{\mathcal{X}^{\mathbb{Z}^d}} \mathbb{I}_A((\underline{\xi}, \xi)) \int_{\mathcal{X}^{\mathbb{Z}^d \setminus \{k\}}} \sigma_{\mathcal{X}^{\mathbb{Z}^d \setminus \{k\}}}(d\underline{\xi}, d\underline{\xi}') \int_{\mathcal{X}} \sigma_{\underline{\xi}, \underline{\xi}'}(d\xi, d\xi') \\ &= \int_{\mathcal{X}^{\mathbb{Z}^d}} \mathbb{I}_A((\underline{\xi}, \xi)) \mu_{\mathcal{X}^{\mathbb{Z}^d \setminus \{k\}}}(d\underline{\xi}) \mu_k(d\xi | \underline{\xi}) = \mu(A). \end{aligned}$$

Similarly, $\tilde{\sigma}(\mathcal{X}^{\mathbb{Z}^d} \times B) = \nu(B)$. As a result, $\tilde{\sigma} \in \Xi_{\mathcal{X}^{\mathbb{Z}^d}}(\mu, \nu)$. Moreover, if $j \neq k$ then

$$\begin{aligned} \int_{\mathcal{X}^{\mathbb{Z}^d} \times \mathcal{X}^{\mathbb{Z}^d}} \rho(\xi_j, \xi'_j) \tilde{\sigma}(d\underline{\xi}, d\underline{\xi}') &= \int_{\mathcal{X}^{\mathbb{Z}^d \setminus \{k\}} \times \mathcal{X}^{\mathbb{Z}^d \setminus \{k\}}} \rho(\xi_j, \xi'_j) \sigma_{\mathcal{X}^{\mathbb{Z}^d \setminus \{k\}}}(d\underline{\xi}, d\underline{\xi}') \int_{\mathcal{X} \times \mathcal{X}} \sigma_{\underline{\xi}, \underline{\xi}'}(d\xi, d\xi') \\ &< \gamma_j + \frac{\epsilon}{2}, \end{aligned}$$

which follows from (4.13), and,

$$\begin{aligned} \int_{\mathcal{X}^{\mathbb{Z}^d} \times \mathcal{X}^{\mathbb{Z}^d}} \rho(\xi_k, \xi'_k) \tilde{\sigma}(d\underline{\xi}, d\underline{\xi}') &= \int_{\mathcal{X}^{\mathbb{Z}^d \setminus \{k\}} \times \mathcal{X}^{\mathbb{Z}^d \setminus \{k\}}} \left(\int_{\mathcal{X} \times \mathcal{X}} \rho(\xi_k, \xi'_k) \sigma_{\underline{\xi}, \underline{\xi}'}(d\xi_k, d\xi'_k) \right) \sigma_{\mathcal{X}^{\mathbb{Z}^d \setminus \{k\}}}(d\underline{\xi}, d\underline{\xi}') \\ &\leq \sum_{\substack{j \in \mathbb{Z}^d \\ j \neq k}} r(j, k) \int_{\mathcal{X}^{\mathbb{Z}^d \setminus \{k\}} \times \mathcal{X}^{\mathbb{Z}^d \setminus \{k\}}} \rho(\xi_j, \xi'_j) \sigma_{\mathcal{X}^{\mathbb{Z}^d \setminus \{k\}}}(d\underline{\xi}, d\underline{\xi}') + \frac{\epsilon}{2} \\ &< \sum_{\substack{j \in \mathbb{Z}^d \\ j \neq k}} r(j, k) \gamma_j + \epsilon, \end{aligned}$$

where we used (4.14) and (4.13) in the first and second inequality respectively. This completes the proof of (4.11)-(4.12). Now, applying the above result to each $j \in \Lambda$, we find that (4.4) holds with

$$\chi(j) = \sum_{\substack{l \in \mathbb{Z}^d \\ l \neq j}} \gamma_l r(l, j), \quad j \in \Lambda.$$

We split the sum over $l \in \mathbb{Z}^d \setminus \{j\}$ into $l \in (\Lambda \setminus \{j\}) \cup \Lambda^c$. Then, for each $l \in \Lambda \setminus \{j\}$, we can replace γ_l by $\chi(l)$ to get an improved bound. In view of (4.5), this yields

$$\begin{aligned}\chi(j) &= \sum_{l \in \Lambda^c} \gamma_l r(l, j) + \sum_{\substack{l \in \Lambda \\ l \neq j}} \sum_{\substack{k \in \mathbb{Z}^d \\ k \neq l}} \gamma_k r(k, l) (R_\Lambda)_{l, j} \\ &= \sum_{l \in \Lambda^c} \gamma_l \sum_{k \in \Lambda} r(l, k) ((\mathbb{I}_\Lambda)_{k, j} + (R_\Lambda)_{k, j}) + \sum_{l \in \Lambda} \gamma_l (R_\Lambda^2)_{l, j}.\end{aligned}$$

Continuing this way, we obtain at the m -th stage

$$\chi(j) = X_m(j) + \mathcal{R}_{m+1}(j),$$

with

$$\begin{aligned}X_m(j) &:= \sum_{l \in \Lambda^c} \gamma_l \sum_{k \in \Lambda} r(l, k) \sum_{n=0}^m (R_\Lambda^n)_{k, j} \\ \mathcal{R}_{m+1}(j) &:= \sum_{l \in \Lambda} \gamma_l (R_\Lambda^{m+1})_{l, j}.\end{aligned}$$

From the matrix norm defined in (4.10), $\|R_\Lambda\| \leq \lambda < 1$ by assumption, and hence $\|R_\Lambda^m\| \rightarrow 0$ as $m \rightarrow \infty$. As a result, $\mathcal{R}_{m+1}(j) \rightarrow 0$ in the limit $m \rightarrow \infty$. \square

4.2 Proof of existence.

exist

Let $\beta > 0$ be fixed. For convenience, we hereafter drop the β -dependence in our notations. Given $\Gamma \in \mathcal{S}$ and $\vartheta \in \mathcal{X}^{\mathbb{Z}^d}$ such that $\sup_{k \in \mathbb{Z}^d} \rho(\xi^*, \vartheta_k) < \infty$, let $\mu_\Gamma(\cdot | \vartheta_{\Gamma^c})$ denote the local Gibbs distribution on $(\mathcal{X}^\Gamma, \mathcal{B}(\mathcal{X}^\Gamma))$ with bounded boundary condition ϑ_{Γ^c} held fixed outside of Γ . Given $\Delta \in \mathcal{S}$ such that $\Delta \subset \Gamma$, we introduce on $(\mathcal{X}^\Delta, \mathcal{B}(\mathcal{X}^\Delta))$ the probability measure

$$\mu_\Delta^\Gamma(\cdot | \vartheta_{\Gamma^c}) := \mu_\Gamma(\cdot | \vartheta_{\Gamma^c}) \circ (\Pi_\Delta^\Gamma)^{-1}, \quad (4.15)$$

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where $\Pi_\Delta^\Gamma : \mathcal{X}^\Gamma \rightarrow \mathcal{X}^\Delta$ is the projection map.

Let $\Lambda \in \mathcal{S}$ be fixed. The first part of the proof consists in showing that

$$\sup_{j \in \Lambda} \int_{\mathcal{X}} \rho(\xi, \xi^*) \mu_j^\Lambda(d\xi | \xi_{\Lambda^c}^*) \leq c_\infty := \frac{c_0}{1 - \lambda}, \quad (4.16)$$

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with $c_0 > 0$ defined in (3.9). Let $j \in \Lambda$ be fixed. By assumption, given $\epsilon > 0$ and $\underline{\eta} \in \mathcal{X}^\Lambda$, there exists a coupling $\sigma_{j; \underline{\eta}, \underline{\xi}^*} \in \Xi_{\mathcal{X}}(\mu_j(\cdot | \underline{\eta}_{\Lambda \setminus \{j\}} \xi_{\Lambda^c}^*), \mu_j(\cdot | \xi_{\mathbb{Z}^d \setminus \{j\}}^*))$ such that

$$\int_{\mathcal{X} \times \mathcal{X}} \rho(\xi, \xi') \sigma_{j; \underline{\eta}, \underline{\xi}^*}(d\xi, d\xi') < \sum_{\substack{l \in \Lambda \\ l \neq j}} r(l, j) \rho(\eta_l, \xi^*) + \epsilon.$$

Then, we have,

$$\begin{aligned}\int_{\mathcal{X}} \rho(\xi, \xi^*) \mu_j(d\xi | \underline{\eta}_{\Lambda \setminus \{j\}} \xi_{\Lambda^c}^*) &= \int_{\mathcal{X} \times \mathcal{X}} \rho(\xi, \xi^*) \sigma_{j; \underline{\eta}, \underline{\xi}^*}(d\xi, d\xi') \\ &\leq \int_{\mathcal{X} \times \mathcal{X}} (\rho(\xi, \xi') + \rho(\xi', \xi^*)) \sigma_{j; \underline{\eta}, \underline{\xi}^*}(d\xi, d\xi') \\ &< \sum_{\substack{l \in \Lambda \\ l \neq j}} r(l, j) \rho(\eta_l, \xi^*) + \epsilon + c_0.\end{aligned}$$

Letting $\epsilon \rightarrow 0$, we are left with

$$\int_{\mathcal{X}} \rho(\xi, \xi^*) \mu_j(d\xi|_{\underline{\eta}_{\Lambda \setminus \{j\}} \underline{\xi}_{\Lambda^c}^*}) \leq \sum_{\substack{l \in \Lambda \\ l \neq j}} r(l, j) \rho(\eta_l, \xi^*) + c_0. \quad (4.17) \quad \boxed{\text{interm1}}$$

Next, let $(K_n)_{n \in \mathbb{N}} \subset \mathcal{X}$ be an increasing sequence of compact sets such that $\mu_{\Lambda}(K_n^{\Lambda} | \underline{\xi}_{\Lambda^c}^*) \rightarrow 1$ when $n \rightarrow \infty$. Define the sequence of K_n -restrictions of (4.15) as $\mu_{\Delta, n}^{\Gamma}(A | \underline{\vartheta}_{\Gamma^c}) := \mu_{\Delta}^{\Gamma}(A \cap K_n^{\Delta} | \underline{\vartheta}_{\Gamma^c})$, $A \in \mathcal{B}(\mathcal{X}^{\Delta})$. Obviously, for all $n \in \mathbb{N}$, we have,

$$\sup_{k \in \Lambda} \int_{\mathcal{X}} \rho(\xi, \xi^*) \mu_{k, n}^{\Lambda}(d\xi | \underline{\xi}_{\Lambda^c}^*) < \infty.$$

Then, we have,

$$\begin{aligned} \int_{\mathcal{X}} \rho(\xi, \xi^*) \mu_{j, n}^{\Lambda}(d\xi | \underline{\xi}_{\Lambda^c}^*) &= \int_{\mathcal{X}^{\Lambda \setminus \{j\}}} \int_{\mathcal{X}} \mathbb{I}_{K_n}(\xi) \rho(\xi, \xi^*) \mu_j(d\xi |_{\underline{\eta}_{\Lambda \setminus \{j\}} \underline{\xi}_{\Lambda^c}^*}) \mu_{\Lambda, n}(d\underline{\eta}_{\Lambda \setminus \{j\}} | \underline{\xi}_{\Lambda^c}^*) \\ &\leq \sum_{\substack{l \in \Lambda \\ l \neq j}} r(l, j) \int_{\mathcal{X}^{\Lambda \setminus \{j\}}} \rho(\eta_l, \xi^*) \mu_{\Lambda, n}(d\underline{\eta}_{\Lambda \setminus \{j\}} | \underline{\xi}_{\Lambda^c}^*) + c_0 \\ &= \sum_{l \in \Lambda} (R_{\Lambda})_{l, j} \int_{\mathcal{X}} \rho(\eta_l, \xi^*) \mu_{l, n}^{\Lambda}(d\underline{\eta}_l | \underline{\xi}_{\Lambda^c}^*) + c_0, \end{aligned} \quad (4.18) \quad \boxed{\text{itterr}}$$

where we used (4.17). Iterating (4.18), we have,

$$\int_{\mathcal{X}} \rho(\xi, \xi^*) \mu_{j, n}^{\Lambda}(d\xi | \underline{\xi}_{\Lambda^c}^*) \leq \sum_{l \in \Lambda} (R_{\Lambda})_{l, j}^p \int_{\mathcal{X}} \rho(\eta_l, \xi^*) \mu_{l, n}^{\Lambda}(d\underline{\eta}_l | \underline{\xi}_{\Lambda^c}^*) + c_{p-1} \quad (4.19) \quad \boxed{\text{fromthr}}$$

where, for all integers $p \geq 2$,

$$c_{p-1} := c_0 \sum_{s=0}^{p-1} \sup_{j \in \Lambda} \sum_{l \in \Lambda} (R_{\Lambda})_{l, j}^s = c_0 \sum_{s=0}^{p-1} \|R_{\Lambda}^s\| \leq \frac{c_0}{1 - \lambda}.$$

Clearly, the first term in the r.h.s. of (4.19) tends to 0 in the limit $p \rightarrow \infty$. To conclude the proof of (4.16), we use the monotone convergence theorem to take $n \rightarrow \infty$.

To complete the proof of existence, consider a sequence of subsets $(\Lambda_n)_{n \in \mathbb{N}}$ ordered by inclusion and exhausting \mathbb{Z}^d and let us show that the sequence of probability measures $(\mu_{\Lambda_n}(\cdot | \underline{\xi}_{\Lambda_n^c}^*))_{n \in \mathbb{N}}$ converges weakly in $\mathcal{P}(\mathcal{X}^{\mathbb{Z}^d})$. Let $\Lambda_0 \in \mathcal{S}$ be fixed. Let $n, n' \in \mathbb{N}$ with $n < n'$ sufficiently large such that $\Lambda_0 \subset \Lambda_n \subset \Lambda_{n'}$. In view of (4.16), by applying Lemma 4.2, for all $\epsilon > 0$, there exists a coupling $\tilde{\sigma} \in \Xi_{\mathcal{X}^{\mathbb{Z}^d}}(\mu_{\Lambda_n}(\cdot | \underline{\xi}_{\Lambda_n^c}^*), \mu_{\Lambda_{n'}}(\cdot | \underline{\xi}_{\Lambda_{n'}^c}^*))$ such that, for all $j \in \mathbb{Z}^d$,

$$\int_{\mathcal{X}^{\mathbb{Z}^d} \times \mathcal{X}^{\mathbb{Z}^d}} \rho(\xi_j, \xi'_j) \tilde{\sigma}(d\underline{\xi}, d\underline{\xi}') < \chi(j) + \frac{\epsilon}{2},$$

where $\chi(j) = 2c_{\infty}$ for $j \notin \Lambda_n$ and where $\chi(j)$ for $j \in \Lambda_n$ is given by

$$\chi(j) = 2c_{\infty} \sum_{l \in \Lambda_n^c} \sum_{k \in \Lambda_n} \sum_{p=0}^{\infty} r(l, j) (R_{\Lambda_n}^p)_{k, j}.$$

As in the proof of Proposition 4.1, we can prove that $\chi(j)$ is arbitrarily small for $j \in \Lambda_0$ provided that $\text{dist}(\Lambda_0, \Lambda_n^c)$ is large enough. Restricting $\tilde{\sigma}$ to $\mathcal{X}^{\Lambda_0} \times \mathcal{X}^{\Lambda_0}$, it follows that, for any $\epsilon > 0$,

$$\rho_W^{(\Lambda_0)} \left(\mu_{\Lambda_0}^{\Lambda_n}(\cdot | \underline{\xi}_{\Lambda_n^c}^*), \mu_{\Lambda_0}^{\Lambda_{n'}}(\cdot | \underline{\xi}_{\Lambda_{n'}^c}^*) \right) < \epsilon,$$

provided that $\text{dist}(\Lambda_0, \Lambda_n^c)$ is large enough. This means that $(\mu_{\Lambda_0}^{\Lambda_n}(\cdot | \xi_{\Lambda_n^c}^*))_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathcal{P}_1(\mathcal{X}^{\Lambda_0})$, and hence converges since $(\mathcal{P}_1(\mathcal{X}^{\Lambda_0}), \rho^{(\Lambda_0)})$ is complete by Theorem 2.1. As this holds for all finite domains Λ_0 , the sequence $(\mu_{\Lambda_n}(\cdot | \xi_{\Lambda_n^c}^*))_{n \in \mathbb{N}}$ converges weakly in $\mathcal{P}(\mathcal{X}^{\mathbb{Z}^d})$.

Let $\bar{\mu}^{(\xi^*)}$ denote the limit. We end the proof of existence by showing that, under our conditions, the equilibrium DLR equation follows from the above, see Remarks 3.4 and 3.8. Let $\Lambda_0, \Lambda \in \mathcal{S}$ such that $\Lambda_0 \subset \Lambda$. Let $(\Lambda_n)_n$ be the sequence of subsets as above. Let $A \in \mathcal{B}(\mathcal{X}^{\Lambda_0})$ and $B \in \mathcal{B}(\mathcal{X}^{\Lambda \setminus \Lambda_0})$ bounded. For all n large enough so that $\Lambda \subset \Lambda_n$, the consistency relation (see Remark 3.4) yields

$$\int_B \mu_{\Lambda_0}(A | \xi_{\Lambda \setminus \Lambda_0}) \mu_{\Lambda \setminus \Lambda_0}^{\Lambda_n}(d\xi_{\Lambda \setminus \Lambda_0} | \xi_{\Lambda_n^c}) = \mu_{\Lambda}^{\Lambda_n}(A \times B | \xi_{\Lambda_n^c}).$$

By using that $(\mu_X^{\Lambda_n}(\cdot | \xi_{\Lambda_n^c}))_n$ converges weakly to $\bar{\mu}_X^{(\xi)} := \bar{\mu}^{(\xi)} \circ \Pi_X^{-1}$, $X \subseteq \Lambda$, (3.13) follows. \square

4.3 Proof of Corollaries 3.11 and 3.12.

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Proof of Corollary 3.11. We have, as in the proof of Proposition 4.1, that there exists a coupling $\tilde{\sigma} \in \Xi_{\mathcal{X}^{\mathbb{Z}^d}}(\mu_{\Delta}^{\Lambda}(\cdot | \vartheta_{\Lambda^c}), \mu_{\Delta}^{\Lambda'}(\cdot | \vartheta_{\Lambda'^c}))$ such that, for all $j \in \mathbb{Z}^d$,

$$\int_{\mathcal{X}^{\mathbb{Z}^d} \times \mathcal{X}^{\mathbb{Z}^d}} \rho(\xi_j, \xi'_j) \tilde{\sigma}(d\xi, d\xi') < \chi(j) + \epsilon,$$

where $\chi(j) = 2c_{\infty}$ for $j \notin \Lambda$, and where $\chi(j)$ for $j \in \Lambda$ is given by

$$\chi(j) = 2c_{\infty} \sum_{l \in \Lambda^c} \sum_{k \in \Lambda} \sum_{p=0}^{\infty} r(l, k) (R_{\Lambda}^p)_{k,j}.$$

In the particular case of nearest-neighbour interactions, $r(l, k) = 0$ unless $|k - l| = 1$, and hence, if $\text{dist}(\Delta, \Lambda^c) = D > 0$, $r(l, k) (R_{\Lambda}^p)_{k,j} = 0$ unless $p \geq D - 1$. Since $\|R_{\Lambda}\| = \lambda < 1$ by assumption, we conclude that

$$\chi(j) \leq 2c_{\infty} \sum_{p=D}^{\infty} \lambda^p = \frac{2c_{\infty}}{1 - \lambda} \lambda^D,$$

from which it follows that

$$\rho_W^{(\Delta)} \left(\mu_{\Delta}^{\Lambda}(\cdot | \vartheta_{\Lambda^c}), \mu_{\Delta}^{\Lambda'}(\cdot | \vartheta_{\Lambda'^c}) \right) \leq \frac{2c_{\infty}}{1 - \lambda} |\Delta| \lambda^D.$$

The second part of the corollary is now straightforward

$$\begin{aligned} \left| \int_{\mathcal{X}^{\Lambda}} f(\xi_{\Delta}) \mu_{\Lambda}^{\beta}(d\xi_{\Delta} | \vartheta_{\Lambda^c}) - \int_{\mathcal{X}^{\Lambda'}} f(\xi'_{\Delta}) \mu_{\Lambda'}^{\beta}(d\xi'_{\Delta} | \vartheta_{\Lambda'^c}) \right| \\ \leq \int_{\mathcal{X}^{\mathbb{Z}^d} \times \mathcal{X}^{\mathbb{Z}^d}} |f(\xi_{\Delta}) - f(\xi'_{\Delta})| \tilde{\sigma}(d\xi, d\xi') \\ \leq c_f \int_{\mathcal{X}^{\mathbb{Z}^d} \times \mathcal{X}^{\mathbb{Z}^d}} \sum_{j \in \Delta} \rho(\xi_j, \xi'_j) \tilde{\sigma}(d\xi, d\xi') \leq |\Delta| (C\lambda^D + \epsilon), \end{aligned}$$

where $c_f > 0$ is the Lipschitz constant. Taking $\epsilon \rightarrow 0$ this proves the second statement. \square

Proof of Corollary 3.12. Let $\Lambda \subset \Delta'^c$ be finite such that $\Delta \subset \Lambda$. We can write

$$\int_{\mathcal{X}^{\Delta \cup \Delta'}} f(\xi_{\Delta}) g(\xi_{\Delta'}) \mu_{\Delta \cup \Delta'}(d\xi_{\Delta \cup \Delta'}) = \int_{\mathcal{X}^{\Lambda^c}} g(\xi_{\Delta'}) \mu_{\Lambda^c}(d\xi_{\Lambda^c}) \int_{\mathcal{X}^{\Lambda}} f(\xi_{\Delta}) \mu_{\Delta}^{\Lambda}(d\xi_{\Delta} | \xi_{\Lambda^c}).$$

Hence, it suffices to show that

$$\rho_W^{(\Delta)} \left(\mu_{\Delta}, \mu_{\Delta}^{\Lambda}(\cdot | \xi_{\Lambda^c}) \right) < C\lambda^{\text{dist}(\Delta, \Lambda^c)}.$$

This is analogous to the previous corollary. \square

5 Applications - Part I: Uniqueness of Gibbs measures at high temperature.

App

app11

5.1 A general result with application to the classical Heisenberg model.

At high temperature, there is in general no phase transition. Dobrushin's uniqueness theorem gives a simple explicit upper bound for the inverse temperature

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Proposition 5.1 *Assume that \mathcal{X} is either a compact convex subset of a normed space with metric ρ , or a compact Riemannian manifold. In the latter case, assume that the metric on \mathcal{X} is given by $\rho(x, y) = \int_0^1 |\zeta(s)| ds$, where ζ is a shortest geodesic from x to y . Consider the following formal Hamiltonian with nearest-neighbour interactions defined on $\mathcal{X}^{\mathbb{Z}^d}$ as*

$$H_{inv}(\underline{\xi}) := \sum_{j \in \mathbb{Z}^d} \mathcal{V}(\xi_j) + \sum_{j \in \mathbb{Z}^d} \sum_{l \in N_1(j)} \mathcal{V}_{l-j}(\xi_j, \xi_l), \quad (5.1)$$

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where $N_1(j) := \{j' \in \mathbb{Z}^d : |j' - j| = 1\}$ denotes the set of nearest-neighbours of $j \in \mathbb{Z}^d$, and where,
(i). $\mathcal{V} : \mathcal{X} \rightarrow \mathbb{R}$ is assumed to be continuous;
(ii). $\mathcal{V}_k : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$, $|k| = 1$ are supposed to be jointly continuous and Lipschitz-continuous in the second variable, i.e. there exists a constant $C_0 > 0$ such that, for all $(\xi, \eta, \eta') \in \mathcal{X} \times \mathcal{X} \times \mathcal{X}$,

$$|\mathcal{V}_k(\xi, \eta) - \mathcal{V}_k(\xi, \eta')| \leq C_0 \rho(\eta, \eta').$$

Further, let $\mu_0 \in \mathcal{P}(\mathcal{X})$ be a probability measure and, for a given $\beta > 0$, let $\mu_j^\beta = \mu^\beta$ denote the associated 1-point Gibbs distribution generated by (5.1). Let $C_1 > 0$ be defined as

$$C_1(\beta) := \sup_{\xi' \in \mathcal{X}} \sup_{\varphi \in \mathcal{X}^{N_1(0)}} \int_{\mathcal{X}} \rho(\xi, \xi') \mu^\beta(d\xi | \varphi) < \infty.$$

Then, provided that $12dC_0C_1(\beta)\beta < 1$, there exists a unique limit Gibbs distribution at inverse temperature $\beta > 0$ in $\mathcal{P}(\mathcal{X}^{\mathbb{Z}^d})$ associated with H_{inv} .

As a direct application of Proposition 5.1, consider the classical Heisenberg model. In this model, the single-spin space is the unit sphere \mathbb{S}^r , $r \geq 1$ with normalized Lebesgue measure, and the pair-interaction potential is

$$\mathcal{V}_{l-j}(s_j, s_l) = -Js_j \cdot s_l, \quad s_j, s_l \in \mathbb{S}^r,$$

for some real coupling constant $J > 0$. Proposition 5.1 guarantees that there is no phase transition for $\beta < (12d\pi J)^{-1}$. Indeed, the Riemann metric is bounded by $\rho(s, s') \leq \pi$ so that $C_1 \leq \pi$, and if θ denotes the angle between s and s' ,

$$|\mathcal{V}_k(s_0, s) - \mathcal{V}_k(s_0, s')| = J|s_0 \cdot (s - s')| \leq J|s - s'| = 2J \sin\left(\frac{\theta}{2}\right) \leq J\theta = J\rho(s, s').$$

In fact, considering \mathbb{S}^r as a subset of \mathbb{R}^{r+1} , the estimate can be slightly improved to $\beta < (24dJ)^{-1}$. Using the symmetry as in the Ising model in Sec. 5.2 below, the estimate can be improved further.

Proof of Proposition 5.1. Given $(\underline{\eta}, \underline{\eta}') \in \mathcal{X}^{N_1(0)} \times \mathcal{X}^{N_1(0)}$ and $M \in \mathbb{N}$, we set

$$\underline{\eta}_k := \left(1 - \frac{k}{M}\right) \underline{\eta} + \frac{k}{M} \underline{\eta}' = \underline{\eta} + \frac{k}{M} (\underline{\eta}' - \underline{\eta}), \quad k = 0, \dots, M, \quad (5.2)$$

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if \mathcal{X} is a convex subset of a normed space, and we set

$$\eta_{k,l} := \zeta_l\left(\frac{k}{M}\right), \quad k = 0, \dots, M, \quad (5.3)$$

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where, for $l \in N_1(0)$, $\zeta_l : [0, 1] \rightarrow \mathcal{X}$ is a minimal-length geodesic from η_l to η'_l if \mathcal{X} is a compact manifold. Note that, since \mathcal{X} is assumed to be compact, there exist $C_* > 0$ such that $\sum_{l \in N_1(0)} \rho(\xi_l, \xi'_l) \leq C_*$ uniformly in $\underline{\xi}, \underline{\xi}' \in \mathcal{X}^{N_1(0)} \times \mathcal{X}^{N_1(0)}$. Our assumptions on \mathcal{X} then imply

$$\sum_{l \in N_1(0)} \rho(\eta_{k,l}, \eta_{k-1,l}) = \frac{1}{M} \sum_{l \in N_1(0)} \rho(\eta_l, \eta'_l) < \frac{C_*}{M}. \quad (5.4) \quad \boxed{\text{kid}}$$

In the following, we choose M in (5.2)-(5.3) large enough so that $C_* < M$. Now suppose that the following inequality holds for some $\lambda > 0$

$$\rho_W \left(\mu^\beta(\cdot | \underline{\eta}_k), \mu^\beta(\cdot | \underline{\eta}_{k-1}) \right) \leq \lambda \sum_{l \in N_1(0)} \rho(\eta_{k,l}, \eta_{k-1,l}), \quad k = 0, \dots, M. \quad (5.5) \quad \boxed{\text{Dobr cd}}$$

Then, by virtue of the the first equality in (5.4), we have,

$$\rho_W \left(\mu^\beta(\cdot | \underline{\eta}), \mu^\beta(\cdot | \underline{\eta}') \right) \leq \sum_{k=1}^M \rho_W \left(\mu^\beta(\cdot | \underline{\eta}_k), \mu^\beta(\cdot | \underline{\eta}_{k-1}) \right) \leq \lambda \sum_{l \in N_1(0)} \rho(\eta_l, \eta'_l). \quad (5.6) \quad \boxed{\text{kift}}$$

In the rest of the proof, we prove an inequality of type (5.5). To do that, we first derive a series of key-estimates for general $\underline{\eta}, \underline{\eta}' \in \mathcal{X}^{N_1(0)}$. For this part of the proof, it is in fact enough that (\mathcal{X}, ρ) is a compact metric space for instance. Given $\xi_0 \in \mathcal{X}$ and $\underline{\eta} \in \mathcal{X}^{N_1(0)}$, define for $\beta > 0$,

$$f^\beta(\xi_0 | \underline{\eta}) := \exp \left(-\beta \left(\mathcal{V}(\xi_0) + \sum_{l \in N_1(0)} \mathcal{V}_l(\xi_0, \eta_l) \right) \right),$$

so that (see (3.10))

$$\mu^\beta(A | \underline{\eta}) := \frac{\int_A f^\beta(\xi_0 | \underline{\eta}) \mu_0(d\xi_0)}{\int_{\mathcal{X}} f^\beta(\xi_0 | \underline{\eta}) \mu_0(d\xi_0)}, \quad A \in \mathcal{B}(\mathcal{X}).$$

Let $(\underline{\eta}, \underline{\eta}') \in \mathcal{X}^{N_1(0)} \times \mathcal{X}^{N_1(0)}$ be fixed. We now show that for all $\beta > 0$ satisfying

$$\theta_\beta := C_0 \beta \sum_{l \in N_1(0)} \rho(\eta_l, \eta'_l) < 1, \quad (5.7) \quad \boxed{\text{cond1beta}}$$

and for all $A \in \mathcal{B}(\mathcal{X})$, the following holds

$$|\mu^\beta(A | \underline{\eta}') - \mu^\beta(A | \underline{\eta})| \leq 2\theta_\beta(1 + \theta_\beta). \quad (5.8) \quad \boxed{\text{intr11}}$$

By using the inequality

$$e^\theta - 1 \leq \theta(1 + \theta), \quad 0 \leq \theta < 1, \quad (5.9) \quad \boxed{\text{expbound}}$$

followed by assumption (ii), we obtain, under conditions (5.7), that

$$|f^\beta(\xi_0 | \underline{\eta}') - f^\beta(\xi_0 | \underline{\eta})| = \left| e^{\beta \sum_{l \in N_1(0)} (\mathcal{V}_l(\xi_0, \eta_l) - \mathcal{V}_l(\xi_0, \eta'_l))} - 1 \right| f^\beta(\xi_0 | \underline{\eta}) \leq \theta_\beta(1 + \theta_\beta) f^\beta(\xi_0 | \underline{\eta}).$$

Denoting

$$Z^\beta(\underline{\eta}) := \int_{\mathcal{X}} f^\beta(\xi_0 | \underline{\eta}) \mu_0(d\xi_0),$$

write

$$\begin{aligned} \mu^\beta(A | \underline{\eta}') - \mu^\beta(A | \underline{\eta}) &= \\ \frac{1}{Z^\beta(\underline{\eta})} \int_A (f^\beta(\xi_0 | \underline{\eta}') - f^\beta(\xi_0 | \underline{\eta})) \mu_0(d\xi_0) &+ \frac{\mu^\beta(A | \underline{\eta}')}{Z^\beta(\underline{\eta})} \int_{\mathcal{X}} (f^\beta(\xi_0 | \underline{\eta}') - f^\beta(\xi_0 | \underline{\eta})) \mu_0(d\xi_0). \end{aligned}$$

Under the conditions (5.7), it then follows from the above that

$$|\mu^\beta(A|\underline{\eta}') - \mu^\beta(A|\underline{\eta})| \leq (1 + \theta_\beta)\theta_\beta (\mu^\beta(A|\underline{\eta}) + \mu^\beta(A|\underline{\eta}')). \quad (5.10) \quad \boxed{\text{folim}}$$

This proves (5.8). Subsequently assume that θ_β in (5.7) is small enough so that

$$\kappa_\beta := \theta_\beta(1 + \theta_\beta) < 1. \quad (5.11) \quad \boxed{\text{cond2beta}}$$

Under this condition, we prove the following upper bound for the Wasserstein distance

$$\rho_W(\mu^\beta(\cdot|\underline{\eta}), \mu^\beta(\cdot|\underline{\eta}')) \leq 6C_1(\beta)\kappa_\beta. \quad (5.12) \quad \boxed{\text{rWdte}}$$

We follow a strategy similar to the one used in the proof of Proposition 7.5, see Section 7.2 below. We cover \mathcal{X} by a number N of open balls $B_i \subset \mathcal{X}$, $i = 1, \dots, N$ of radius $\frac{1}{2}\epsilon$ with $0 < \epsilon < 1$, and define

$$A_i := B_i \setminus \bigcup_{j=1}^{i-1} B_j, \quad i = 1, \dots, N.$$

Note that the A_i 's form a partition of \mathcal{X} . We may assume that $\mu^\beta(A_i|\underline{\eta}') > 0$ for all $i = 1, \dots, N$, and define, for any $E \in \mathcal{B}(\mathcal{X})$, the measures

$$\nu_{\underline{\eta}, \underline{\eta}'}^\beta(E) := \mu^\beta(E|\underline{\eta}) - \frac{1 - \kappa_\beta}{1 + \kappa_\beta} \sum_{i=1}^N \mu^\beta(E \cap A_i|\underline{\eta}) = \frac{2\kappa_\beta}{1 + \kappa_\beta} \mu^\beta(E|\underline{\eta}), \quad (5.13) \quad \boxed{\text{nuetetp}}$$

$$\tilde{\nu}_{\underline{\eta}, \underline{\eta}'}^\beta(E) := \mu^\beta(E|\underline{\eta}') - \frac{1 - \kappa_\beta}{1 + \kappa_\beta} \sum_{i=1}^N \mu^\beta(A_i|\underline{\eta}) \frac{\mu^\beta(E \cap A_i|\underline{\eta}')}{\mu^\beta(A_i|\underline{\eta}')}. \quad (5.14) \quad \boxed{\text{tnuetetp}}$$

Note that (5.13) and (5.14) are both positive measures since (5.10) yields, for all $i = 1, \dots, N$,

$$\frac{1 + \kappa_\beta}{1 - \kappa_\beta} \mu^\beta(A_i|\underline{\eta}) \geq \mu^\beta(A_i|\underline{\eta}') \geq \frac{1 - \kappa_\beta}{1 + \kappa_\beta} \mu^\beta(A_i|\underline{\eta}).$$

Then define in $\mathcal{P}(\mathcal{X} \times \mathcal{X})$

$$\sigma_{\underline{\eta}, \underline{\eta}'}^\beta(E \times F) = \frac{1 - \kappa_\beta}{1 + \kappa_\beta} \sum_{i=1}^N \mu^\beta(E \cap A_i|\underline{\eta}) \frac{\mu^\beta(F \cap A_i|\underline{\eta}')}{\mu^\beta(A_i|\underline{\eta}')} + \frac{\nu_{\underline{\eta}, \underline{\eta}'}^\beta(E) \tilde{\nu}_{\underline{\eta}, \underline{\eta}'}^\beta(F)}{\nu_{\underline{\eta}, \underline{\eta}'}^\beta(\mathcal{X})}.$$

Note that $\sigma_{\underline{\eta}, \underline{\eta}'}^\beta(\mathcal{X} \times F) = \mu^\beta(F|\underline{\eta}')$ and $\sigma_{\underline{\eta}, \underline{\eta}'}^\beta(E \times \mathcal{X}) = \mu^\beta(E|\underline{\eta})$ so that $\sigma_{\underline{\eta}, \underline{\eta}'}^\beta \in \Xi_{\mathcal{X}}(\mu^\beta(\cdot|\underline{\eta}), \mu^\beta(\cdot|\underline{\eta}'))$.

(This follows from the fact that $\nu_{\underline{\eta}, \underline{\eta}'}^\beta(\mathcal{X}) = \tilde{\nu}_{\underline{\eta}, \underline{\eta}'}^\beta(\mathcal{X}) = \frac{2\kappa_\beta}{1 + \kappa_\beta}$.) Now,

$$\begin{aligned} \int_{\mathcal{X} \times \mathcal{X}} \rho(\xi, \xi') \sigma_{\underline{\eta}, \underline{\eta}'}^\beta(d\xi, d\xi') &= \frac{1 - \kappa_\beta}{1 + \kappa_\beta} \sum_{i=1}^N \frac{1}{\mu^\beta(A_i|\underline{\eta}')} \int_{A_i} \int_{A_i} \rho(\xi, \xi') \mu^\beta(d\xi|\underline{\eta}) \mu^\beta(d\xi'|\underline{\eta}') \\ &\quad + \frac{1}{\nu_{\underline{\eta}, \underline{\eta}'}^\beta(\mathcal{X})} \int_{\mathcal{X}} \int_{\mathcal{X}} \rho(\xi, \xi') \nu_{\underline{\eta}, \underline{\eta}'}^\beta(d\xi) \tilde{\nu}_{\underline{\eta}, \underline{\eta}'}^\beta(d\xi'). \end{aligned}$$

Using the fact that $\text{diam}(A_i) \leq \epsilon$ in the first term and the triangle inequality in the second term, we have,

$$\int_{\mathcal{X} \times \mathcal{X}} \rho(\xi, \xi') \sigma_{\underline{\eta}, \underline{\eta}'}^\beta(d\xi, d\xi') \leq \epsilon + \int_{\mathcal{X}} \rho(\xi, \tilde{\xi}) \nu_{\underline{\eta}, \underline{\eta}'}^\beta(d\xi) + \int_{\mathcal{X}} \rho(\tilde{\xi}, \xi') \tilde{\nu}_{\underline{\eta}, \underline{\eta}'}^\beta(d\xi'),$$

for some $\tilde{\xi} \in \mathcal{X}$. Since the A_i 's form a partition of \mathcal{X} , the last two terms can be bounded as follows

$$\begin{aligned} \int_{\mathcal{X}} \rho(\xi, \tilde{\xi}) \nu_{\underline{\eta}, \underline{\eta}'}^\beta(d\xi) &\leq 2\kappa_\beta \int_{\mathcal{X}} \rho(\xi, \tilde{\xi}) \mu^\beta(d\xi|\underline{\eta}) \leq 2C_1(\beta)\kappa_\beta, \\ \int_{\mathcal{X}} \rho(\tilde{\xi}, \xi') \tilde{\nu}_{\underline{\eta}, \underline{\eta}'}^\beta(d\xi') &\leq \left(1 - \left(\frac{1 - \kappa_\beta}{1 + \kappa_\beta}\right)^2\right) \int_{\mathcal{X}} \rho(\xi, \tilde{\xi}) \mu^\beta(d\xi|\underline{\eta}') \leq 4C_1(\beta)\kappa_\beta. \end{aligned}$$

Gathering the above estimates, the upper bound (5.12) follows after taking the limit $\epsilon \rightarrow 0$. To conclude the proof of the Proposition, substitute in the right-hand side of (5.12) κ_β with its definition in (5.11) and the explicit expression of θ_β in (5.7), and then replace $(\underline{\eta}, \underline{\eta}')$ by $(\underline{\eta}_k, \underline{\eta}_{k-1})$ (see (5.2)-(5.3)) in both the left-hand side and right-hand side of (5.12). This gives

$$\rho_W(\mu^\beta(\cdot | \underline{\eta}_k), \mu^\beta(\cdot | \underline{\eta}_{k-1})) \leq 6C_1(\beta)C_0\beta \sum_{l \in N_1(0)} \rho(\eta_{k,l}, \eta_{k-1,l}) \left(1 + C_0\beta \sum_{l \in N_1(0)} \rho(\eta_{k,l}, \eta_{k-1,l}) \right).$$

By virtue of the upper bound in (5.4), M can be chosen large enough so that the condition in (5.11), implying the one in (5.7), is satisfied. Moreover, in view of (5.6), the following upper bound

$$\rho_W(\mu^\beta(\cdot | \underline{\eta}), \mu^\beta(\cdot | \underline{\eta}')) \leq 6C_0C_1(\beta)\beta \left(1 + C_0\beta \frac{C_*}{M} \right) \sum_{l \in N_1(0)} \rho(\eta_l, \eta'_l),$$

holds for any $(\underline{\eta}, \underline{\eta}') \in \mathcal{X}^{N_1(0)} \times \mathcal{X}^{N_1(0)}$. It remains to take the limit $M \rightarrow \infty$ and the upper bound in Proposition 5.1 follows from the condition (3.14) in Remark 3.9. \square

Remark 5.2 *The uniqueness result for sufficiently high temperatures in Proposition 5.1 can be extended to the case of more general interactions, as well as unbounded spins. The conditions of Proposition 5.1 have to be modified accordingly.*

5.2 The Ising and Potts models.

IsPo

Theorem 5.1 does not apply to the Potts model or the Ising model.

Potts model in dimension $d \geq 2$. The single-spin space is a finite set $\mathcal{X} = \{1, \dots, q\}$ with $q \in \mathbb{N}$, $q > 1$ and the pair-interaction potential is

$$\mathcal{V}_k(s_j, s_k) = -J\delta_{s_j, s_k}, \quad s_j, s_k \in \{1, \dots, q\},$$

for some real coupling constant $J > 0$. The discrete metric is given by

$$\rho(s, s') := 1 - \delta_{s, s'}, \quad s, s' \in \{1, \dots, q\},$$

where $\delta_{s, s'}$ is the Kronecker delta, i.e., $\delta_{s, s'} = 1$ when $s = s'$ and $\delta_{s, s'} = 0$ otherwise. Given two probability measures $\mu, \mu' \in \mathcal{P}(\mathcal{X})$, the minimizing measure $\sigma \in \mathcal{P}(\mathcal{X} \times \mathcal{X})$ for $\rho_W(\mu, \mu')$ satisfies

$$\begin{aligned} \sigma(s, s') &= p_{s, s'}, \quad s, s' \in \{1, \dots, q\}; \\ \sum_{s'=1}^q p_{s, s'} &= \mu(\{s\}), \quad \sum_{s=1}^q p_{s, s'} = \mu'(\{s'\}); \end{aligned} \tag{5.15} \quad \text{consss}$$

and minimizes

$$\sum_{s, s'=1}^q \rho(s, s') \sigma(s, s') = \sum_{\substack{s, s'=1 \\ s \neq s'}}^q p_{s, s'} = 1 - \sum_{s=1}^q p_{s, s}, \tag{5.16} \quad \text{tominimz}$$

where, to derive the last identity in the above right-hand side, we used that

$$\sum_{s, s'=1}^q p_{s, s'} = \sum_{s=1}^q \mu(\{s\}) = \sum_{s'=1}^q \mu'(\{s'\}) = 1. \tag{5.17} \quad \text{simplif}$$

We now claim that

$$\rho_W(\mu, \mu') = \frac{1}{2} \sum_{s=1}^q |\mu(\{s\}) - \mu'(\{s\})|. \tag{5.18} \quad \text{idrowpot}$$

For convenience's sake, the proof of (5.18) is deferred to the next subsection. Denoting

$$E_{s_j}(\underline{s}) := -J \sum_{l \in N_1(j)} \delta_{s_j, s_l},$$

by using (5.18), we have,

$$\rho_W \left(\mu_j^\beta(\cdot | \underline{s}), \mu_j^\beta(\cdot | \underline{s}') \right) = \frac{1}{2} \sum_{s_j=1}^q \left| \frac{e^{-\beta E_{s_j}(\underline{s})}}{\sum_{s_j=1}^q e^{-\beta E_{s_j}(\underline{s})}} - \frac{e^{-\beta E_{s_j}(\underline{s}')}}{\sum_{s_j=1}^q e^{-\beta E_{s_j}(\underline{s}')}} \right|.$$

Notice that

$$\left| \sum_{l \in N_1(j)} (\delta_{s_j, s'_l} - \delta_{s_j, s_l}) \right| \leq \sum_{l \in N_1(j)} (1 - \delta_{s_l, s'_l}) = \sum_{l \in N_1(j)} \rho(s_l, s'_l).$$

Assume now that $\beta > 0$ is small enough so that

$$\beta J \sum_{l \in N_1(j)} \rho(s_l, s'_l) < 1. \quad (5.19) \quad \boxed{\text{conbetaaa}}$$

By (5.9), we obtain the upper bound

$$\left| e^{-\beta E_{s_j}(\underline{s})} - e^{-\beta E_{s_j}(\underline{s}')} \right| = e^{-\beta E_{s_j}(\underline{s})} \left| 1 - e^{\beta J \sum_{l \in N_1(j)} (\delta_{s_j, s'_l} - \delta_{s_j, s_l})} \right| \leq 2\beta J e^{-\beta E_{s_j}(\underline{s})} \sum_{l \in N_1(j)} \rho(s_l, s'_l).$$

It remains to use that, for any $\underline{\sigma} \in \{1, \dots, q\}^{N_1(j)}$,

$$\sum_{s_j=1}^q e^{-\beta E_{s_j}(\underline{\sigma})} = \prod_{k \in N_1(j)} \sum_{s_j=1}^q e^{\beta J \delta_{s_j, \sigma_k}} = \prod_{k \in N_1(j)} (q - 1 + e^{\beta J}) = (q - 1 + e^{\beta J})^{2d},$$

which implies

$$\rho_W \left(\mu_j^\beta(\cdot | \underline{s}), \mu_j^\beta(\cdot | \underline{s}') \right) \leq 2\beta J \sum_{l \in N_1(j)} \rho(s_l, s'_l).$$

Note that if $2d\beta J < 1$, then (5.19) is satisfied. By (3.14) in Remark 3.9, we conclude that there is no phase transition if $\beta < (4dJ)^{-1}$.

Proof of (5.18). In view of (5.16), we need to maximize the diagonal elements of the matrix $(p_{s,s'})_{1 \leq s, s' \leq q}$ where the s -th row and s' -th column correspond to $\mu(\{s\})$ and $\mu'(\{s'\})$ respectively. If $\mu'(\{s\}) \geq \mu(\{s\})$ for some $s \in \{1, \dots, q\}$, $p_{s,s}$ is maximized when $p_{s,s} = \mu(\{s\})$. Conversely, if $\mu'(\{s\}) \leq \mu(\{s\})$ for some $s \in \{1, \dots, q\}$, $p_{s,s}$ is maximized when $p_{s,s} = \mu'(\{s\})$. Assume for the moment that, for such a choice of $p_{s,s}$'s, we can always find some off-diagonal elements $p_{s,s'} \geq 0$, $s \neq s'$ so that the conditions in (5.15) and (5.17) are fulfilled. Let $r := |\{s \in \{1, \dots, q\} : \mu'(\{s\}) \geq \mu(\{s\})\}|$ with $1 \leq r < q$ (as a result of (5.17)). Even if it means renaming the $\mu(\{s\})$'s, we can always assume that $\mu'(\{s\}) \geq \mu(\{s\})$, $s \in \{1, \dots, r\}$. Then, by using (5.17), we can write,

$$\begin{aligned} 1 - \sum_{s=1}^q p_{s,s} &= \frac{1}{2} \left(1 - \sum_{s=r+1}^q \mu'(\{s\}) - \sum_{s=1}^r \mu(\{s\}) \right) + \frac{1}{2} \left(1 - \sum_{s=1}^r \mu(\{s\}) - \sum_{s=r+1}^q \mu'(\{s\}) \right) \\ &= \frac{1}{2} \sum_{s=1}^r (\mu'(\{s\}) - \mu(\{s\})) + \frac{1}{2} \sum_{s=r+1}^q (\mu(\{s\}) - \mu'(\{s\})) = \frac{1}{2} \sum_{s=1}^r |\mu'(\{s\}) - \mu(\{s\})|. \end{aligned}$$

This proves (5.18). We now show that we can always construct a $q \times q$ lower-triangular matrix $(p_{s,s'})_{1 \leq s, s' \leq q}$ with diagonal elements $p_{s,s} = \mu(\{s\})$ for $s \in \{1, \dots, r\}$, $p_{s,s} = \mu'(\{s\})$ for $s \in \{r+1, \dots, q\}$ and off-diagonal elements $p_{s,s'} \geq 0$, $s \neq s'$ such that (5.15) and (5.17) are fulfilled.

Set $p_s := \mu(\{s\})$ and $p'_s := \mu'(\{s\})$, $s \in \{1, \dots, q\}$. Hereafter, we use (i, j) in place of (s, s') for the matrix indices. Let $r := |\{i \in \{1, \dots, q\} : p'_i \geq p_i\}|$ with $1 \leq r < q$. As before, even if it means renaming the p_i 's, we can always assume that $p'_i \geq p_i$, $i \in \{1, \dots, r\}$. Recall the conditions

$$\begin{aligned} \sum_{j=1}^r p_{k,j} &= p_k - p'_k, \quad k = r+1, \dots, q, \\ \sum_{k=r+1}^q p_{k,j} &= p'_j - p_j, \quad j = 1, \dots, r. \end{aligned} \tag{5.20} \quad \boxed{\text{condrefr}}$$

We start by reordering the rows and columns so that $p'_j - p_j$ is in increasing order for $j = 1, \dots, r$, and $p_k - p'_k$ is in decreasing order for $k = r+1, \dots, q$. We then proceed by induction on q , eliminating the q -th row and column. We want to determine $p_{q,j}$ for $1 \leq j \leq r$ such that

$$\sum_{j=1}^r p_{q,j} = p_q - p'_q \quad \text{and} \quad p_{q,j} + \sum_{k=r+1}^{q-1} p_{k,j} = p'_j - p_j.$$

Now since $p_1 + \dots + p_q = p'_1 + \dots + p'_q$,

$$\sum_{j=1}^r (p'_j - p_j) = \sum_{k=r+1}^q (p_k - p'_k) \geq p_q - p'_q,$$

so that in principle this is possible. In fact, if $r = q-1$, then we can simply put $p_{q,j} = p'_j - p_j$ and no induction is needed. Otherwise, let j_0 be the maximal value such that $\sum_{j=j_0}^r (p'_j - p_j) \geq p_q - p'_q$. Put $p_{q,j} = 0$ for $1 \leq j < j_0$, $p_{q,j} = p'_j - p_j$ for $j_0 < j \leq r$ and $p_{q,j_0} = p_q - p'_q - \sum_{j=j_0+1}^r (p'_j - p_j) \geq 0$. To prove the claim by induction, we then need to show that we can continue this procedure for $k = q-1$. From (5.20), the reduced set of equations reads

$$\begin{aligned} \sum_{j=1}^{j_0} p_{k,j} &= p_k - p'_k, \quad k = r+1, \dots, q-1; \\ p_{k,j} &= 0, \quad j = j_0+1, \dots, r; \quad k = r+1, \dots, q-1; \\ \sum_{k=r+1}^{q-1} p_{k,j} &= p'_j - p_j, \quad j = 1, \dots, j_0-1; \\ \sum_{k=r+1}^{q-1} p_{k,j_0} + p_q - p'_q &= p'_{j_0} - p_{j_0} + \sum_{j=j_0+1}^r (p'_j - p_j). \end{aligned}$$

We therefore need that

$$p'_{j_0} - p_{j_0} + \sum_{j=j_0+1}^r (p'_j - p_j) \geq p_q - p'_q.$$

This holds by definition of j_0 . Therefore, the procedure can be continued.

Ising model in dimension $d \geq 2$. The single-spin space is $\mathcal{X} = \{-1, 1\}$ and the pair-interaction potential is

$$\mathcal{V}_{l-j}(s_j, s_l) = -J(|j-l|)s_j s_l, \quad s_j, s_l \in \{-1, 1\}, \quad l \neq j,$$

where J is a sufficiently fast decreasing positive function satisfying $\sum_{\substack{k \in \mathbb{Z}^d \\ k \neq 0}} J(|k|) < \infty$.

For $\mathcal{X} = \{-1, 1\}$, the discrete metric is given by

$$\rho(s, s') := 1 - \delta_{s,s'}.$$

By virtue of (5.18) with $q = 2$, given two probability measures $\mu, \mu' \in \mathcal{P}(\mathcal{X})$, we have,

$$\rho_W(\mu, \mu') = \frac{1}{2} \sum_{s \in \{-, +\}} |\mu(\{s\}) - \mu'(\{s\})| = |\mu(\{+\}) - \mu'(\{+\})|. \quad (5.21) \quad \text{rhoIsing}$$

Denoting

$$S_0(\underline{s}) := \sum_{\substack{j \in \mathbb{Z}^d \\ j \neq 0}} J(|j|) s_j,$$

by using (5.21), we have,

$$\begin{aligned} \rho_W(\mu_0^\beta(\cdot | \underline{s}), \mu_0^\beta(\cdot | \underline{s}')) &= \left| \frac{e^{\beta S_0(\underline{s})}}{e^{\beta S_0(\underline{s})} + e^{-\beta S_0(\underline{s})}} - \frac{e^{\beta S_0(\underline{s}')}}{e^{\beta S_0(\underline{s}')} + e^{-\beta S_0(\underline{s}')}} \right| \\ &= \frac{1}{2} |\tanh(\beta S_0(\underline{s})) - \tanh(\beta S_0(\underline{s}'))|. \end{aligned} \quad (5.22) \quad \text{Isingid}$$

This implies that

$$\rho_W(\mu_0^\beta(\cdot | \underline{s}), \mu_0^\beta(\cdot | \underline{s}')) \leq \frac{1}{2} \beta |S_0(\underline{s}) - S_0(\underline{s}')| \leq \frac{1}{2} \beta \sum_{\substack{j \in \mathbb{Z}^d \\ j \neq 0}} J(|j|) |s_j - s'_j| \leq \beta \sum_{\substack{j \in \mathbb{Z}^d \\ j \neq 0}} J(|j|) \rho(s_j, s'_j).$$

By (4.3), we conclude that there is no phase transition if $\beta(\sum_{\substack{j \in \mathbb{Z}^d \\ j \neq 0}} J(|j|)) < 1$.

Remark 5.3 *In the 1-dimensional Ising model with nearest-neighbour interactions, it can be inferred from (5.22) that there is no phase transition for all $\beta > 0$. Indeed, (5.22) is non-zero when: i) $s_{-1} = s_1$ and $s'_j = -s_j$; ii) $s_{-1} = s_1$ and $s'_{-1} \neq s'_1$; iii) $s_{-1} \neq s_1$ and $s'_{-1} = s_1$. However, only the situations ii) and iii) can occur for which (5.22) equals to $\frac{1}{2} \tanh(2\beta J) < \frac{1}{2}$ (here $J > 0$ is the coupling constant). It remains to use the condition in (3.14) (with $d = 1$) to conclude.*

6 Applications - Part II: Convex perturbation of the Gaussian free field model.

classiC

In this section, we focus on some models of Euclidean lattice field theories, see, e.g., [25, 46].

Notations. The state space is \mathbb{R}^n with its standard topology and Lebesgue measure. The standard inner product and Euclidean norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ respectively. Below, the norm $\|\cdot\|$ on the space $\mathbb{M}_n(\mathbb{C})$ of $n \times n$ matrices denotes the operator norm.

6.1 The Gaussian free field model with n -dimensional spins.

Gaussfr
class

Proposition 6.1 *For any integer $n > 1$, let $L \in \mathbb{M}_n(\mathbb{R})$ be a non-negative symmetric matrix. Consider the following formal Hamiltonians with nearest-neighbour interactions defined on $(\mathbb{R}^n)^{\mathbb{Z}^d}$ as*

$$H_n^{cla}(\underline{x}) := \sum_{j \in \mathbb{Z}^d} \frac{1}{2} \alpha \|x_j\|^2 - \sum_{j \in \mathbb{Z}^d} \sum_{l \in N_1(j)} \langle x_j, L x_l \rangle. \quad (6.1) \quad \text{Hn}$$

Then, provided that $\alpha > 2d\|L\|$, there exists, for all $\beta > 0$, a unique limit Gibbs distribution in $\mathcal{P}((\mathbb{R}^n)^{\mathbb{Z}^d})$ associated with H_n^{cla} .

Remark 6.2 *The interactions between particles are of ferromagnetic type since L is non-negative.*

Proof. Fix $j \in \mathbb{Z}^d$ and $\underline{y}, \underline{y}' \in (\mathbb{R}^n)^{N_1(j)}$ distinct. Set $\underline{y} := \sum_{l \in N_1(j)} y_l \in \mathbb{R}^n$, $\underline{y}' := \sum_{l \in N_1(j)} y'_l \in \mathbb{R}^n$. In view of (6.1), the corresponding 1-point Gibbs distribution reads

$$\mu_j^{\beta, (n)}(A | \underline{y}) := \frac{1}{Z_j^{\beta, (n)}(\underline{y})} \int_A \exp \left(-\beta \left(\frac{1}{2} \alpha \|x\|^2 - \langle x, L \underline{y} \rangle \right) \right) d^n x, \quad A \in \mathcal{B}(\mathbb{R}^n),$$

with

$$Z_j^{\beta,(n)}(\underline{y}) := \int_{\mathbb{R}^n} \exp \left(-\beta \left(\frac{1}{2} \alpha \|x\|^2 - \langle x, Ly \rangle \right) \right) d^n x.$$

We now construct a coupling in $\mathcal{P}(\mathbb{R}^n \times \mathbb{R}^n)$ such that the marginals coincide with the 1-point Gibbs distribution above with the different boundary conditions, see (6.2) and (6.3) below.

Set $\theta_{y'-y}^{(n)} := \frac{1}{\alpha} L(y' - y)$. Define $\sigma_{j;\underline{y},\underline{y}'}^{\beta,(n)} \in \mathcal{P}(\mathbb{R}^n \times \mathbb{R}^n)$ as

$$\sigma_{j;\underline{y},\underline{y}'}^{\beta,(n)}(A \times B) := \frac{1}{Z_j^{\beta,(n)}(\underline{y})} \int_{A \times B} \exp \left(-\beta \left(\frac{1}{2} \alpha \|x\|^2 - \langle x, Ly \rangle \right) \right) \delta \left(x' - x - \theta_{y'-y}^{(n)} \right) d^n x d^n x', \quad (6.2) \quad \text{mean}$$

for any $A, B \in \mathcal{B}(\mathbb{R}^n)$. Here, δ denotes the Dirac measure. It is easily seen that

$$\sigma_{j;\underline{y},\underline{y}'}^{\beta,(n)}(A \times \mathbb{R}^n) = \mu_j^{\beta,(n)}(A|\underline{y}) \quad \text{and} \quad \sigma_{j;\underline{y},\underline{y}'}^{\beta,(n)}(\mathbb{R}^n \times B) = \mu_j^{\beta,(n)}(B|\underline{y}'). \quad (6.3) \quad \text{measn}$$

From (6.2), we have, by direct calculation,

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} \|x - x'\| \sigma_{j;\underline{y},\underline{y}'}^{\beta,(n)}(d^n x, d^n x') = \frac{1}{\alpha} \|L(y' - y)\|.$$

The proposition now follows from the condition in (3.14) since the above identity yields

$$\rho_W \left(\mu_j^{\beta,(n)}(\cdot|\underline{y}), \mu_j^{\beta,(n)}(\cdot|\underline{y}') \right) \leq \frac{1}{\alpha} \|L\| \sum_{l \in N_1(j)} \|y_l - y'_l\|. \quad \square$$

casen1

Remark 6.3 In the special case $n = 1$, the formal Hamiltonian on $\mathbb{R}^{\mathbb{Z}^d}$ commonly considered is

$$H_1^{\text{cla}}(\underline{x}) := \sum_{j \in \mathbb{Z}^d} \frac{1}{2} \alpha x_j^2 + \sum_{j \in \mathbb{Z}^d} \sum_{l \in N_1(j)} \frac{1}{2} (x_j - x_l)^2. \quad (6.4) \quad \text{H1}$$

We can mimic the arguments used in the proof of Proposition 6.1 by gathering the quadratic terms together, giving the factor $\alpha_d := \alpha + 4d$. The existence and uniqueness, for all $\beta > 0$, of a limit Gibbs distribution in $\mathcal{P}(\mathbb{R}^{\mathbb{Z}^d})$ is guaranteed whenever $\alpha > 0$.

remmm

Remark 6.4 The method used in the proof given above allows us to extend the uniqueness result of Proposition 6.1 to long-range pair-interaction potentials, typically of the form

$$\sum_{j \in \mathbb{Z}^d} \sum_{\substack{l \in \mathbb{Z}^d \\ l \neq j}} J(|j - l|) (x_j - x_l)^2, \quad (6.5) \quad \text{forpot}$$

where J is a sufficiently fast decreasing function satisfying $\sum_{j \in \mathbb{Z}^d} J(|j|) < \infty$.

6.2 Convex perturbation of the Gaussian free field model.

convxx

We extend the uniqueness result to the Gaussian free field model perturbed by a convex self-interaction potential. Proposition 6.5 covers the case of 1-dimensional spins while Proposition 6.9 covers the case of n -dimensional spins for a particular class of convex potentials.

6.2.1 The case of 1-dimensional spins.

convex

Proposition 6.5 Consider the following formal Hamiltonian with nearest-neighbour interactions defined on $\mathbb{R}^{\mathbb{Z}^d}$, which is a perturbation of (6.4) in the sense

$$\tilde{H}_1^{\text{cla}}(\underline{x}) := \sum_{j \in \mathbb{Z}^d} \left(\frac{1}{2} \alpha x_j^2 + g(x_j) \right) + \sum_{j \in \mathbb{Z}^d} \sum_{l \in N_1(j)} \frac{1}{2} (x_j - x_l)^2. \quad (6.6) \quad \text{Hconv}$$

Assume that $g : \mathbb{R} \rightarrow \mathbb{R}$ is a convex function. Then provided that $\alpha > 0$, there exists, for all $\beta > 0$, a unique limit Gibbs distribution in $\mathcal{P}(\mathbb{R}^{\mathbb{Z}^d})$ associated with \tilde{H}_1^{cla} .

Remark 6.6 In the lattice approximation of quantum field theory, the ϕ^4 model corresponds to $g(x) = cx^4$, $c > 0$ in (6.6). Further models are covered by our assumptions, see, e.g., [25, 46].

Remark 6.7 The uniqueness result of Proposition 6.5 can be extended to long-range pair interaction potentials of the form (6.5), see Remark 6.4.

To prove Proposition 6.5, we need the following lemma.

incre

Lemma 6.8 Let $\nu \in \mathcal{P}(\mathbb{R})$ be a probability measure. Then, for any ν -integrable and non-decreasing function f on \mathbb{R} , the function

$$z \mapsto w_\nu(z) := \frac{1}{\nu((-\infty, z])} \int_{-\infty}^z f(x) \nu(dx)$$

is non-decreasing on \mathbb{R} .

Proof. Suppose that $z < z'$. We have

$$w_\nu(z') - w_\nu(z) = \frac{1}{\nu((-\infty, z'])} \left(\int_z^{z'} f(x) \nu(dx) - \frac{\nu((z, z'])}{\nu((-\infty, z])} \int_{-\infty}^z f(x) \nu(dx) \right).$$

Since f is non-decreasing by assumption, $f(x) \geq f(z)$ on $(z, z']$ and $f(x) \leq f(z)$ on $(-\infty, z]$. As a result,

$$\nu((-\infty, z']) (w_\nu(z') - w_\nu(z)) \geq f(z) \left(\nu((z, z']) - \frac{\nu((z, z']) \nu((-\infty, z])}{\nu((-\infty, z])} \right) = 0. \quad \square$$

We now turn to

Proof of Proposition 6.5. Fix $j \in \mathbb{Z}^d$ and $y, y' \in \mathbb{R}^{N_1(j)}$ distinct. Set $y := \sum_{l \in N_1(j)} y_l \in \mathbb{R}$ and $y' := \sum_{l \in N_1(j)} y'_l \in \mathbb{R}$. From (6.6), the corresponding 1-point Gibbs distribution reads

$$\mu_j^\beta(A|\underline{y}) := \frac{1}{Z_j^\beta(\underline{y})} \int_A \exp \left(-\beta \left(\frac{1}{2} \alpha_d x^2 + g(x) - xy \right) \right) dx, \quad A \in \mathcal{B}(\mathbb{R}),$$

where $\alpha_d := \alpha + 4d$, and with

$$Z_j^\beta(\underline{y}) := \int_{\mathbb{R}} \exp \left(-\beta \left(\frac{1}{2} \alpha_d x^2 + g(x) - xy \right) \right) dx.$$

We now construct a coupling in $\mathcal{P}(\mathbb{R} \times \mathbb{R})$ such that the marginals coincide with the 1-point Gibbs distribution above with the different boundary conditions, see (6.8) and (6.9) below. Let $(z_m)_{m \in \mathbb{Z}}$ be the increasing sequence of points $z_m := m\delta$ with $0 < \delta < 1$. Define the sequence $(z'_m)_{m \in \mathbb{Z}}$ such that

$$\mu_j^\beta((-\infty, z'_m]|\underline{y}') = \mu_j^\beta((-\infty, z_m]|\underline{y}). \quad (6.7) \quad \text{eqq1}$$

This implies

$$\mu_j^\beta((z'_{m-1}, z'_m]|\underline{y}') = \mu_j^\beta((z_{m-1}, z_m]|\underline{y}).$$

From the foregoing, define the coupling $\sigma_{j;\underline{y},\underline{y}'}^\beta \in \mathcal{P}(\mathbb{R} \times \mathbb{R})$ as

$$\sigma_{j;\underline{y},\underline{y}'}^\beta(A \times B) := \sum_{m \in \mathbb{Z}} \frac{\mu_j^\beta(A \cap (z_{m-1}, z_m]|\underline{y}) \mu_j^\beta(B \cap (z'_{m-1}, z'_m]|\underline{y}')}{\mu_j^\beta((z_{m-1}, z_m]|\underline{y})}, \quad (6.8) \quad \text{sigmyy'}$$

for any $A, B \in \mathcal{B}(\mathbb{R})$. It is easily seen that

$$\sigma_{j;\underline{y},\underline{y}'}^\beta(A \times \mathbb{R}) = \mu_j^\beta(A|\underline{y}) \quad \text{and} \quad \sigma_{j;\underline{y},\underline{y}'}^\beta(\mathbb{R} \times B) = \mu_j^\beta(B|\underline{y}'). \quad (6.9) \quad \text{coinn}$$

Furthermore,

$$\begin{aligned} \int_{\mathbb{R} \times \mathbb{R}} |x - x'| \sigma_{j;\underline{y},\underline{y}'}^\beta(dx, dx') &= \sum_{m \in \mathbb{Z}} \int_{(z_{m-1}, z_m] \times (z'_{m-1}, z'_m]} |x - x'| \frac{\mu_j^\beta(dx|\underline{y}) \mu_j^\beta(dx'|\underline{y}')}{\mu_j^\beta((z_{m-1}, z_m]|\underline{y})} \\ &\leq \sum_{m \in \mathbb{Z}} \left(\sup_{(z, z') \in (z_{m-1}, z_m] \times (z'_{m-1}, z'_m]} |z - z'| \right) \frac{\mu_j^\beta((z_{m-1}, z_m]|\underline{y}) \mu_j^\beta((z'_{m-1}, z'_m]|\underline{y}')}{\mu_j^\beta((z_{m-1}, z_m]|\underline{y})}. \end{aligned} \quad (6.10) \quad \text{temp}$$

Next, we claim that under our conditions (a proof is given below)

$$\sup_{(z, z') \in (z_{m-1}, z_m] \times (z'_{m-1}, z'_m]} |z - z'| \leq \delta + \frac{1}{\alpha_d} |y - y'|. \quad (6.11) \quad \text{claim1}$$

Inserting (6.11) into the right-hand side of (6.10) and letting $\delta \rightarrow 0$, we get,

$$\rho_W \left(\mu_j^\beta(\cdot|\underline{y}), \mu_j^\beta(\cdot|\underline{y}') \right) \leq \frac{1}{\alpha_d} |y - y'| \leq \frac{1}{\alpha_d} \sum_{l \in N_1(j)} |y_l - y'_l|,$$

and the Proposition follows from the condition in (3.14). To complete the proof of the Proposition, we now prove (6.11). Suppose that $y < y'$ (the case $y > y'$ can be treated by similar arguments). First, we show that $z_m \leq z'_m$ for all $m \in \mathbb{Z}$. It suffices to remark that

$$\mu_j^\beta((-\infty, z]|\underline{y}') = \frac{\int_{-\infty}^z e^{\beta x(y' - y)} \mu_j^\beta(dx|\underline{y})}{\int_{-\infty}^{+\infty} e^{\beta x(y' - y)} \mu_j^\beta(dx|\underline{y})} \leq \mu_j^\beta((-\infty, z]|\underline{y}),$$

as a result of Lemma 6.8 since $x \mapsto e^{\beta x(y' - y)}$ is increasing. Set $z = z_m$ and then use (6.7). Secondly, we show that $z'_m \leq z_m + \theta_{y' - y}$ with $\theta_{y' - y} := \frac{1}{\alpha_d}(y' - y) > 0$ proving (6.11) when $y < y'$. Due to (6.7), it suffices to prove that

$$\mu_j^\beta((-\infty, z + \theta_{y' - y}]|\underline{y}') \geq \mu_j^\beta((-\infty, z]|\underline{y}). \quad (6.12) \quad \text{inh}$$

By a change of variables, we have the rewriting

$$\mu_j^\beta((-\infty, z + \theta_{y' - y}]|\underline{y}') = \frac{\int_{-\infty}^z e^{-\frac{1}{2}\beta\alpha_d(x - \alpha_d^{-1}y)^2 - \beta g(x + \theta_{y' - y})} dx}{\int_{-\infty}^{+\infty} e^{-\frac{1}{2}\beta\alpha_d(x - \alpha_d^{-1}y)^2 - \beta g(x + \theta_{y' - y})} dx}.$$

This yields the following identity

$$\begin{aligned} &\frac{\int_{-\infty}^{+\infty} e^{-\frac{1}{2}\beta\alpha_d(x - \alpha_d^{-1}y)^2 - \beta g(x + \theta_{y' - y})} dx}{\int_{-\infty}^z e^{-\frac{1}{2}\beta\alpha_d(x - \alpha_d^{-1}y)^2 - \beta g(x)} dx} \left(\mu_j^\beta((-\infty, z + \theta_{y' - y}]|\underline{y}') - \mu_j^\beta((-\infty, z]|\underline{y}) \right) \\ &= \frac{1}{\mu_j^\beta((-\infty, z]|\underline{y})} \int_{-\infty}^z e^{-\beta(g(x + \theta_{y' - y}) - g(x))} \mu_j^\beta(dx|\underline{y}) - \int_{-\infty}^{\infty} e^{-\beta(g(x + \theta_{y' - y}) - g(x))} \mu_j^\beta(dx|\underline{y}). \end{aligned} \quad (6.13) \quad \text{cleq}$$

Note that $x \mapsto g(x + \theta) - g(x)$, $\theta > 0$ is non-decreasing since g is convex. By multiplying the right-hand side of (6.13) by (-1) then applying Lemma 6.8 with $f(x) = -e^{-\beta[g(x + \theta_{y' - y}) - g(x)]}$ and $\nu = \mu_j^\beta(\cdot|\underline{y})$, we conclude that (6.13) is non-negative. Therefore, (6.12) follows. \square

6.2.2 The case of n -dimensional spins.

convex2

Proposition 6.9 *For any integer $n > 1$, let $L \in \mathbb{M}_n(\mathbb{R})$ be a non-negative symmetric matrix. Consider the following formal Hamiltonians with nearest-neighbour interactions defined on $(\mathbb{R}^n)^{\mathbb{Z}^d}$, which are a perturbations of (6.1) given by*

$$\tilde{H}_n^{\text{cla}}(\underline{x}) := \sum_{j \in \mathbb{Z}^d} \left(\frac{1}{2} \alpha \|x_j\|^2 + \sum_{r=1}^n g_r(x_{j,r}) \right) - \sum_{j \in \mathbb{Z}^d} \sum_{l \in N_1(j)} \langle x_j, L x_l \rangle. \quad (6.14) \quad \text{Hconvn}$$

Assume that $g_r : \mathbb{R} \rightarrow \mathbb{R}$, $r = 1 \dots, n$ are convex functions. Then, provided that $\alpha > 2d\|L\|$, there exists, for all $\beta > 0$, a unique limit Gibbs distribution in $\mathcal{P}((\mathbb{R}^n)^{\mathbb{Z}^d})$ associated with \tilde{H}_n^{cla} .

Remark 6.10 The uniqueness result of Proposition 6.9 can be extended to long-range pair interaction potentials of the form (6.5), see Remark 6.4.

Proof of Proposition 6.9. Fix $j \in \mathbb{Z}^d$ and $\underline{y}, \underline{y}' \in (\mathbb{R}^n)^{N_1(j)}$ distinct. Set $\underline{y} := \sum_{l \in N_1(j)} y_l \in \mathbb{R}^n$, $\underline{y}' := \sum_{l \in N_1(j)} y'_l \in \mathbb{R}^n$. In view of (6.14), the corresponding 1-point Gibbs distribution reads

$$\mu_j^{\beta, (n)}(A|\underline{y}) := \frac{1}{Z_j^{\beta, (n)}(\underline{y})} \int_A \prod_{r=1}^n \exp \left(-\beta \left(\frac{1}{2} \alpha x_r^2 + g_r(x_r) - x_r (Ly)_r \right) \right) dx_1 \cdots dx_n, \quad A \in \mathcal{B}(\mathbb{R}^n),$$

with

$$Z_j^{\beta, (n)}(\underline{y}) := \int_{\mathbb{R}^n} \prod_{r=1}^n \exp \left(-\beta \left(\frac{1}{2} \alpha x_r^2 + g_r(x_r) - x_r (Ly)_r \right) \right) dx_1 \cdots dx_n.$$

Pick $0 < \delta < 1$. Let $(z_{1, m_1})_{m_1 \in \mathbb{Z}}$ be the sequence $z_{1, m_1} := m_1 \kappa_n \delta^2$ with

$$\kappa_n := \frac{1}{3 \max(1, \alpha^{-1} \|L(y - y')\|) \sqrt{n}}.$$

Define the sequence $(z'_{1, m_1})_{m_1 \in \mathbb{Z}}$ such that

$$\mu_j^{\beta, (n)}(\{z \in \mathbb{R}^n : z_1 \in (-\infty, z_{1, m_1}]\}|\underline{y}) = \mu_j^{\beta, (n)}(\{z' \in \mathbb{R}^n : z'_1 \in (-\infty, z'_{1, m_1}]\}|\underline{y}').$$

Set $A_{m_1}^{(1)} := \{z \in \mathbb{R}^n : z_1 \in (z_{1, m_1-1}, z_{1, m_1}]\}$ and $A_{m_1}'^{(1)} := \{z' \in \mathbb{R}^n : z'_1 \in (z'_{1, m_1-1}, z'_{1, m_1}]\}$. Then, for $r = 2, \dots, n$, let $(z_{r, m_r})_{m_r \in \mathbb{Z}}$ be the same increasing sequences $z_{r, m_r} := m_r \kappa_n \delta^2$. Define recursively the sets $A_{m_1, \dots, m_r}^{(r)} := \{z \in A_{m_1, \dots, m_{r-1}}^{(r-1)} : z_r \in (z_{r, m_r-1}, z_{r, m_r}]\}$, $2 \leq r \leq n$. Define the sequences $(z'_{r, m_r})_{m_r \in \mathbb{Z}}$, $r = 2, \dots, n$ such that

$$\mu_j^{\beta, (n)}(A_{m_1, \dots, m_r}^{(r)}|\underline{y}) = \mu_j^{\beta, (n)}(A_{m_1, \dots, m_r}'^{(r)}|\underline{y}'), \quad 2 \leq r \leq n,$$

where the sets $A_{m_1, \dots, m_r}'^{(r)}$ are defined similarly to $A_{m_1, \dots, m_r}^{(r)}$ but with $(z'_r)_r$ and $(z'_{r, m_r})_r$. Define

$$\sigma_{j; \underline{y}, \underline{y}'}^{\beta, (n)}(A \times B) := \sum_{m_1, \dots, m_n \in \mathbb{Z}} \frac{\mu_j^{\beta, (n)}(A \cap A_{m_1, \dots, m_n}^{(n)}|\underline{y}) \mu_j^{\beta, (n)}(B \cap A_{m_1, \dots, m_n}'^{(n)}|\underline{y}')}{\mu_j^{\beta, (n)}(A_{m_1, \dots, m_n}^{(n)}|\underline{y})},$$

for any $A, B \in \mathcal{B}(\mathbb{R}^n)$. It is easily seen that

$$\sigma_{j; \underline{y}, \underline{y}'}^{\beta, (n)}(A \times \mathbb{R}^n) = \mu_j^{\beta, (n)}(A|\underline{y}) \quad \text{and} \quad \sigma_{j; \underline{y}, \underline{y}'}^{\beta, (n)}(\mathbb{R}^n \times B) = \mu_j^{\beta, (n)}(B|\underline{y}').$$

Furthermore,

$$\begin{aligned} \int_{\mathbb{R}^n \times \mathbb{R}^n} \|x - x'\| \sigma_{j; \underline{y}, \underline{y}'}^{\beta, (n)}(d^n x, d^n x') &\leq \sum_{m_1, \dots, m_n \in \mathbb{Z}} \left(\sup_{(z, z') \in A_{m_1, \dots, m_n}^{(n)} \times A_{m_1, \dots, m_n}'^{(n)}} \|z - z'\| \right) \\ &\quad \times \frac{\mu_j^{\beta, (n)}(A_{m_1, \dots, m_n}^{(n)}|\underline{y}) \mu_j^{\beta, (n)}(A_{m_1, \dots, m_n}'^{(n)}|\underline{y}')}{\mu_j^{\beta, (n)}(A_{m_1, \dots, m_n}^{(n)}|\underline{y})}. \end{aligned} \quad (6.15) \quad \boxed{\text{togty}}$$

We now claim that, under our conditions, the following holds

$$\sup_{(z, z') \in A_{m_1, \dots, m_n}^{(n)} \times A_{m_1, \dots, m_n}'^{(n)}} \|z - z'\| \leq \delta + \frac{1}{\alpha} \|L(y - y')\|. \quad (6.16) \quad \boxed{\text{claimn}}$$

Inserting (6.16) into the right-hand side of (6.15) and letting $\delta \rightarrow 0$, we get,

$$\rho_W \left(\mu_j^{\beta, (n)}(\cdot | \underline{y}), \mu_j^{\beta, (n)}(\cdot | \underline{y}') \right) \leq \frac{1}{\alpha} \|L(y - y')\| \leq \frac{1}{\alpha} \|L\| \sum_{l \in N_1(j)} \|y_l - y'_l\|,$$

and the Proposition follows from (3.14). To prove (6.16), it is enough to show that

$$\sup_{(x_r, x'_r) \in (z_{r, m_r-1}, z_{r, m_r}] \times (z'_{r, m_r-1}, z'_{r, m_r}]} |x_r - x'_r| \leq \kappa_n \delta^2 + \frac{1}{\alpha} |(L(y' - y))_r|, \quad r = 1, \dots, n.$$

Since this is now similar to the case of 1-dimensional spins, it suffices to repeat the arguments used to prove (6.11) in the proof of Proposition 6.5. \square

Remark 6.11 *Some translation-invariant Hamiltonians with infinite-range pair-interactions are considered in [33, 12, 7]. A typical pair-interaction potential is of the form (6.5). In [33, 12, 7], the self-interaction potentials diverge at least quadratically and the conditions set on the pair-interaction potentials assure the superstability of the Hamiltonians. For a wide class of boundary conditions, existence of 'superstable' limit Gibbs distributions (see [43]) is proven in [33] (the state space is \mathbb{R}^n) and existence of tempered limit Gibbs distributions is proven in [12, 7] (the state space is \mathbb{R}). Conditions for uniqueness are also discussed in [33, 12, 7]. In [12], sufficient conditions are derived from Dobrushin's uniqueness theorem using an explicit expression for the Wasserstein distance between probability measures on \mathbb{R} , see [12, Thm. 2.2].*

7 Proof of Theorem 2.1.

compl

7.1 Some technical results.

We start with two lemmas:

lem1

Lemma 7.1 *Let \mathcal{X} be a completely regular Hausdorff space. Let $K \subset \mathcal{X}$ be a non-empty compact set and $O \subset \mathcal{X}$ an open set such that $K \subset O$. Let $(\mu_n)_{n \in \mathbb{N}} \subset \mathcal{P}(\mathcal{X})$ be a sequence of probability measures converging weakly to $\mu \in \mathcal{P}(\mathcal{X})$. Then, given $\epsilon > 0$, there exists an open set $V \subset O$ such that $K \subset V$, and for all n large enough, $\mu_n(V) < \mu(K) + \epsilon$.*

Proof. Given $0 < \delta < 1$, we may replace O by an open set U such that $K \subset U$ and $\mu(U \setminus K) < \delta$. By Urysohn's lemma, there exists a continuous function $f : \mathcal{X} \rightarrow [0, 1]$ such that $\mathbf{1}_K \leq f \leq \mathbf{1}_U$. Since $(\mu_n)_n$ converges weakly to μ , then it follows that, for n sufficiently large,

$$\left| \int_{\mathcal{X}} f(x) \mu(dx) - \int_{\mathcal{X}} f(x) \mu_n(dx) \right| < \delta.$$

Now define $V := \{x \in \mathcal{X} : f(x) > 1 - \delta\}$. Clearly, $K \subset V \subset U$. From the above, we have, for n large enough,

$$\mu_n(V) < \frac{1}{1 - \delta} \int_{\mathcal{X}} f(x) \mu_n(dx) < \frac{1}{1 - \delta} \left(\int_{\mathcal{X}} f(x) \mu(dx) + \delta \right) < \frac{1}{1 - \delta} (\mu(K) + 2\delta),$$

where we used that $\mu(U) < \mu(K) + \delta$. Set $\epsilon = 3\delta/(1 - \delta)$ and the lemma follows. \square

We recall the following well-known result

lemm2

Lemma 7.2 *Let \mathcal{X} be a completely regular Hausdorff space. Let $(\mu_n)_{n \in \mathbb{N}} \subset \mathcal{P}(\mathcal{X})$ be a sequence of probability measures converging weakly to $\mu \in \mathcal{P}(\mathcal{X})$. Then, for all open sets $O \subset \mathcal{X}$,*

$$\liminf_{n \rightarrow \infty} \mu_n(O) \geq \mu(O).$$

We continue with the following two technical lemmas

lem3

Lemma 7.3 *Let \mathcal{X} be a completely regular Hausdorff space. Let $K \subset \mathcal{X}$ be a non-empty compact set and $O_1, O_2 \subset \mathcal{X}$ be two open sets such that $K \subset (O_1 \cup O_2)$. Let $(\mu_n)_{n \in \mathbb{N}} \subset \mathcal{P}(\mathcal{X})$ be a sequence of probability measures converging weakly to $\mu \in \mathcal{P}(\mathcal{X})$. Then, given $0 < \delta < 1$ and $0 < \epsilon < 1$, there exist two open sets $V_1 \subset O_1$ and $V_2 \subset O_2$ such that $K \subset (V_1 \cup V_2)$, with $(K \setminus O_2) \subset V_1$ and $(K \setminus V_1) \subset V_2$, such that, for all n large enough, all the following hold*

- (i). *Either $\mu(K \setminus O_2) = 0$ and $\mu_n(V_1) < \delta$, or $\mu(K \setminus O_2) > 0$ and $\mu_n(V_1) < (1 + \epsilon)\mu(K \setminus O_2)$;*
- (ii). *Either $\mu(K \setminus V_1) = 0$ and $\mu_n(V_2) < \delta$, or $\mu(K \setminus V_1) > 0$ and $\mu_n(V_2) < (1 + \epsilon)\mu(K \setminus V_1)$;*
- (iii). *$\mu_n(V_1 \cap V_2) < \frac{1}{2}\epsilon\mu(K \setminus V_1)$ whenever $\mu(K \setminus V_1) > 0$.*

Proof. Set $K_1 := K \setminus O_2$. Assume first that $\mu(K) = 0$. Clearly, there exists an open set $V_1 \subset O_1$ such that $K_1 \subset V_1$ and $\mu_n(V_1) < \delta$ for n large enough. Set $K_2 := K \setminus V_1$. Similarly, there exists an open set $V_2 \subset O_2$ such that $K_2 \subset V_2$ and $\mu_n(V_2) < \delta$ for n large enough. So we can now assume that $\mu(K) > 0$. If $\mu(K_1) = 0$, then as above, there exists an open set $V_1 \subset O_1$ such that $K_1 \subset V_1$ and $\mu_n(V_1) < \delta$ for n sufficiently large. If $\mu(K_1) > 0$ then by Lemma 7.1, there exists an open set $V_1 \subset O_1$ such that $K_1 \subset V_1$ and, for n large enough, $\mu_n(V_1) - \mu(K_1) < \epsilon\mu(K_1)$. This proves (i). We point out that, if $\mu(K \setminus V_1) > 0$, we may assume that, for n large enough, we also have,

$$\mu_n(V_1) - \mu(K_1) < \frac{1}{4}\epsilon\mu(K \setminus V_1). \quad (7.1)$$

im2

Indeed, we can replace V_1 by an open set $V'_1 \subset V_1$ such that $K_1 \subset V'_1$ and for n sufficiently large, $\mu_n(V'_1) < \mu(K_1) + \frac{1}{4}\epsilon\mu(K \setminus V_1) < \mu(K_1) + \frac{1}{4}\epsilon\mu(K \setminus V'_1)$. Write V_1 instead of V'_1 and (7.1) follows. We next turn to (ii). Set $K_2 := K \setminus V_1$. Notice that $K_1 \cap K_2 = \emptyset$. Let $U_1 \subset V_1$ and $U_2 \subset O_2$ be two open sets such that $U_1 \cap U_2 = \emptyset$ and $K_1 \subset U_1$ and $K_2 \subset U_2$. By the same arguments used to prove (i), we may choose an open set $V_2 \subset U_2$ such that $K_2 \subset V_2$ and for n sufficiently large, $\mu_n(V_2) < \delta$ if $\mu(K_2) = 0$ and $\mu_n(V_2) - \mu(K_2) < \epsilon\mu(K_2)$ if $\mu(K_2) > 0$. To prove (iii), first note that $\mu_n(V_1 \cap V_2) \leq \mu_n(V_1) - \mu_n(U_1)$ because $V_2 \subset U_2 \subset ((O_1 \cup O_2) \setminus U_1)$. Assuming $\mu(K_2) > 0$, it follows from Lemma 7.2 that $\mu_n(V_1 \cap V_2) \leq \mu_n(V_1) - \mu(U_1) + \frac{1}{4}\epsilon\mu(K_2)$ for n sufficiently large. It remains to use (7.1) noting that $K_1 \subset U_1$. \square

t3

Lemma 7.4 *Let \mathcal{X} be a completely regular Hausdorff space. Let $K \subset \mathcal{X}$ be a non-empty compact set and $B_1, \dots, B_N \subset \mathcal{X}$ $N \geq 2$ open sets such that $K \subset \bigcup_{i=1}^N B_i$. Let $(\mu_n)_{n \in \mathbb{N}} \subset \mathcal{P}(\mathcal{X})$ be a sequence of probability measures converging weakly to $\mu \in \mathcal{P}(\mathcal{X})$. Then, given $0 < \epsilon < 1$, there exist N open sets V_1, \dots, V_N with $V_i \subset B_i$, $i = 1, \dots, N$ such that $K \subset \bigcup_{j=1}^N V_j$, and, setting*

$$\begin{aligned} K_1 &:= K \setminus \bigcup_{j=2}^N B_j; \\ K_i &:= K \setminus \left(\bigcup_{j=1}^{i-1} V_j \cup \bigcup_{j=i+1}^N B_j \right), \quad i = 2, \dots, N-1; \\ K_N &:= K \setminus \bigcup_{j=1}^{N-1} V_j, \end{aligned} \quad (7.2)$$

Ki

such that $K_1 \subset V_1$ and $K_i \subset V_i \setminus \bigcup_{j=1}^{i-1} V_j$, $i = 2, \dots, N$, and, for all n sufficiently large, either $\mu(K_i) = 0$ and $\mu_n(V_i) < \frac{\epsilon}{N}$, or $\mu(K_i) > 0$ and $\mu_n(V_i) < (1 + \epsilon)\mu(K_i)$ for all $i = 1, \dots, N$, and moreover,

$$\mu_n \left(V_i \cap \bigcup_{j=1}^{i-1} V_j \right) < \frac{1}{2}\epsilon\mu(K_i). \quad (7.3)$$

intermaj

Proof. We start by applying Lemma 7.3 to $K^{(2)} := K \setminus \bigcup_{j=3}^N B_j \subset (B_1 \cup B_2)$. Hence, there exist two open sets $V'_1 \subset B_1$ and $V'_2 \subset B_2$ such that $K^{(2)} \subset (V'_1 \cup V'_2)$, and such that, for n large enough, denoting $K'_1 := K^{(2)} \setminus B_2 \subset V'_1$ and $K'_2 := K^{(2)} \setminus V'_1 \subset V'_2$, either $\mu(K'_i) = 0$ and $\mu_n(V'_i) < \frac{\epsilon}{N}$, or $\mu(K'_i) > 0$ and $\mu_n(V'_i) < (1 + \epsilon)\mu(K'_i)$, $i = 1, 2$, and, moreover, $\mu_n(V'_1 \cap V'_2) < \frac{1}{2}\epsilon\mu(K'_2)$. We now proceed by induction on N , assuming that the statement holds for $N-1$ and modifying the previous sets V'_i , $i = 1, \dots, N-2$ in the process (the sets V'_i , $i = 1, \dots, j$ constructed at the induction step j are different from the sets V'_i , $i = 1, \dots, j$ constructed at the step $j+1$). Assume thus that we have constructed V'_i , $i = 1, \dots, N-1$ for the set $K^{(N-1)} := K \setminus B_N \subset \bigcup_{i=1}^{N-1} V'_i$. We then apply Lemma 7.3 to $K \subset (\bigcup_{i=1}^{N-1} V'_i \cup B_N)$. Hence, there exist two open sets $U_1 \subset \bigcup_{i=1}^{N-1} V'_i$ and $U_2 \subset B_N$ such that $K \subset (U_1 \cup U_2)$ with $K^{(N-1)} \subset U_1$ and $K \setminus U_1 \subset U_2$, and such that, for n large enough, either $\mu(K^{(N-1)}) = 0$ and $\mu_n(U_1) < \frac{\epsilon}{N}$ or $\mu(K^{(N-1)}) > 0$ and $\mu_n(U_1) < (1 + \epsilon)\mu(K^{(N-1)})$, and also, either $\mu(K \setminus U_1) = 0$ and $\mu_n(U_2) < \frac{\epsilon}{N}$ or $\mu(K \setminus U_1) > 0$ and $\mu_n(U_2) < (1 + \epsilon)\mu(K \setminus U_1)$, and moreover, $\mu_n(U_1 \cap U_2) < \frac{1}{2}\epsilon\mu(K \setminus U_1)$. Now let us set $V_N := U_2$ and $V_i := V'_i \cap U_1$, $i = 1, \dots, N-1$. With this definition, $V_i \subset V'_i$ and $U_1 = \bigcup_{i=1}^{N-1} V_i$. Set $K_N := K \setminus U_1$,

$$K_i := K \setminus \left(\bigcup_{j=1}^{i-1} V_j \cup \bigcup_{j=i+1}^N B_j \right), \quad i = 2, \dots, N-1,$$

and $K_1 := K \setminus \bigcup_{j=2}^N B_j$. Set also

$$K^{(i)} := K \setminus \bigcup_{j=i+1}^N B_j, \quad i = 2, \dots, N-1.$$

By construction of the sets V'_i , we have $K^{(i)} \setminus \bigcup_{j=1}^{i-1} V'_j \subset V'_i$. Since $K^{(N-1)} \subset U_1 = \bigcup_{i=1}^{N-1} V_i$, we have $K^{(i)} \setminus \bigcup_{j=1}^{i-1} V'_j = K^{(i)} \setminus \bigcup_{j=1}^{i-1} V_j = K_i$. It remains to use that $K_i \subset K^{(N-1)}$ to conclude that $K_1 \subset V_1$ and $K_i \subset V_i \setminus \bigcup_{j=1}^{i-1} V_j \subset V_i$, $i = 2, \dots, N-1$. Hence, if $\mu(K_i) = 0$ then $\mu_n(V_i) \leq \mu_n(V'_i) < \frac{\epsilon}{N}$, while if $\mu(K_i) > 0$ then $\mu_n(V_i) \leq \mu_n(V'_i) < (1 + \epsilon)\mu(K_i)$. Moreover, $\mu_n(V_i \cap \bigcup_{j=1}^{i-1} V_j) < \mu_n(V'_i \cap \bigcup_{j=1}^{i-1} V'_j) < \frac{1}{2}\epsilon\mu(K_i)$. When $i = N$, $K_N \subset V_N$ since $K \setminus U_1 \subset U_2$. Hence, if $\mu(K_N) = 0$ then $\mu_n(V_N) = \mu_n(U_2) < \frac{\epsilon}{N}$, otherwise $\mu_n(V_N) = \mu_n(U_2) < (1 + \epsilon)\mu(K \setminus U_1) = (1 + \epsilon)\mu(K_N)$. Moreover, $\mu_n(V_N \cap \bigcup_{j=1}^{N-1} V_j) = \mu_n(U_2 \cap U_1) < \frac{1}{2}\epsilon\mu(K_N)$. This proves the lemma. \square

7.2 Equivalence of weak convergence and convergence in Wasserstein metric.

equivcvg

The following Proposition contains the key-results for the proof of Theorem 2.1 (ii).

prop1

Proposition 7.5 *Let (\mathcal{X}, ρ) be a metric space. Let $\mu \in \mathcal{P}_1(\mathcal{X})$ and $(\mu_n)_{n \in \mathbb{N}} \subset \mathcal{P}(\mathcal{X})$ be a sequence of probability measures such that $\mu_n \in \mathcal{P}_1(\mathcal{X})$ for n large enough.*

(i). *Suppose that $\rho_W(\mu_n, \mu) \rightarrow 0$ when $n \rightarrow \infty$. Then (μ_n) converges weakly to μ .*

(ii). *Suppose that (μ_n) converges weakly to μ . Further, assume that a Prokhorov-like condition holds, i.e. for every $\epsilon > 0$, there exists a non-empty compact set $K \subset \mathcal{X}$ such that for all n large enough and for some $x_0 \in \mathcal{X}$,*

$$\int_{K^c} \rho(x, x_0) \mu_n(dx) < \epsilon. \quad (7.4)$$

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Then $\rho_W(\mu_n, \mu) \rightarrow 0$ when $n \rightarrow \infty$.

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Remark 7.6 *The proof of Proposition 7.5 relies on Lemma 7.4 which holds for general completely regular Hausdorff spaces \mathcal{X} . This enables the extension to uniform spaces, replacing the Wasserstein metric by a Wasserstein uniformity.*

Proof. (i). Fix $\epsilon > 0$. Let $f : \mathcal{X} \rightarrow \mathbb{R}$ be a bounded uniformly continuous function. By definition, there exists $\delta > 0$ such that for every $x, y \in \mathcal{X}$ with $\rho(x, y) < \delta$, $|f(x) - f(y)| < \frac{\epsilon}{2}$. Besides, in

view of the definition (2.1), there exists a sequence of probability measures $\sigma_n \in \Xi_{\mathcal{X}}(\mu_n, \mu)$ on $\mathcal{X} \times \mathcal{X}$ such that for all n large enough,

$$\int_{\mathcal{X} \times \mathcal{X}} \rho(x, y) \sigma_n(dx, dy) < \rho_W(\mu_n, \mu) + \frac{\delta^2}{2}. \quad (7.5) \quad \text{funde}$$

Let $\Delta_\delta := \{(x, y) \in \mathcal{X} \times \mathcal{X} : \rho(x, y) < \delta\}$ be a δ -neighbourhood of the diagonal. Then (7.5) implies that, for all n large enough,

$$\sigma_n(\Delta_\delta^c) < \frac{1}{\delta} \int_{\Delta_\delta^c} \rho(x, y) \sigma_n(dx, dy) < \frac{1}{\delta} \left(\rho_W(\mu_n, \mu) + \frac{\delta^2}{2} \right). \quad (7.6) \quad \text{funde2}$$

Since by assumption $\rho_W(\mu_n, \mu) \rightarrow 0$, (7.6) yields $\sigma_n(\Delta_\delta^c) < \delta$ for n large enough. This implies

$$\left| \int_{\mathcal{X}} f(x) \mu_n(dx) - \int_{\mathcal{X}} f(x) \mu(dx) \right| \leq \int_{\Delta_\delta \cup \Delta_\delta^c} |f(x) - f(y)| \sigma_n(dx, dy) < \frac{\epsilon}{2} + 2\|f\|_\infty \delta. \quad (7.7) \quad \text{funde3}$$

It remains to replace δ above by $\min(\delta, \frac{\epsilon}{4\|f\|_\infty})$. As (7.7) holds for all bounded uniformly continuous functions $f : \mathcal{X} \rightarrow \mathbb{R}$, the weak convergence follows, see, e.g., [8, Thm 2.1].

We now prove (ii). Fix $x_0 \in \mathcal{X}$ and $0 < \epsilon < 1$. By assumption, there exists a non-empty compact set $K \subset \mathcal{X}$ such that, for n sufficiently large,

$$\int_{K^c} \rho(x, x_0) \mu(dx) + \int_{K^c} \rho(x, x_0) \mu_n(dx) < \frac{\epsilon}{6}.$$

We cover K with a number N of open balls $B_i \subset \mathcal{X}$ of radius $0 < r < 1$ to be chosen hereafter. Applying Lemma 7.4, given $0 < \kappa < 1$, there exist N open sets V_1, \dots, V_N with $V_i \subset B_i$ and also N compact sets K_1, \dots, K_N which are defined in (7.2) such that, setting

$$\begin{aligned} A_1 &:= V_1, \\ A_i &:= V_i \setminus \bigcup_{j=1}^{i-1} V_j, \quad i = 2, \dots, N, \end{aligned} \quad (7.8) \quad \text{Ai}$$

we have $K_i \subset A_i \subset V_i$, $i = 1, \dots, N$ and, for all n sufficiently large, either $\mu(K_i) = 0$ and $\mu_n(A_i) \leq \mu_n(V_i) < \frac{\kappa}{N}$, or $\mu(K_i) > 0$ and

$$\mu_n(A_i) \leq \mu_n(V_i) < (1 + \kappa)\mu(K_i), \quad i = 1, \dots, N. \quad (7.9) \quad \text{upbbnd}$$

Next, set $I := \{i \in \{1, \dots, N\} : \mu(K_i) > 0\}$ and define a measure $\nu \in \mathcal{M}^+(\mathcal{X})$ by

$$\nu(E) := \mu(E) - (1 - \kappa) \sum_{i \in I} \mu(E \cap A_i), \quad E \in \mathcal{B}(\mathcal{X}), \quad (7.10) \quad \text{nu}$$

and, also, introduce the sequence of measures $(\nu_n)_{n \in \mathbb{N}}$ by

$$\nu_n(E) := \mu_n(E) - (1 - \kappa) \sum_{i \in I} \mu(A_i) \frac{\mu_n(E \cap A_i)}{\mu_n(A_i)}, \quad E \in \mathcal{B}(\mathcal{X}). \quad (7.11) \quad \text{nun}$$

Note that $\nu_n \in \mathcal{M}^+(\mathcal{X})$ for all n large enough since for $i \in I$,

$$\mu_n(A_i) = \mu_n(V_i) - \mu_n \left(V_i \cap \bigcup_{j=1}^{i-1} V_j \right) > \left(\mu(V_i) - \frac{1}{2} \kappa \mu(K_i) \right) - \frac{1}{2} \kappa \mu(K_i) > (1 - \kappa) \mu(A_i).$$

Here, we used Lemma 7.2 combined with the upper bound (7.3). In view of (7.10) and (7.11), define for all n large enough $\sigma_n \in \mathcal{P}(\mathcal{X} \times \mathcal{X})$ as follows

$$\sigma_n(E \times F) := (1 - \kappa) \sum_{i \in I} \mu(E \cap A_i) \frac{\mu_n(F \cap A_i)}{\mu_n(A_i)} + \frac{\nu(E) \nu_n(F)}{\nu(\mathcal{X})}, \quad E, F \in \mathcal{B}(\mathcal{X}).$$

It follows from the fact

$$\nu(\mathcal{X}) = 1 - (1 - \kappa) \sum_{i \in I} \mu(A_i) = \nu_n(\mathcal{X}),$$

that we have $\sigma_n \in \Xi_{\mathcal{X}}(\mu, \mu_n)$. Moreover, we have,

$$\begin{aligned} \int_{\mathcal{X} \times \mathcal{X}} \rho(x, y) \sigma_n(dx, dy) \\ = (1 - \kappa) \sum_{i \in I} \frac{1}{\mu_n(A_i)} \int_{A_i \times A_i} \rho(x, y) \mu(dx) \mu_n(dy) + \frac{1}{\nu(\mathcal{X})} \int_{\mathcal{X} \times \mathcal{X}} \rho(x, y) \nu(dx) \nu_n(dy). \end{aligned}$$

To conclude the proof of the proposition, it suffices to show that, for all n large enough,

$$\int_{\mathcal{X} \times \mathcal{X}} \rho(x, y) \sigma_n(dx, dy) < \epsilon. \quad (7.12) \quad \boxed{\text{2sho}}$$

On the one hand, since $\text{diam}(A_i) \leq \text{diam}(B_i) = 2r$, we have,

$$(1 - \kappa) \sum_{i \in I} \frac{1}{\mu_n(A_i)} \int_{A_i \times A_i} \rho(x, y) \mu(dx) \mu_n(dy) \leq 2r(1 - \kappa) \sum_{i \in I} \mu(A_i) \leq 2r.$$

On the other hand, by the triangle inequality, we have, for all n large enough,

$$\frac{1}{\nu(\mathcal{X})} \int_{\mathcal{X} \times \mathcal{X}} \rho(x, y) \nu(dx) \nu_n(dy) \leq \int_{\mathcal{X}} \rho(x, x_0) \nu(dx) + \int_{\mathcal{X}} \rho(x_0, y) \nu_n(dy). \quad (7.13) \quad \boxed{\text{splitit}}$$

In view of (7.10), the first term in the right-hand side of (7.13) can be rewritten as

$$\int_{\mathcal{X}} \rho(x, x_0) \nu(dx) = \int_{\mathcal{X} \setminus \cup_{i \in I} A_i} \rho(x, x_0) \mu(dx) + \kappa \sum_{i \in I} \int_{A_i} \rho(x, x_0) \mu(dx),$$

and can be bounded as follows

$$\begin{aligned} \int_{\mathcal{X}} \rho(x, x_0) \nu(dx) \\ \leq \int_{\mathcal{X} \setminus \cup_{i \in I} A_i} \rho(x, x_0) \mu(dx) + \kappa \left(\int_{(\cup_{i \in I} A_i) \setminus K} \rho(x, x_0) \mu(dx) + \sum_{i \in I} \int_{A_i \cap K} \rho(x, x_0) \mu(dx) \right) \\ \leq \int_{K^c} \rho(x, x_0) \mu(dx) + \kappa \int_K \rho(x, x_0) \mu(dx). \end{aligned}$$

For the second term in the right-hand side of (7.13), in view of the definition (7.11), we have,

$$\int_{\mathcal{X}} \rho(x, x_0) \nu_n(dx) \leq \int_{\mathcal{X}} \rho(x, x_0) \mu_n(dx) - \frac{1 - \kappa}{1 + \kappa} \sum_{i \in I} \int_{A_i} \rho(x, x_0) \mu_n(dx),$$

where we used the bound in (7.9). It then follows that,

$$\begin{aligned} \int_{\mathcal{X}} \rho(x, x_0) \nu_n(dx) \leq \int_{\mathcal{X} \setminus \cup_{i \in I} A_i} \rho(x, x_0) \mu_n(dx) \\ + 2\kappa \left(\int_{(\cup_{i \in I} A_i) \setminus K} \rho(x, x_0) \mu_n(dx) + \sum_{i \in I} \int_{A_i \cap K} \rho(x, x_0) \mu_n(dx) \right). \end{aligned}$$

Gathering the above estimates together, we eventually get, for all n large enough,

$$\int_{\mathcal{X} \times \mathcal{X}} \rho(x, y) \sigma_n(dx, dy) < 2r + 2 \left(\int_{K^c} \rho(x, x_0) \mu(dx) + \int_{K^c} \rho(x, x_0) \mu_n(dx) \right) + 3\kappa \sup_{x \in K} \rho(x, x_0).$$

(7.12) follows by taking $r = \frac{1}{6}\epsilon$ and $\kappa = \frac{1}{9 \max(1, \sup_{x \in K} \rho(x, x_0))} \epsilon$. \square

7.3 Proof of Theorem 2.1 (i).

Non-degeneracy. Let us show that $\rho_W(\mu, \nu) = 0 \implies \mu = \nu$ (the converse is obvious). Pick $\epsilon > 0$. From (2.1), there exists a probability measure $\sigma \in \Xi_{\mathcal{X}}(\mu, \nu)$ such that,

$$\int_{\mathcal{X} \times \mathcal{X}} \rho(x, y) \sigma(dx, dy) < \epsilon. \quad (7.14) \quad \text{funde2'}$$

Denote $\Delta_{\sqrt{\epsilon}} := \{(x, y) \in \mathcal{X} \times \mathcal{X} : \rho(x, y) < \sqrt{\epsilon}\}$. From (7.14), we have,

$$\sigma(\Delta_{\sqrt{\epsilon}}^c) < \frac{1}{\sqrt{\epsilon}} \int_{\Delta_{\sqrt{\epsilon}}^c} \rho(x, y) \sigma(dx, dy) < \sqrt{\epsilon}. \quad (7.15) \quad \text{fundeq3'}$$

Suppose now that $\mu \neq \nu$. Then there exists a non-empty compact set $K \subset \mathcal{X}$ such that $\mu(K) > \nu(K)$ (without loss of generality). Setting $\delta := \frac{1}{2}(\mu(K) - \nu(K)) > 0$, there exists an open set $O \subset \mathcal{X}$ such that $K \subset O$ and $\mu(K) > \nu(O) + \delta$. We may assume that O is the δ -neighbourhood of K . Indeed, if $K_\delta \subset O$ then $\mu(K) > \nu(K_\delta) + \delta$, and if $\rho(K, O^c) < \delta$, then we can replace δ by $\rho(K, O^c)$. For the set $(K \times O^c) \subset \Delta_\delta^c$, we have,

$$\sigma(K \times O^c) = \sigma(K \times \mathcal{X}) - \sigma(K \times O) > \sigma(K \times \mathcal{X}) - \sigma(\mathcal{X} \times O) = \mu(K) - \nu(O) > \delta.$$

It follows that $\sigma(\Delta_\delta^c) > \delta$ which contradicts (7.15) if we set $\delta = \sqrt{\epsilon}$.

Triangle inequality. Let $\mu_l \in \mathcal{P}_1(\mathcal{X})$, $l = 1, 2, 3$. We need to show that

$$\rho_W(\mu_1, \mu_3) \leq \rho_W(\mu_1, \mu_2) + \rho_W(\mu_2, \mu_3). \quad (7.16) \quad \text{trine}$$

Given $\epsilon > 0$, there exist $\sigma_{1,2} \in \Xi_{\mathcal{X}}(\mu_1, \mu_2)$ and $\sigma_{2,3} \in \Xi_{\mathcal{X}}(\mu_2, \mu_3)$ such that,

$$\int_{\mathcal{X} \times \mathcal{X}} \rho(x_l, x_{l+1}) \sigma_{l,l+1}(dx_l, dx_{l+1}) < \rho_W(\mu_l, \mu_{l+1}) + \epsilon, \quad l = 1, 2. \quad (7.17) \quad \text{rhoWtwo}$$

Denote by $\mu_{1,2}$ the conditional probability measure defined by (see Section 8.1)

$$\sigma_{1,2}(A_1 \times A_2) = \int_{A_2} \mu_{1,2}(A_1|x_2) \mu_2(dx_2), \quad A_1, A_2 \in \mathcal{B}(\mathcal{X}). \quad (7.18) \quad \text{sigma12}$$

Subsequently, we put

$$\sigma_{1,3}(A_1 \times A_3) := \int_{\mathcal{X} \times A_3} \mu_{1,2}(A_1|x_2) \sigma_{2,3}(dx_2, dx_3), \quad A_1, A_3 \in \mathcal{B}(\mathcal{X}). \quad (7.19) \quad \text{sigma13}$$

Using the triangle inequality and the definition in (7.19), we have,

$$\begin{aligned} & \int_{\mathcal{X} \times \mathcal{X}} \rho(x_1, x_3) \sigma_{1,3}(dx_1, dx_3) \\ & \leq \int_{\mathcal{X} \times \mathcal{X} \times \mathcal{X}} \rho(x_1, x_2) \mu_{1,2}(dx_1|x_2) \sigma_{2,3}(dx_2, dx_3) + \int_{\mathcal{X} \times \mathcal{X}} \rho(x_2, x_3) \mu_{1,2}(\mathcal{X}|x_2) \sigma_{2,3}(dx_2, dx_3) \\ & = \int_{\mathcal{X} \times \mathcal{X}} \rho(x_1, x_2) \mu_{1,2}(dx_1|x_2) \mu_2(dx_2) + \int_{\mathcal{X} \times \mathcal{X}} \rho(x_2, x_3) \sigma_{2,3}(dx_2, dx_3) \end{aligned}$$

Using (7.18) in the first term of the above right-hand side, this yields

$$\int_{\mathcal{X} \times \mathcal{X}} \rho(x_1, x_3) \sigma_{1,3}(dx_1, dx_3) \leq \int_{\mathcal{X} \times \mathcal{X}} \rho(x_1, x_2) \sigma_{1,2}(dx_1, dx_2) + \int_{\mathcal{X} \times \mathcal{X}} \rho(x_2, x_3) \sigma_{2,3}(dx_2, dx_3).$$

To obtain (7.16), it remains to use (7.17) and take the limit $\epsilon \rightarrow 0$.

7.4 Proof of Theorem 2.1 (ii).

Given a non-empty compact set Q and a real $\alpha > 0$, in the following we denote

$$Q_\alpha := \{x \in \mathcal{X} : \text{dist}(x, Q) < \alpha\}. \quad (7.20)$$

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Let $(\mu_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in $\mathcal{P}_1(\mathcal{X})$. This means that, given $\delta > 0$, there exists $N \in \mathbb{N}$ such that $\rho_W(\mu_n, \mu_m) < \frac{\delta^2}{2}$ for all $n, m \geq N$. As a result, there exists a coupling $\sigma_{n,m} \in \Xi_{\mathcal{X}}(\mu_n, \mu_m)$ for each $n, m \geq N$ such that,

$$\int_{\mathcal{X} \times \mathcal{X}} \rho(x, y) \sigma_{n,m}(dx, dy) \leq \delta^2. \quad (7.21)$$

goo1

By setting $\Delta_\delta := \{(x, y) \in \mathcal{X} \times \mathcal{X} : \rho(x, y) < \delta\}$, (7.21) implies that $\sigma_{n,m}(\Delta_\delta^c) < \delta$, see (7.15). Further, for any uniformly continuous function $f : \mathcal{X} \rightarrow \mathbb{R}$ satisfying

$$|f(x) - f(y)| < C\rho(x, y), \quad (7.22)$$

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for some constant $C > 0$, we have, for all $n, m \geq N$,

$$\left| \int_{\mathcal{X}} f(x) \mu_n(dx) - \int_{\mathcal{X}} f(y) \mu_m(dy) \right| = \left| \int_{\Delta_\delta \cup \Delta_\delta^c} (f(x) - f(y)) \sigma_{n,m}(dx, dy) \right| < (C + 2\|f\|_\infty)\delta. \quad (7.23)$$

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Fix $0 < \epsilon < 1$. Let $\delta_0 := \frac{\epsilon^2}{12} > 0$ and $n_0 \in \mathbb{N}$ such that $\rho_W(\mu_n, \mu_m) < \frac{\delta_0^2}{2}$ for all $n, m \geq n_0$. Choose a compact set K_0 such that $\mu_{n_0}(K_0^c) < \frac{\epsilon}{2}$. Put $O_0 := K_{0,\epsilon}$ (see (7.20)) and introduce

$$f_0(x) := \begin{cases} 1 - \frac{1}{\epsilon} \text{dist}(x, K_0), & \text{if } \text{dist}(x, K_0) \leq \epsilon \\ 0, & \text{if } \text{dist}(x, K_0) > \epsilon \end{cases},$$

where it is understood that $f_0 = 1$ on K_0 . Clearly, f_0 is uniformly continuous and satisfies (7.22) with $C = \frac{1}{\epsilon}$. It follows from (7.23) that, for all $n \geq n_0$,

$$\mu_n(O_0) > \int_{\mathcal{X}} f_0(x) \mu_n(dx) > \int_{\mathcal{X}} f_0(x) \mu_{n_0}(dx) - \left(2 + \frac{1}{\epsilon}\right) \delta_0 > \mu_{n_0}(K_0) - \left(2 + \frac{1}{\epsilon}\right) \delta_0.$$

Since $\mu_{n_0}(K_0) > 1 - \frac{\epsilon}{2}$, we find that $\mu_n(O_0^c) < \frac{3}{4}\epsilon$ for all $n \geq n_0$.

Next, we proceed by induction to construct a non-decreasing subsequence $(n_i)_{i \in \mathbb{N}}$ and compact sets K_i , $i \in \mathbb{N}$ such that, denoting $O_i := K_{i,2^{-i}\epsilon}$ (see (7.20)), we have, for all $n \geq n_i$,

$$\mu_{n_i}(K_i^c) < \left(1 - \frac{1}{2^{i+1}}\right) \epsilon \quad \text{and} \quad \mu_n(O_i^c) < \left(1 - \frac{1}{2^{i+2}}\right) \epsilon.$$

Suppose that we have found $n_{i-1} \in \mathbb{N}$ and we have constructed a compact set K_{i-1} such that $\mu_{n_{i-1}}(K_{i-1}^c) < (1 - \frac{1}{2^i})\epsilon$ and $\mu_n(O_{i-1}^c) < (1 - \frac{1}{2^{i+1}})\epsilon$ for all $n \geq n_{i-1}$. Let $\delta_i := \frac{\epsilon^2}{2^{i+3}(1+2^{i-1})} > 0$ and $n_i \in \mathbb{N}$ such that $\rho_W(\mu_n, \mu_m) < \frac{\delta_i^2}{2}$ for all $n, m \geq n_i$. We choose $K_i \subset O_{i-1}$ compact such that $K_{i-1} \subset K_i$ and $\mu_{n_i}(K_i^c) < (1 - \frac{1}{2^{i+1}})\epsilon$, and we put $O_i = K_{i,2^{-i}\epsilon}$. Then, setting

$$f_i(x) := \begin{cases} 1 - \frac{2^i}{\epsilon} \text{dist}(x, K_i), & \text{if } \text{dist}(x, K_i) \leq \frac{\epsilon}{2^i} \\ 0, & \text{if } \text{dist}(x, K_i) > \frac{\epsilon}{2^i} \end{cases},$$

it follows from (7.23) that, for all $n \geq n_i$,

$$\mu_n(O_i) > \int_{\mathcal{X}} f_i(x) \mu_n(dx) > \int_{\mathcal{X}} f_i(x) \mu_{n_i}(dx) - \left(2 + \frac{2^i}{\epsilon}\right) \delta_i > \mu_{n_i}(K_i) - \left(2 + \frac{2^i}{\epsilon}\right) \delta_i.$$

It remains to use that $\mu_n(K_i) > 1 - (1 - \frac{1}{2^{i+1}})\epsilon$ to conclude this induction step.

Define $K := \bigcup_{i \in \mathbb{N}} K_i$ and let us show that K is compact. Fix $0 < \eta < 1$ and let $i_0 \in \mathbb{N}$ such

that $\frac{1}{2^{i_0-3}} < \eta$. We can cover K_{i_0} by finitely many balls of radius $\frac{\eta}{2}$ denoted by $B_{\frac{\eta}{2}}^{(l)}$, $l = 1, \dots, M$. Since $K_j \subset O_{j-1}$, $\text{dist}(K_j, K_{j-1}) < \frac{1}{2^{j-1}}$. It follows by induction that $\text{dist}(K_j, K_i) < \frac{1}{2^{i-1}}$ for all $j > i$. Now, if $x' \in K$, there exists $j \geq i_0$ such that $\text{dist}(x', K_j) < \frac{\eta}{4}$ and hence $\text{dist}(x', K_{i_0}) < \frac{\eta}{4} + \frac{1}{2^{i_0-1}} < \frac{\eta}{2}$. Therefore, there exists $x \in K_{i_0}$ with $\rho(x, x') < \frac{\eta}{2}$, and there is a ball $B_{\frac{\eta}{2}}^{(l_0)}$ with $x \in B_{\frac{\eta}{2}}^{(l_0)}$ so that $x' \in B_{\eta}^{(l_0)}$. We conclude that K is covered by the balls $B_{\eta}^{(l)}$ with double the radius. This means that K is totally bounded, and since \mathcal{X} is complete, K is compact.

Consider now the sequence of probability measures $(\mu_{n_i})_{i \in \mathbb{N}}$. From the foregoing, we have found a compact set K such that $\mu_{n_i}(K^c) < \epsilon$ for all $i \in \mathbb{N}$. By Prokhorov's theorem, see e.g. [10, Sec. IX.5.5], there exists a subsequence $(\mu_{m_k})_{k \in \mathbb{N}}$, with $\mu_{m_k} = \mu_{n_{i_k}}$ converging weakly to a probability measure $\mu \in \mathcal{P}(\mathcal{X})$. Let us show that $\mu \in \mathcal{P}_1(\mathcal{X})$. Note that $\mu_{m_k} \in \mathcal{P}_1(\mathcal{X})$ for all $k \in \mathbb{N}$ by assumption. Further, since $\rho_W(\mu_m, \mu_{m_1}) < \frac{\delta_1^2}{2}$ for all $m \geq m_1 \geq n_1$, by mimicking the arguments leading to (7.21), there exists, for each m , a coupling $\sigma_{m, m_1} \in \Xi_{\mathcal{X}}(\mu_m, \mu_{m_1})$ such that,

$$\int_{\mathcal{X} \times \mathcal{X}} \rho(x, y) \sigma_{m, m_1}(dx, dy) \leq \delta_1^2 < 1.$$

As a result, given $x_0 \in \mathcal{X}$, there exists a constant $c_{m_1} > 0$ such that, for $k \in \mathbb{N}$, $k \geq 2$,

$$\int_{\mathcal{X}} \rho(x, x_0) \mu_{m_k}(dx) \leq \int_{\mathcal{X} \times \mathcal{X}} \rho(x, y) \sigma_{m_k, m_1}(dx, dy) + \int_{\mathcal{X}} \rho(y, x_0) \mu_{m_1}(dy) < 1 + c_{m_1}. \quad (7.24)$$

goo2

This means, in particular, that the left-hand side of (7.24) is uniformly bounded for all $k \in \mathbb{N}$. By virtue of the monotone convergence theorem, we conclude that

$$\int_{\mathcal{X}} \rho(x, x_0) \mu(dx) \leq 1 + c_{m_1},$$

and hence $\mu \in \mathcal{P}_1(\mathcal{X})$. We can now apply Proposition 7.5 (ii) giving that $\rho_W(\mu_{m_k}, \mu) \rightarrow 0$ when $k \rightarrow \infty$. Using the triangle inequality, we conclude that $\rho_W(\mu_m, \mu) \rightarrow 0$ when $m \rightarrow \infty$. \square

8 Appendix.

appX

8.1 Disintegration theorem.

appdx3

We recall the disintegration theorem for the existence of conditional probability measures, see, e.g., [10, Sec. IX.2.7]. Given a topological space (\mathcal{X}, τ) , let $\mathcal{B}(\mathcal{X})$ denote the Borel σ -algebra of subsets of \mathcal{X} .

existence

Theorem 8.1 *Let \mathcal{X} be a completely regular Hausdorff space and assume that the compact subspaces of \mathcal{X} are metrizable. Let \mathcal{Y} be a Hausdorff space. Let $\mu \in \mathcal{P}(\mathcal{X})$ be a Radon probability measure and $\pi : \mathcal{X} \rightarrow \mathcal{Y}$ be a μ -measurable function. Let $\nu \in \mathcal{P}(\mathcal{Y})$ be the image measure $\nu = \pi_*(\mu) = \mu \circ \pi^{-1}$. Then there exists a map $\mu_y : \mathcal{Y} \rightarrow \mathcal{P}(\mathcal{X})$, $y \mapsto \mu_y$ such that for all $A \in \mathcal{B}(\mathcal{Y})$ and all bounded Borel-measurable functions $f : \mathcal{X} \rightarrow \mathbb{R}$,*

$$\int_A \int_{\mathcal{X}} f(x) \mu_y(dx) \nu(dy) = \int_{\pi^{-1}(A)} f(x) \mu(dx). \quad (8.1)$$

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The above map is a.e. unique and μ_y is concentrated on $\pi^{-1}(\{y\})$ for a.e. y . One usually writes $\mu_y(\cdot) = \mu(\cdot | y)$. Further, if $f \in \mathcal{L}^1(\mu)$ then $f \in \mathcal{L}^1(\mu_y)$ for a.e. y , and the identity (8.1) still holds.

Remark 8.2 *Taking the indicator function $f = \mathbf{1}_E$, $E \subseteq \mathcal{X}$ and choosing $A = \mathcal{Y}$ in (8.1) yields*

$$\int_{\mathcal{Y}} \mu_y(E) \nu(dy) = \mu(E).$$

The special case where \mathcal{X} is a product space is particularly important. Considering $\mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2$ and considering the projection map onto one of the factors, for instance $\pi_2 : \mathcal{X} \mapsto \mathcal{X}_2$, all spaces $\pi_2^{-1}(\{y\})$ are equivalent to \mathcal{X}_1 . Given $\mu \in \mathcal{P}(\mathcal{X})$ a Radon probability measure, we can thus define $\mu'_y(A) := \mu_y(A \times \{y\})$ for any $A \in \mathcal{B}(\mathcal{X}_1)$. In the case of probability measures, we have in fact:

Corollary 8.3 *Let $\mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2$ be a product of completely regular Hausdorff spaces. Assume in addition that the compact subspaces of \mathcal{X}_1 are metrizable. Let $\mu \in \mathcal{P}(\mathcal{X})$ be a Radon probability measure and $\pi_2 : \mathcal{X} \rightarrow \mathcal{X}_2$ be the projection map. Let $\nu \in \mathcal{P}(\mathcal{X}_2)$ be the image measure $\nu = \mu \circ \pi_2^{-1}$. Then there exists a map $\mu_y : \mathcal{X}_2 \rightarrow \mathcal{P}(\mathcal{X}_1)$, $y \mapsto \mu_y$ such that for all $B \in \mathcal{B}(\mathcal{X}_2)$ and all Borel-measurable functions $f \in \mathcal{L}^1(\mu)$,*

$$\int_B \nu(dy) \int_{\mathcal{X}_1} f(x, y) d\mu_y(dx) = \int_{\mathcal{X}_1 \times B} f(x, y) d\mu(dx, dy). \quad (8.2)$$

desintr

The probability measure μ_y is called the conditional measure on \mathcal{X}_1 and is usually denoted $\mu(\cdot | y)$, i.e. $\mu(A | y)$ is the probability of $A \in \mathcal{B}(\mathcal{X}_1)$ given that $\pi_2(\underline{x}) = y$.

8.2 The quantum harmonic crystal model revisited.

appdx1

In this section, we give an application of Dobrushin's uniqueness theorem to the quantum harmonic crystal model. In this lattice model, we associate with each site $j \in \mathbb{Z}^d$ a one-particle Hilbert space $L^2(\mathbb{R}) = L^2(\mathbb{R}, dx_j)$ where dx_j is the Lebesgue measure on \mathbb{R} .

Notations. Hereafter, we identify τ -periodic functions on \mathbb{R} with functions on the 1-dimensional torus $\mathbb{T}_\tau := \mathbb{R}/(\tau\mathbb{Z})$ which we define by identifying points in \mathbb{R} that differ by τn for some $n \in \mathbb{Z}$. The state space is the Banach space $\Omega_\beta := \mathcal{C}(\mathbb{T}_\beta)$ of β -periodic continuous parametrised paths, endowed with the supremum norm $\|\cdot\|_\infty$ and equipped with the Borel σ -algebra $\mathcal{B}(\Omega_\beta)$ of its subsets. We introduce the real Hilbert space $L^2(\mathbb{T}_\beta)$. The standard inner product and norm are denoted by $\langle \cdot, \cdot \rangle_\beta$ and $\|\cdot\|_2$ respectively. By $\mathcal{B}(L^2(\mathbb{T}_\beta))$ we denote the Borel σ -algebra of subsets of $L^2(\mathbb{T}_\beta)$. Note that $\Omega_\beta \in \mathcal{B}(L^2(\mathbb{T}_\beta))$ and $\mathcal{B}(\Omega_\beta) = \mathcal{B}(L^2(\mathbb{T}_\beta)) \cap \Omega_\beta$. We refer to the beginning of Sec. 3.1 for notations related to the configuration spaces.

The quantum harmonic crystal is described by the formal translation-invariant Hamiltonian

$$H^{\text{qua}} := - \sum_{j \in \mathbb{Z}^d} \frac{1}{2} \frac{d^2}{dx_j^2} + \sum_{j \in \mathbb{Z}^d} \frac{1}{2} \alpha x_j^2 + \sum_{j \in \mathbb{Z}^d} \sum_{l \in N_1(j)} \frac{1}{2} (x_j - x_l)^2, \quad (8.3)$$

HquaZ

for some $\alpha > 0$. (8.3) may be represented by the family $\{H_\Lambda\}_{\Lambda \in \mathcal{S}}$ of local Hamiltonians

$$H_\Lambda := - \sum_{j \in \Lambda} \frac{1}{2} \frac{d^2}{dx_j^2} + \sum_{j \in \Lambda} \frac{1}{2} \alpha x_j^2 + \sum_{j \in \Lambda} \sum_{l \in N_1(j) \cap \Lambda} \frac{1}{2} (x_j - x_l)^2. \quad (8.4)$$

HLqt

By standard arguments, (8.4) defines a family of bounded below essentially self-adjoint operators acting in $L^2(\mathbb{R}^{|\Lambda|})$ with discrete spectrum. The definition (8.4) corresponds to the free (or zero) boundary conditions. The system described by the family $\{H_\Lambda\}_{\Lambda \in \mathcal{S}}$ of local Schrödinger operators can be equivalently described by the family of local path measures $\{\mu_\Lambda^\beta\}_{\Lambda \in \mathcal{S}}$ defined as follows. The semi-group $\{\exp(-\tau H_\Lambda), \tau > 0\}$ associated with H_Λ is positivity preserving and of trace class, i.e. $\text{Tr}[\exp(-\tau H_\Lambda)] < \infty$ for all $\tau > 0$. Thus, for every $\beta > 0$, the semi-group $\exp(-\beta H_\Lambda)$ generates a stationary β -periodic Markov process, see, e.g., [1, Sec. 3]. This stochastic process has a canonical realisation on $(\Omega_\beta^\Lambda, \mathcal{B}(\Omega_\beta^\Lambda))$ described by the measure $\mu_\Lambda^\beta \in \mathcal{P}(\Omega_\beta^\Lambda)$, the marginal distributions of which are given by the integral kernels of the operator $\exp(-\tau H_\Lambda)$, $\tau \in [0, \beta]$. By

means of the Feynman-Kac formula, the measure μ_Λ^β on $(\Omega_\beta^\Lambda, \mathcal{B}(\Omega_\beta^\Lambda))$ is then defined as

$$\mu_\Lambda^\beta(d\omega_\Lambda) := \frac{1}{Z_\Lambda^\beta} \exp \left(- \sum_{j \in \Lambda} \frac{1}{2} \alpha \int_0^\beta \omega_j(\tau)^2 d\tau \right) \times \exp \left(- \sum_{j \in \Lambda} \sum_{l \in N_1(j) \cap \Lambda} \frac{1}{2} \int_0^\beta (\omega_j(\tau) - \omega_l(\tau))^2 d\tau \right) \prod_{j \in \Lambda} \mu_0^\beta(d\omega_j), \quad (8.5) \quad \text{FKmea}$$

where Z_Λ^β is a normalisation constant, and μ_0^β denotes the Brownian bridge measure on $(\Omega_\beta, \mathcal{B}(\Omega_\beta))$ defined by means of the conditional Wiener measures with the condition $\omega_j(0) = \omega_j(\beta)$, see, e.g., [11, Sec. 6.3.2]. Thus defined, $\{\mu_\Lambda^\beta\}_{\Lambda \in \mathcal{S}}$ forms the family of local (Euclidean) Gibbs distributions with zero boundary conditions.

Next, define the embedding $\iota : \Omega_\beta \hookrightarrow L^2(\mathbb{T}_\beta)$, $\iota(f) = f$. Since $\|\iota(f)\|_2 \leq \beta \|f\|_\infty$, then ι is a continuous injection. $\mu_0^\beta \circ \iota^{-1}$ is the image measure of μ_0^β on $L^2(\mathbb{T}_\beta)$ and $\mu_0^\beta \circ \iota^{-1}(\Omega_\beta) = 1$. The extension of the family of local Gibbs distributions to $((L^2(\mathbb{T}_\beta))^\Lambda, \mathcal{B}((L^2(\mathbb{T}_\beta))^\Lambda))$ is then defined similarly to (8.5) but with $\mu_0^\beta \circ \iota^{-1}$. Unless otherwise specified, we will not hereafter distinguish in our notation measures on $(\Omega_\beta^\Lambda, \mathcal{B}(\Omega_\beta^\Lambda))$ from their extensions to $((L^2(\mathbb{T}_\beta))^\Lambda, \mathcal{B}((L^2(\mathbb{T}_\beta))^\Lambda))$.

For the quantum harmonic crystal model we have the well-known result

quantc

Proposition 8.4 *Consider the following formal energy functional (Hamiltonian) with nearest-neighbour interactions defined on $(L^2(\mathbb{T}_\beta))^{\mathbb{Z}^d}$ as*

$$h^{qua}(\underline{\omega}) := \sum_{j \in \mathbb{Z}^d} \frac{1}{2} \alpha \int_0^\beta \omega_j(\tau)^2 d\tau + \sum_{j \in \mathbb{Z}^d} \sum_{l \in N_1(j)} \frac{1}{2} \int_0^\beta (\omega_j(\tau) - \omega_l(\tau))^2 d\tau. \quad (8.6) \quad \text{Hqt}$$

Then, provided that $\alpha > 0$, there exists, for all $\beta > 0$, a unique limit (Euclidean) Gibbs distribution in $\mathcal{P}((L^2(\mathbb{T}_\beta))^{\mathbb{Z}^d})$ associated with (8.6).

Proof of Proposition 8.4. Fix $j \in \mathbb{Z}$ and $\underline{\eta}, \underline{\eta}' \in (L^2(\mathbb{T}_\beta))^{N_1(j)}$ distinct. Set $\eta := \sum_{l \in N_1(j)} \eta_l \in L^2(\mathbb{T}_\beta)$ and $\eta' := \sum_{l \in N_1(j)} \eta'_l \in L^2(\mathbb{T}_\beta)$. In view of (8.6), the 1-point Gibbs distribution reads

$$\mu_j^\beta(d\omega|\underline{\eta}) := \frac{1}{Z_j^\beta(\underline{\eta})} \exp \left(- \frac{1}{2} \alpha_d \|\omega\|_2^2 + \langle \omega, \eta \rangle_\beta \right) \mu_0^\beta(d\omega),$$

where we set $\alpha_d := \alpha + 2d$, and with

$$Z_j^\beta(\underline{\eta}) := \int_{L^2(\mathbb{T}_\beta)} \exp \left(- \frac{1}{2} \alpha_d \|\omega\|_2^2 + \langle \omega, \eta \rangle_\beta \right) \mu_0^\beta(d\omega).$$

We now construct a coupling in $\mathcal{P}(L^2(\mathbb{T}_\beta) \times L^2(\mathbb{T}_\beta))$ such that the marginals coincide with the 1-point Gibbs distribution above with the different boundary conditions, see (8.9) and (8.10) below. To do so, introduce the 1-point correlation function defined by

$$\rho_j^\beta(f|\underline{\eta}) := \int_{L^2(\mathbb{T}_\beta)} \exp(i \langle f, \omega \rangle_\beta) \mu_j^\beta(d\omega|\underline{\eta}), \quad f \in L^2(\mathbb{T}_\beta). \quad (8.7) \quad \text{1pointcorr}$$

Define the Fourier coefficients of f as follows

$$\hat{f}_\ell := \frac{1}{\beta} \left\langle f, \exp \left(- \frac{2i\pi}{\beta} \ell \cdot \right) \right\rangle_\beta, \quad \ell \in \mathbb{Z}.$$

We claim that, under the above conditions, (8.7) can be rewritten as

$$\rho_j^\beta(f|\underline{\eta}) = \exp \left(\sum_{\ell \in \mathbb{Z}} \frac{1}{(2\pi\beta^{-1}\ell)^2 + \alpha_d} \left(- \frac{1}{2} |\hat{f}_\ell|^2 + i \left[\Re(\hat{f}_\ell) \Re(\hat{\eta}_\ell) + \Im(\hat{f}_\ell) \Im(\hat{\eta}_\ell) \right] \right) \right). \quad (8.8) \quad \text{rerhoj}$$

For reader's convenience, the proof of (8.8) is deferred to Sec. 8.3.2. Note that (8.8) is nothing but the characteristic function of a product of shifted Gaussian measures on \mathbb{R}^2 when $\ell \neq 0$ and \mathbb{R} when $\ell = 0$, centered at $c_\ell(\beta)(\Re(\hat{\eta}_\ell), \Im(\hat{\eta}_\ell))$ and at $c_0(\beta)\hat{\eta}_0$ respectively, and with covariance

$$c_\ell(\beta) := \frac{1}{(2\pi\beta^{-1}\ell)^2 + \alpha_d}.$$

More precisely, $\mu_j^\beta(\cdot | \underline{\eta}) = \bigotimes_{\ell \in \mathbb{Z}} \gamma_\ell^\beta$, where, when $\ell = 0$ and $\ell \neq 0$ respectively,

$$\begin{aligned} \gamma_0^\beta(dx) &:= \exp\left(-\frac{1}{2c_0(\beta)}(x - c_0(\beta)\hat{\eta}_0)^2\right) \frac{dx}{\sqrt{2\pi c_0(\beta)}}; \\ \gamma_\ell^\beta(dx, dy) &:= \exp\left(-\frac{1}{2c_\ell(\beta)}(x - c_\ell(\beta)\Re(\hat{\eta}_\ell))^2\right) \frac{dx}{\sqrt{2\pi c_\ell(\beta)}} \\ &\quad \times \exp\left(-\frac{1}{2c_\ell(\beta)}(y - c_\ell(\beta)\Im(\hat{\eta}_\ell))^2\right) \frac{dy}{\sqrt{2\pi c_\ell(\beta)}}. \end{aligned}$$

Note that x and y above denote respectively the real and imaginary parts of $\hat{\omega}_\ell$. Subsequently, we define the coupling $\sigma_{j;\underline{\eta},\underline{\eta}'}^\beta \in \mathcal{P}(L^2(\mathbb{T}_\beta) \times L^2(\mathbb{T}_\beta))$ as

$$\sigma_{j;\underline{\eta},\underline{\eta}'}^\beta(d\omega, d\omega') := \bigotimes_{\ell \in \mathbb{Z}} \gamma_\ell^\beta(d\hat{\omega}_\ell) \delta(\hat{\omega}'_\ell - \hat{\omega}_\ell - c_\ell(\beta)(\hat{\eta}'_\ell - \hat{\eta}_\ell)), \quad (8.9) \quad \boxed{\text{sigjet}}$$

for any $A, B \in \mathcal{B}(L^2(\mathbb{T}_\beta))$. Here, δ denotes the Dirac measure. We can readily check that

$$\sigma_{j;\underline{\eta},\underline{\eta}'}^\beta(A \times L^2(\mathbb{T}_\beta)) = \mu_j^\beta(A | \underline{\eta}) \quad \text{and} \quad \sigma_{j;\underline{\eta},\underline{\eta}'}^\beta(L^2(\mathbb{T}_\beta) \times B) = \mu_j^\beta(B | \underline{\eta}'). \quad (8.10) \quad \boxed{\text{coin-0}}$$

By Cauchy-Schwarz inequality, we have

$$\begin{aligned} \int_{L^2(\mathbb{T}_\beta) \times L^2(\mathbb{T}_\beta)} \|\omega - \omega'\|_2 d\sigma_{j;\underline{\eta},\underline{\eta}'}^\beta(d\omega, d\omega') \\ \leq \sqrt{\int_{L^2(\mathbb{T}_\beta) \times L^2(\mathbb{T}_\beta)} \|\omega - \omega'\|_2^2 d\sigma_{j;\underline{\eta},\underline{\eta}'}^\beta(d\omega, d\omega')} = \sqrt{\sum_{\ell \in \mathbb{Z}} c_\ell(\beta)^2 |\hat{\eta}_\ell - \hat{\eta}'_\ell|^2}. \end{aligned}$$

To derive the right-hand side, we used Parseval's identity followed by a direct computation from (8.9) and the definition of the γ_ℓ^β 's. Finally, Parseval's identity yields

$$\rho_W\left(\mu_j^\beta(\cdot | \underline{\eta}), \mu_j^\beta(\cdot | \underline{\eta}')\right) \leq c_0(\beta) \|\eta - \eta'\|_2 \leq \frac{1}{\alpha_d} \sum_{l \in N_1(j)} \|\eta_l - \eta'_l\|_2,$$

and the Proposition follows from the condition in (3.14). \square

remmm3

Remark 8.5 In lattice models of quantum anharmonic crystals, see, e.g., [5, 6, 30] and references therein, the whole system is formally described by

$$\tilde{H}^{qua} := - \sum_{j \in \mathbb{Z}^d} \frac{1}{2} \frac{d^2}{dx_j^2} + \sum_{j \in \mathbb{Z}^d} \frac{1}{2} \alpha x_j^2 + \sum_{j \in \mathbb{Z}^d} g(x_j) + \sum_{j \in \mathbb{Z}^d} \sum_{l \in N_1(j)} \frac{1}{2} (x_j - x_l)^2. \quad (8.11) \quad \boxed{\text{HquaZ2}}$$

(8.11) may be represented by the corresponding family $\{\tilde{H}_\Lambda\}_{\Lambda \in \mathcal{S}}$ of local Hamiltonians

$$\tilde{H}_\Lambda := - \sum_{j \in \Lambda} \frac{1}{2} \frac{d^2}{dx_j^2} + \sum_{j \in \Lambda} \left(\frac{1}{2} \alpha x_j^2 + g(x_j)\right) + \sum_{j \in \Lambda} \sum_{l \in N_1(j) \cap \Lambda} \frac{1}{2} (x_j - x_l)^2. \quad (8.12) \quad \boxed{\text{HLqt2}}$$

Assuming that g is continuous a.e. and bounded from below, (8.12) defines a family of bounded below essentially self-adjoint operators acting in $L^2(\mathbb{R}^{|\Lambda|})$ with compact resolvent. By mimicking the arguments above (8.5), $\{\tilde{H}_\Lambda\}_{\Lambda \in \mathcal{S}}$ can be equivalently described by the family $\{\tilde{\mu}_\Lambda^\beta\}_{\Lambda \in \mathcal{S}}$ of measures

$$\begin{aligned} \tilde{\mu}_\Lambda^\beta(d\omega_\Lambda) &:= \frac{1}{Z_\Lambda^\beta} \exp\left[-\sum_{j \in \Lambda} \left(\frac{1}{2}\alpha \int_0^\beta \omega_j(\tau)^2 d\tau + \int_0^\beta g(\omega_j(\tau)) d\tau\right)\right] \\ &\quad \times \exp\left[-\sum_{j \in \Lambda} \sum_{l \in N_1(j) \cap \Lambda} \frac{1}{2} \int_0^\beta (\omega_j(\tau) - \omega_l(\tau))^2 d\tau\right] \prod_{j \in \Lambda} \mu_0^\beta(d\omega_j), \end{aligned}$$

where μ_0^β the Brownian bridge measure. Thus defined, $\{\tilde{\mu}_\Lambda^\beta\}_{\Lambda \in \mathcal{S}}$ on $((L^2(\mathbb{T}_\beta))^\Lambda, \mathcal{B}((L^2(\mathbb{T}_\beta))^\Lambda))$ forms the actual family of local (Euclidean) Gibbs distributions with zero boundary conditions. We refer to [2, 4] and reference therein for the uniqueness problem in some lattice models of quantum anharmonic crystals with translation-invariant Hamiltonians of type (8.11).

8.3 Gaussian correlations functions.

appdx2

In this section, we give a direct proof of the exponential decay of correlation functions for the Gaussian free-field model with 1-dimensional spins and the quantum harmonic crystal model in the high temperature regime. We refer to [24, 31] and references therein for further related results.

8.3.1 The classical case.

For the notations used in this subsection, we refer to Sec. 6. Let $\Delta, \Lambda \in \mathcal{S}$ such that $\Delta \subset \Lambda$. Given $\beta > 0$ and $\underline{t}_\Delta \in \mathbb{R}^\Delta$, we define the correlation functions by

$$\rho_{\Delta, \Lambda}^\beta(\underline{t}_\Delta) := \int_{\mathbb{R}^\Lambda} \exp\left(i\beta \sum_{j \in \Delta} t_j x_j\right) \mu_\Lambda^\beta(d\underline{x}_\Lambda). \quad (8.13) \quad \text{correl}$$

The local Gibbs distribution in $\mathcal{P}(\mathbb{R}^\Lambda)$ for the Gaussian free-field model with 1-dimensional spin generated by the Hamiltonian in (6.4) reads

$$\mu_\Lambda^\beta(d\underline{x}_\Lambda) := \frac{1}{Z_\Lambda^\beta} \exp\left(-\beta \left(\sum_{j \in \Lambda} \frac{1}{2} \alpha x_j^2 + \sum_{j \in \Lambda} \sum_{l \in N_1(j) \cap \Lambda} \frac{1}{2} (x_j - x_l)^2\right)\right) \prod_{j \in \Lambda} dx_j,$$

where $\alpha > 0$ and Z_Λ^β is the corresponding normalisation constant.

decaycorr

Proposition 8.6 *For every $\beta_0 > 0$ there exist two positive constants $c = c(\beta_0, \alpha, d)$ and $C = C(\beta_0, \alpha, d)$ such that, for any $\Delta, \Lambda, \Lambda' \in \mathcal{S}$ such that $\Delta \subset \Lambda \subset \Lambda'$ and for any $0 < \beta \leq \beta_0$,*

$$\left\| \rho_{\Delta, \Lambda}^\beta - \rho_{\Delta, \Lambda'}^\beta \right\| := \sup_{\underline{t}_\Delta \in \mathbb{R}^\Delta : \|\underline{t}_\Delta\| = 1} \left| \rho_{\Delta, \Lambda}^\beta(\underline{t}_\Delta) - \rho_{\Delta, \Lambda'}^\beta(\underline{t}_\Delta) \right| \leq C e^{-c\beta \text{dist}(\Delta, \partial\Lambda)}. \quad (8.14) \quad \text{rhocla}$$

Remark 8.7 *The above result implies in particular that the measures μ_Λ^β converge to a measure μ^β as Λ tends to \mathbb{Z}^d .*

Proof. Introduce the matrix M_Λ with elements $(M_{j,l})_{j,l \in \Lambda}$ defined as

$$M_{j,l} := \begin{cases} |\{j' \in \Lambda : |j - j'| = 1\}|, & \text{if } j = l; \\ -1 & \text{if } |j - l| = 1; \\ 0 & \text{otherwise.} \end{cases} \quad (8.15) \quad \text{MLambda}$$

Then we can write

$$Z_\Lambda^\beta = \int_{\mathbb{R}^\Lambda} \exp\left(-\beta \frac{1}{2} \langle \underline{x}_\Lambda, (\alpha \mathbb{I}_\Lambda + M_\Lambda) \underline{x}_\Lambda \rangle\right) \prod_{j \in \Lambda} dx_j,$$

and,

$$Z_{\Lambda}^{\beta} \rho_{\Delta, \Lambda}^{\beta}(t_{\Delta}) = \int_{\mathbb{R}^{\Lambda}} \exp \left(-\beta \left(\frac{1}{2} \langle \underline{x}_{\Lambda}, (\alpha \mathbb{I}_{\Lambda} + M_{\Lambda}) \underline{x}_{\Lambda} \rangle - i \langle \tilde{t}_{\Lambda}, \underline{x}_{\Lambda} \rangle \right) \right) \prod_{j \in \Lambda} dx_j,$$

where $\tilde{t}_j := t_j$ if $j \in \Delta$, $\tilde{t}_j := 0$ if $j \in \Lambda \setminus \Delta$. Now, the corresponding Gaussian integrals can be computed using the well-known formula:

$$\int_{\mathbb{R}^m} \exp \left(-\frac{1}{2} \langle \underline{x}, M \underline{x} \rangle + \langle \underline{b}, \underline{x} \rangle \right) \prod_{k=1}^m dx_k = \frac{(2\pi)^{\frac{m}{2}}}{\sqrt{\det(M)}} \exp \left(\frac{1}{2} \langle \underline{b}, M^{-1} \underline{b} \rangle \right), \quad (8.16) \quad \boxed{\text{fundam}}$$

where $\underline{x} \in \mathbb{R}^m$, $\underline{b} \in \mathbb{R}^m$ and $M \in \mathbb{M}_m(\mathbb{R})$ is an invertible matrix. It follows from (8.16) that (below the subscript Δ indicates the restriction to Δ),

$$\rho_{\Delta, \Lambda}^{\beta}(t_{\Delta}) = \exp \left(-\beta \frac{1}{2} \langle \underline{t}_{\Delta}, (\alpha \mathbb{I}_{\Lambda} + M_{\Lambda})_{\Delta}^{-1} \underline{t}_{\Delta} \rangle \right). \quad (8.17) \quad \boxed{\text{rhoWW}}$$

To prove (8.14), it thus suffices to prove that

$$\|(\alpha \mathbb{I}_{\Lambda} + M_{\Lambda})_{\Delta}^{-1} - (\alpha \mathbb{I}_{\Lambda'} + M_{\Lambda'})_{\Delta}^{-1}\| \leq C' e^{-c \text{dist}(\Delta, \partial \Lambda)}, \quad (8.18)$$

for some constant $C' > 0$. Set $\alpha_d := \alpha + 2d$ and $R_{\Lambda} := 2d \mathbb{I}_{\Lambda} - M_{\Lambda}$. We then have,

$$(\alpha \mathbb{I}_{\Lambda} + M_{\Lambda})^{-1} = (\alpha_d \mathbb{I}_{\Lambda} - R_{\Lambda})^{-1} = \alpha_d^{-1} \sum_{k=0}^{\infty} \alpha_d^{-k} (R_{\Lambda})^k,$$

and the above series converges since $\|R_{\Lambda}\| \leq 2d < \alpha_d$. For $(\alpha \mathbb{I}_{\Lambda'} + M_{\Lambda'})^{-1}$, a similar expansion obviously holds true. We now claim that $(R_{\Lambda'})_{\Delta}^k = (R_{\Lambda})_{\Delta}^k$ if $k \leq \text{dist}(\Delta, \partial \Lambda)$. We proceed by induction to show, in fact, that $(R_{\Lambda'})_{j,l}^k = (R_{\Lambda})_{j,l}^k$ if $j, l \in \Lambda$ and $k \leq \text{dist}(\{j, l\}, \partial \Lambda)$. For $k = 1$, clearly $(R_{\Lambda'})_{j,l} = (R_{\Lambda})_{j,l}$ if $\{j, l\} \subset \Lambda$. For $k \geq 1$ and $j, l \in \Lambda$ with $\text{dist}(\{j, l\}, \partial \Lambda) \geq k + 1$,

$$(R_{\Lambda'})_{j,l}^{k+1} = \sum_{j' \in \Lambda'} (R_{\Lambda'})_{j,j'}^k (R_{\Lambda'})_{j',l}.$$

The only non-zero terms are those with $|j' - l| \leq 1$. Then $j' \in \Lambda$ and $\text{dist}(j', \partial \Lambda) \geq \text{dist}(l, \partial \Lambda) - |j' - l| \geq k$, so $(R_{\Lambda'})_{j,j'}^k = (R_{\Lambda})_{j,j'}^k$. Therefore,

$$(R_{\Lambda'})_{j,l}^{k+1} = \sum_{j' \in \Lambda'} (R_{\Lambda'})_{j,j'}^k (R_{\Lambda'})_{j',l} = \sum_{j' \in \Lambda'} (R_{\Lambda})_{j,j'}^k (R_{\Lambda})_{j',l} = (R_{\Lambda})_{j,l}^{k+1}.$$

The proposition eventually follows from the following identity

$$(\alpha \mathbb{I}_{\Lambda} + M_{\Lambda})_{\Delta}^{-1} - (\alpha \mathbb{I}_{\Lambda'} + M_{\Lambda'})_{\Delta}^{-1} = \alpha_d^{-1} \sum_{k=\text{dist}(\Delta, \partial \Lambda)+1}^{\infty} \alpha_d^{-k} ((R_{\Lambda})_{\Delta}^k - (R_{\Lambda'})_{\Delta}^k). \quad \square$$

Remark 8.8 By the same arguments, there exists a constant $c > 0$ such that, for any $\Lambda \in \mathcal{S}$,

$$|(\alpha \mathbb{I}_{\Lambda} + M_{\Lambda})_{j,l}^{-1}| \leq e^{-c|j-l|}.$$

Remark 8.9 The result of Proposition 8.6 can be generalized to higher spin dimensions.

8.3.2 The quantum case.

seccant

To extend the above arguments to the quantum harmonic crystal model, an expression for the correlation functions analogous to (8.17) is derived. For the notations used in this section, we refer

to Sec. 8.2. Let $\Delta, \Lambda \in \mathcal{S}$ such that $\Delta \subset \Lambda$. Given $\beta > 0$ and a family of functions $\underline{f}_\Delta := (f_j)_{j \in \Delta}$ with $f_j \in L^2(\mathbb{T}_\beta)$, the correlation functions are defined by

$$\rho_\Lambda^\beta(\underline{f}_\Delta) := \int_{(L^2(\mathbb{T}_\beta))^\Lambda} \left(\prod_{j \in \Delta} \exp(i \langle f_j, \omega_j \rangle_\beta) \right) \mu_\Lambda^\beta(d\omega_\Lambda). \quad (8.19) \quad \boxed{\text{correlq}}$$

Remember that, for any $\beta > 0$, the local Gibbs distribution μ_Λ^β in $\mathcal{P}((L^2(\mathbb{T}_\beta))^\Lambda)$ reads

$$\mu_\Lambda^\beta(d\omega_\Lambda) := \frac{1}{Z_\Lambda^\beta} \exp \left(- \sum_{j \in \Lambda} \frac{1}{2} \alpha \|\omega_j\|_2^2 \right) \exp \left(- \frac{1}{2} \sum_{j \in \Lambda} \sum_{l \in N_1(j) \cap \Lambda} \|\omega_j - \omega_l\|_2^2 \right) \prod_{j \in \Lambda} \mu_0^\beta(d\omega_j),$$

where $\alpha > 0$, Z_Λ^β is the normalisation constant and μ_0^β is the standard Brownian bridge measure. The Fourier coefficients of the f_j 's are defined as

$$\hat{f}_{j,\ell} := \frac{1}{\beta} \left\langle f_j, \exp \left(- \frac{2i\pi}{\beta} \ell \cdot \right) \right\rangle_\beta, \quad \ell \in \mathbb{Z}. \quad (8.20) \quad \boxed{\text{Fourier}}$$

Here is the counterpart of (8.17) in the quantum case

rewrq **Lemma 8.10** *Let M_Λ be the matrix defined through (8.15). Then, under the above conditions,*

$$\rho_\Lambda^\beta(\underline{f}_\Delta) = \exp \left(- \frac{1}{2} \beta \sum_{\ell \in \mathbb{Z}} \left\langle \overline{(\hat{f}_\Delta)_\ell}, ((2\pi\beta^{-1}\ell)^2 + \alpha) \mathbb{I}_\Lambda + M_\Lambda \right\rangle_\Delta^{-1} (\hat{f}_\Delta)_\ell \right), \quad (8.21)$$

where the subscript Δ for the inverse matrix stands for its restriction to Δ .

To prove Lemma 8.10, we need the following technical lemma

singh **Lemma 8.11** *For any $\gamma > 0$,*

$$\lim_{n \rightarrow \infty} \sqrt{\prod_{j=1}^{n-1} \left(2 \left(1 - \cos \left(\frac{2\pi}{n} j \right) \right) + \left(\frac{\gamma}{n} \right)^2 \right)} = 2 \sinh \left(\frac{\gamma}{2} \right). \quad (8.22) \quad \boxed{\sinh}$$

Proof. First note the following identity

$$\prod_{j=0}^{n-1} \left(2 \left(1 - \cos \left(\frac{2\pi}{n} j \right) \right) + \left(\frac{\gamma}{n} \right)^2 \right) = 2^n \prod_{j=0}^{n-1} \left(\cosh(\vartheta_n) - \cos \left(\frac{2\pi}{n} j \right) \right),$$

where we set $\vartheta_n := \text{arcosh}(1 + \frac{\gamma^2}{2n^2})$. Recall that $\text{arcosh}(x) = \ln(x + \sqrt{x^2 - 1})$, $x \geq 1$. Next, use

$$2^{m-1} \prod_{k=0}^{m-1} \left(\cosh(x) - \cos \left(y + \frac{2\pi}{m} k \right) \right) = \cosh(mx) - \cos(my), \quad m \in \mathbb{N}.$$

Letting $y = 0$ and $m = n$ in the above formula, we then obtain,

$$\prod_{j=0}^{n-1} \left(2 \left(1 - \cos \left(\frac{2\pi}{n} j \right) \right) + \left(\frac{\gamma}{n} \right)^2 \right) = 2(\cosh(n\vartheta_n) - 1).$$

It remains to use that $\lim_{n \rightarrow \infty} n\vartheta_n = \lim_{n \rightarrow \infty} n \cdot \text{arcosh}(1 + \frac{\gamma^2}{2n^2}) = \gamma$, together with the identity $\sinh(\frac{x}{2}) = \text{sign}(x) \sqrt{\frac{\cosh(x)-1}{2}}$ and the lemma follows. \square

Proof of Lemma 8.10. Let $g_\Lambda := (g_j)_{j \in \Lambda}$ with $g_j := f_j$ if $j \in \Delta$ and $g_j := 0$ otherwise. Given $n \in \mathbb{N}$, let $\underline{g}_{\Lambda,k} := (g_{j,k})_{j \in \Lambda}$ with $g_{j,k} := g_j(\frac{\beta}{n}k) \forall k \leq n$ (hereafter, we use that $f_j(0) = f_j(\beta)$). Introduce the sequence of functions $(W_n^{\beta,\zeta})_{n \in \mathbb{N}}$, with $W_n^{\beta,\zeta}(\cdot) = W_n^\beta(\cdot, (\underline{g}_{\Lambda,k})_{k=1}^n, \zeta) : \mathbb{R}^{n|\Lambda|} \rightarrow \mathbb{C}$ defined as

$$W_n^{\beta,\zeta}((\underline{x}_{\Lambda,k})_{k=1}^n) := \left(\frac{2\pi\beta}{n} \right)^{-\frac{n|\Lambda|}{2}} \left(\prod_{j \in \Lambda} \exp \left(-\frac{n}{2\beta} \left(\sum_{k=2}^n (x_{j,k} - x_{j,k-1})^2 + (x_{j,n} - x_{j,1})^2 \right) \right) \right) \\ \times \left(\prod_{j \in \Lambda} \prod_{k=1}^n \exp \left(-\frac{\beta}{2n} \left(\alpha x_{j,k}^2 + \sum_{l \in N_1(j) \cap \Lambda} (x_{j,k} - x_{l,k})^2 \right) \right) \right) \exp \left(i\zeta \frac{\beta}{n} \sum_{k=1}^n \langle \underline{g}_{\Lambda,k}, \underline{x}_{\Lambda,k} \rangle \right).$$

Below, we only consider the values $\zeta \in \{0, 1\}$. By definition of Wiener measure, see [11, Sec. 6.3.2],

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^{n|\Lambda|}} \left(\prod_{j \in \Lambda} \prod_{k=1}^n dx_{j,k} \right) W_n^{\beta,\zeta}((\underline{x}_{\Lambda,k})_{k=1}^n) = \begin{cases} Z_\Lambda^\beta, & \text{if } \zeta = 0; \\ Z_\Lambda^\beta \rho_\Lambda^\beta(f_\Delta), & \text{if } \zeta = 1. \end{cases} \quad (8.23) \quad \boxed{\text{ZLam}}$$

To investigate the limit $n \rightarrow \infty$, we need a suitable rewriting. Define $A_n \in \mathcal{M}_n(\mathbb{R})$ as follows

$$A_n := \begin{pmatrix} \frac{2n}{\beta} & -\frac{n}{\beta} & 0 & \dots & 0 & -\frac{n}{\beta} \\ -\frac{n}{\beta} & \frac{2n}{\beta} & -\frac{n}{\beta} & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & -\frac{n}{\beta} & \frac{2n}{\beta} & -\frac{n}{\beta} \\ -\frac{n}{\beta} & 0 & \dots & 0 & -\frac{n}{\beta} & \frac{2n}{\beta} \end{pmatrix}. \quad (8.24) \quad \boxed{\text{An}}$$

Note that, for any $j \in \Lambda$, we have,

$$\exp \left(-\frac{n}{2\beta} \left(\sum_{k=2}^n (x_{j,k} - x_{j,k-1})^2 + (x_{j,n} - x_{j,1})^2 \right) \right) \prod_{k=1}^n \exp \left(-\frac{\beta}{2n} \alpha x_{j,k}^2 \right) \\ = \exp \left(-\frac{1}{2} \left\langle \underline{x}_j, \left(A_n + \frac{\beta}{n} \alpha \mathbb{I}_n \right) \underline{x}_j \right\rangle \right).$$

Denoting $\underline{x}_\Lambda := (\underline{x}_{\Lambda,k})_{k=1}^n$ and $\underline{g}_\Lambda := (\underline{g}_{\Lambda,k})_{k=1}^n$, we actually have the compact rewriting

$$W_n^{\beta,\zeta}(\underline{x}_\Lambda) = \left(\frac{2\pi\beta}{n} \right)^{-\frac{n|\Lambda|}{2}} \exp \left(-\frac{1}{2} \langle \underline{x}_\Lambda, \mathbb{A} \underline{x}_\Lambda \rangle \right) \exp \left(i\zeta \frac{\beta}{n} \langle \underline{g}_\Lambda, \underline{x}_\Lambda \rangle \right), \quad (8.25) \quad \boxed{\text{Wcomp}}$$

where the matrix $\mathbb{A} \in \mathbb{M}_{n|\Lambda|}(\mathbb{R})$ is defined by the Kronecker sum

$$\mathbb{A} := \mathbb{I}_{|\Lambda|} \otimes A_n + \frac{\beta}{n} (\alpha \mathbb{I}_{|\Lambda|} + M_\Lambda) \otimes \mathbb{I}_n. \quad (8.26) \quad \boxed{\text{defbba}}$$

Note that M_Λ is a symmetric positive definite matrix, see (8.15). As for A_n , it is a symmetric circulant matrix. The latter can be diagonalised by the use of discrete Fourier transform. Define

$$U_{\ell,k} := \frac{1}{\sqrt{n}} \exp \left(\frac{2i\pi}{n} \ell k \right), \quad k = 1, \dots, n; \ell = 0, \dots, n-1.$$

Putting $U := (U_{\ell,k=1}, \dots, U_{\ell,k=n})_{0 \leq \ell \leq n-1}$, we have,

$$U A_n U^* = \text{diag} \left(\lambda_0^{(n)}, \dots, \lambda_{n-1}^{(n)} \right), \quad (8.27) \quad \boxed{\text{unita}}$$

where the eigenvalues read

$$\lambda_\ell^{(n)} := \frac{2n}{\beta} \left(1 - \cos \left(\frac{2\pi\ell}{n} \right) \right), \quad \ell = 0, \dots, n-1. \quad (8.28) \quad \boxed{\text{lambda1}}$$

Consider the case when n is odd. Then $\lambda_\ell^{(n)} = \lambda_{n-\ell}^{(n)}$, i.e. the eigenvalues $\lambda_\ell^{(n)}$ with $\ell = 1, \dots, \frac{n-1}{2}$ are two-fold degenerate. Corresponding real-valued eigenvectors lead to the transformation

$$u_\ell = \sqrt{\frac{2}{n}} \sum_{k=1}^n x_k \cos \left(\frac{2\pi k\ell}{n} \right) \quad \text{and} \quad v_\ell = \sqrt{\frac{2}{n}} \sum_{k=1}^n x_k \sin \left(\frac{2\pi k\ell}{n} \right), \quad (8.29) \quad \boxed{\text{uv1}}$$

and, in the case of $\ell = 0$, we set

$$u_0 := \frac{1}{\sqrt{n}} \sum_{k=1}^n x_k.$$

Note that, by writing $\omega_n := \frac{2\pi}{n}$, the matrix

$$O_n := \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & \dots & 1 & 1 \\ \sqrt{2} \cos(\omega_n) & \sqrt{2} \cos(2\omega_n) & \dots & \sqrt{2} \cos((n-1)\omega_n) & 1 \\ \sqrt{2} \sin(\omega_n) & \sqrt{2} \sin(2\omega_n) & \dots & \sqrt{2} \sin((n-1)\omega_n) & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ \sqrt{2} \cos(\frac{n-1}{2}\omega_n) & \sqrt{2} \cos(2\frac{n-1}{2}\omega_n) & \dots & \sqrt{2} \cos((n-1)\frac{n-1}{2}\omega_n) & 1 \\ \sqrt{2} \sin(\frac{n-1}{2}\omega_n) & \sqrt{2} \sin(2\frac{n-1}{2}\omega_n) & \dots & \sqrt{2} \sin((n-1)\frac{n-1}{2}\omega_n) & 0 \end{pmatrix},$$

is orthogonal, and the inverse transformation is given by

$$x_k = \frac{1}{\sqrt{n}} \left(u_0 + \sqrt{2} \sum_{\ell=1}^{\frac{n-1}{2}} \left(u_\ell \cos \left(\frac{2\pi}{n} k\ell \right) + v_\ell \sin \left(\frac{2\pi}{n} k\ell \right) \right) \right). \quad (8.30) \quad \boxed{\text{uv2}}$$

For any $\underline{x} \in \mathbb{R}^n$, set

$$z_\ell := \sum_{k=1}^n U_{\ell,k} x_k, \quad \ell = 0, \dots, n-1.$$

Note that, in view of (8.29), we have the identities

$$z_\ell = \begin{cases} \frac{1}{\sqrt{2}}(u_\ell + iv_\ell), & \text{if } \ell = 1, \dots, \frac{n-1}{2}; \\ \frac{1}{\sqrt{2}}(u_{n-\ell} - iv_{n-\ell}), & \text{if } \ell = \frac{n+1}{2}, \dots, n. \end{cases} \quad (8.31) \quad \boxed{\text{oddin}}$$

From the above features, we get,

$$\langle \underline{x}, A_n \underline{x} \rangle = \sum_{k,k'=1}^n (A_n)_{k,k'} \left(\sum_{\ell=0}^{n-1} \overline{U_{k,\ell}} z_\ell \right) \left(\sum_{\ell'=0}^n \overline{U_{k',\ell'}} z_{\ell'} \right) = \sum_{\ell,\ell'=0}^{n-1} \overline{z}_\ell (U A_n U^*)_{\ell,\ell'} z_{\ell'},$$

and by using (8.27) along with (8.31),

$$\langle \underline{x}, A_n \underline{x} \rangle = \sum_{\ell=0}^{n-1} \lambda_\ell^{(n)} |z_\ell|^2 = \sum_{\ell=1}^{\frac{n-1}{2}} \lambda_\ell^{(n)} (u_\ell^2 + v_\ell^2). \quad (8.32) \quad \boxed{\text{intermr1}}$$

Defining similarly, for any $\underline{y} \in \mathbb{R}^n$,

$$\tilde{y}_\ell := \sum_{k=1}^n U_{\ell,k} y_k, \quad \ell = 0, \dots, n-1, \quad (8.33) \quad \boxed{\text{defy}}$$

we also have,

$$\langle \underline{y}, \underline{x} \rangle = \sum_{k=1}^n \left(\sum_{\ell=0}^{n-1} \overline{U_{k,\ell}} \tilde{y}_\ell \right) \left(\sum_{\ell'=0}^{n-1} \overline{U_{k,\ell'}} z_{\ell'} \right) = \sum_{\ell,\ell'=0}^{n-1} \tilde{y}_\ell \bar{z}_{\ell'} \sum_{k=1}^n \frac{1}{\sqrt{n}} U_{\ell'-\ell,k} = \sum_{\ell=0}^{n-1} \tilde{y}_\ell \bar{z}_\ell,$$

and by using (8.31), it holds

$$\langle \underline{y}, \underline{x} \rangle = u_0 \tilde{y}_0 + \frac{1}{\sqrt{2}} \sum_{\ell=1}^{\frac{n-1}{2}} (u_\ell (\tilde{y}_\ell + \tilde{y}_{n-\ell}) - i v_\ell (\tilde{y}_\ell - \tilde{y}_{n-\ell})). \quad (8.34) \quad \boxed{\text{intermr2}}$$

From (8.25) and (8.26) combined with (8.29)-(8.30) and (8.32)-(8.34), we obtain,

$$\begin{aligned} \int_{\mathbb{R}^{n|\Lambda|}} \left(\prod_{k=1}^n \prod_{j \in \Lambda} dx_{j,k} \right) W_n^{\beta,\zeta} ((\underline{x}_{\Lambda,k})_{k=1}^n) = \\ \left(\frac{2\pi\beta}{n} \right)^{-\frac{n|\Lambda|}{2}} \left(\int_{\mathbb{R}^{|\Lambda|}} \left(\prod_{j \in \Lambda} du_0 \right) \exp \left(-\frac{1}{2} \left\langle \underline{u}_{0\Lambda}, \frac{\beta}{n} (\alpha \mathbb{I}_\Lambda + M_\Lambda) \underline{u}_{0\Lambda} \right\rangle \right) \exp \left(i\zeta \frac{\beta}{n} \left\langle (\tilde{\underline{g}}_\Lambda)_0, \underline{u}_{0\Lambda} \right\rangle \right) \right) \\ \times \left(\int_{\mathbb{R}^{\frac{n-1}{2}|\Lambda|}} \left(\prod_{j \in \Lambda} \prod_{\ell=1}^{\frac{n-1}{2}} du_\ell \right) \prod_{\ell=1}^{\frac{n-1}{2}} \exp \left(-\frac{1}{2} \left\langle \underline{u}_{\ell\Lambda}, \frac{\beta}{n} \left(\left(\frac{n}{\beta} \lambda_\ell^{(n)} + \alpha \right) \mathbb{I}_\Lambda + M_\Lambda \right) \underline{u}_{\ell\Lambda} \right\rangle \right) \right. \\ \times \exp \left(i\zeta \frac{1}{\sqrt{2}} \frac{\beta}{n} \left\langle (\tilde{\underline{g}}_\Lambda)_\ell + (\tilde{\underline{g}}_\Lambda)_{n-\ell}, \underline{u}_{\ell\Lambda} \right\rangle \right) \\ \times \left(\int_{\mathbb{R}^{\frac{n-1}{2}|\Lambda|}} \left(\prod_{j \in \Lambda} \prod_{\ell=1}^{\frac{n-1}{2}} dv_\ell \right) \prod_{\ell=1}^{\frac{n-1}{2}} \exp \left(-\frac{1}{2} \left\langle \underline{v}_{\ell\Lambda}, \frac{\beta}{n} \left(\left(\frac{n}{\beta} \lambda_\ell^{(n)} + \alpha \right) \mathbb{I}_\Lambda + M_\Lambda \right) \underline{v}_{\ell\Lambda} \right\rangle \right) \right. \\ \times \exp \left(\zeta \frac{1}{\sqrt{2}} \frac{\beta}{n} \left\langle (\tilde{\underline{g}}_\Lambda)_\ell - (\tilde{\underline{g}}_\Lambda)_{n-\ell}, \underline{v}_{\ell\Lambda} \right\rangle \right) \Bigg), \end{aligned}$$

where $\underline{u}_{\ell\Lambda} := (u_\ell)_{j \in \Lambda}$, $\underline{v}_{\ell\Lambda} := (v_\ell)_{j \in \Lambda}$ and $(\tilde{\underline{g}}_\Lambda)_\ell := (\tilde{g}_{j,\ell})_{j \in \Lambda}$ with $\tilde{g}_{j,\ell}$ as in (8.33). By using (8.16) for each one of the three integrals, and then rearranging the $\tilde{g}_{j,\ell}$'s, we are left with

$$\begin{aligned} \int_{\mathbb{R}^{n|\Lambda|}} \left(\prod_{k=1}^n \prod_{j \in \Lambda} dx_{j,k} \right) W_n^{\beta,\zeta} ((\underline{x}_{\Lambda,k})_{k=1}^n) = \\ \left(\frac{\beta}{n} \right)^{-\frac{n|\Lambda|}{2}} \prod_{\ell=0}^{n-1} \frac{\exp \left(-\zeta^2 \frac{1}{2} \frac{\beta}{n} \left\langle (\tilde{\underline{g}}_\Lambda)_\ell, \left(\left(\frac{n}{\beta} \lambda_\ell^{(n)} + \alpha \right) \mathbb{I}_\Lambda + M_\Lambda \right)^{-1} (\tilde{\underline{g}}_\Lambda)_{n-\ell} \right\rangle \right)}{\sqrt{\det \left(\frac{\beta}{n} \left(\left(\frac{n}{\beta} \lambda_\ell^{(n)} + \alpha \right) \mathbb{I}_\Lambda + M_\Lambda \right) \right)}}. \quad (8.35) \quad \boxed{\text{finwr}} \end{aligned}$$

Let $\{\varsigma_p\}_{p=1}^{|\Lambda|}$ be the eigenvalues of M_Λ counting with multiplicities. Note that $\varsigma_p \geq 0$, and moreover,

$$\begin{aligned} \left(\frac{\beta}{n} \right)^{\frac{n|\Lambda|}{2}} \prod_{\ell=0}^{n-1} \sqrt{\det \left(\frac{\beta}{n} \left(\left(\frac{n}{\beta} \lambda_\ell^{(n)} + \alpha \right) \mathbb{I}_\Lambda + M_\Lambda \right) \right)} \\ = \prod_{p=1}^{|\Lambda|} \sqrt{\prod_{\ell=0}^{n-1} \left(2 \left(1 - \cos \left(\frac{2\pi}{n} \ell \right) \right) + \left(\frac{\beta \sqrt{\alpha + \varsigma_p}}{n} \right)^2 \right)}, \end{aligned}$$

where we used (8.28) in the above r.h.s. In view of (8.23) along with (8.35), Lemma 8.11 yields

$$Z_\Lambda^\beta = \prod_{p=1}^{|\Lambda|} \left(2 \sinh \left(\beta \frac{\sqrt{\alpha + \varsigma_p}}{2} \right) \right)^{-1};$$

and in view of (8.35) again along with (8.20), it remains to use the identity

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \tilde{g}_{j,n-\ell} = \lim_{n \rightarrow \infty} \frac{1}{\beta} \frac{\beta}{n} \sum_{k=1}^n \exp \left(i \frac{2\pi(n-\ell)}{\beta} \frac{\beta k}{n} \right) f_j \left(\frac{\beta}{n} k \right) = \hat{f}_{j,\ell}, \quad j \in \Delta, \quad (8.36) \quad \text{bete}$$

to conclude the proof of the lemma. \square

We end this section with the proof of the identity in (8.8)

Proof of (8.8). We follow the strategy used in the above proof. By abuse of notation, we set $\underline{\eta} := (\eta_k)_{k=1}^n$ with $\eta_k := \eta(\frac{\beta}{n}k)$. Similarly, set $\underline{f} := (f_k)_{k=1}^n$ with $f_k := f(\frac{\beta}{n}k)$. Introduce the sequence of functions $w_n^{\beta,\zeta}(\cdot | \underline{\eta}) = w_n^{\beta,\zeta}(\cdot, \underline{f} | \underline{\eta}) : \mathbb{R}^n \rightarrow \mathbb{C}$ defined as

$$w_n^{\beta,\zeta}(\underline{x} | \underline{\eta}) := \left(\frac{2\pi\beta}{n} \right)^{-\frac{n}{2}} \exp \left(-\frac{n}{2\beta} \left(\sum_{k=2}^n (x_k - x_{k-1})^2 + (x_n - x_1)^2 \right) \right) \\ \times \prod_{k=1}^n \exp \left(-\frac{\beta}{2n} (\alpha_d x_k^2 - x_k \eta_k) \right) \exp \left(i\zeta \frac{\beta}{n} \langle \underline{f}, \underline{x} \rangle \right).$$

By definition of Wiener measures, see, e.g., [11, Sec. 6.3.2],

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^n} w_n^{\beta,\zeta}(\underline{x} | \underline{\eta}) \prod_{k=1}^n dx_k = \begin{cases} Z_j^\beta(\underline{\eta}), & \text{if } \zeta = 0; \\ Z_j^\beta(\underline{\eta}) \rho_j^\beta(\underline{f} | \underline{\eta}), & \text{if } \zeta = 1. \end{cases} \quad (8.37) \quad \text{ZLam2}$$

By using (8.24), note the following rewriting

$$w_n^{\beta,\zeta}(\underline{x} | \underline{\eta}) = \left(\frac{2\pi\beta}{n} \right)^{-\frac{n}{2}} \exp \left(-\frac{1}{2} \left\langle \underline{x}, \left(A_n + \frac{\beta}{n} \alpha_d \mathbb{I}_n \right) \underline{x} \right\rangle \right) \exp \left(\frac{\beta}{n} \langle \underline{\eta} + i\zeta \underline{f}, \underline{x} \rangle \right). \quad (8.38) \quad \text{smallw}$$

Define

$$\tilde{\eta}_\ell := \frac{1}{\sqrt{n}} \sum_{k=1}^n \exp \left(\frac{2i\pi}{n} k\ell \right) \eta_k; \quad \tilde{f}_\ell := \frac{1}{\sqrt{n}} \sum_{k=1}^n \exp \left(\frac{2i\pi}{n} k\ell \right) f_k, \quad \ell = 0, \dots, n-1.$$

Using (8.34), note also the identity

$$\langle \underline{\eta} + i\zeta \underline{f}, \underline{x} \rangle = u_0(\tilde{\eta}_0 + i\zeta \tilde{f}_0) + \sqrt{2} \sum_{\ell=1}^{\frac{n-1}{2}} \left(u_\ell \left(\Re(\tilde{\eta}_\ell) + i\zeta \Re(\tilde{f}_\ell) \right) + v_\ell \left(\Im(\tilde{\eta}_\ell) + i\zeta \Im(\tilde{f}_\ell) \right) \right). \quad (8.39) \quad \text{intermr3}$$

From (8.38), (8.32) together with (8.39), we obtain,

$$\int_{\mathbb{R}^n} w_n^{\beta,\zeta}(\underline{x} | \underline{\eta}) \prod_{k=1}^n dx_k = \\ \left(\frac{2\pi\beta}{n} \right)^{-\frac{n}{2}} \int_{\mathbb{R}^n} du_0 \left(\prod_{\ell=1}^{\frac{n-1}{2}} du_\ell \right) \left(\prod_{\ell=1}^{\frac{n-1}{2}} dv_\ell \right) \exp \left(-\frac{1}{2} \frac{\beta}{n} \left(\frac{n}{\beta} \lambda_0^{(n)} + \alpha_d \right) u_0^2 + \frac{\beta}{n} (\tilde{\eta}_0 + i\zeta \tilde{f}_0) u_0 \right) \\ \times \prod_{\ell=1}^{\frac{n-1}{2}} \exp \left(-\frac{1}{2} \frac{\beta}{n} \left(\frac{n}{\beta} \lambda_\ell^{(n)} + \alpha_d \right) u_\ell^2 + \frac{\beta}{n} \sqrt{2} \left(\Re(\tilde{\eta}_\ell) + i\zeta \Re(\tilde{f}_\ell) \right) u_\ell \right) \\ \times \prod_{\ell=1}^{\frac{n-1}{2}} \exp \left(-\frac{1}{2} \frac{\beta}{n} \left(\frac{n}{\beta} \lambda_\ell^{(n)} + \alpha_d \right) v_\ell^2 + \frac{\beta}{n} \sqrt{2} \left(\Im(\tilde{\eta}_\ell) + i\zeta \Im(\tilde{f}_\ell) \right) v_\ell \right).$$

Next, we use (8.16). By noticing that $\Re(\tilde{\eta}_\ell) = \Re(\tilde{\eta}_{n-\ell})$ and $\Im(\tilde{\eta}_\ell) = -\Im(\tilde{\eta}_{n-\ell})$ (and similarly for the \tilde{f} s), we are eventually left with

$$\int_{\mathbb{R}^n} w_n^{\beta, \zeta}(\underline{x}|\underline{\eta}) \prod_{k=1}^n dx_k = \prod_{\ell=0}^{n-1} \frac{\exp\left(\frac{\beta}{4n} \left(\frac{n}{\beta} \lambda_\ell^{(n)} + \alpha_d\right)^{-1} \left(2|\tilde{\eta}_\ell|^2 - 2\zeta^2|\tilde{f}_\ell|^2 + i4\zeta \left(\Re(\tilde{\eta}_\ell)\Re(\tilde{f}_\ell) + \Im(\tilde{\eta}_\ell)\Im(\tilde{f}_\ell)\right)\right)\right)}{\sqrt{\frac{\beta^2}{n^2} \left(\frac{n}{\beta} \lambda_\ell^{(n)} + \alpha_d\right)}}.$$

In view of (8.37), (8.8) follows from (8.36) together with the identity

$$Z_j^\beta(\underline{\eta}) = \frac{\exp\left(\frac{1}{2} \sum_{\ell \in \mathbb{Z}} ((2\pi\beta^{-1}\ell)^2 + \alpha_d)^{-1} |\hat{\eta}_\ell|^2\right)}{2 \sinh\left(\frac{1}{2}\beta\sqrt{\alpha_d}\right)},$$

which is derived from Lemma 8.11 along with (8.20). \square

References

- AHK [1] Alberverio S., Høegh-Krohn R., *Homogeneous random fields and statistical mechanics*, J. Funct. Anal. **19**, 242–272 (1975)
- AKRT1 [2] Alberverio S., Kondratiev Yu.G., Röckner M., Tsikalenko T.V., *Uniqueness of Gibbs states for quantum lattice systems with non-local interaction*, Proba. Theory. Relat. Fields **108**, 193–218 (1997)
- AKR [3] Alberverio S., Kondratiev Yu.A., Rebenko A.L., *Peierls argument and long-range order behavior of quantum lattice systems with unbounded spins*, . Stat. Phys **92**, 1137–1152 (1998)
- AKRT2 [4] Alberverio S., Kondratiev Yu.G., Röckner M., Tsikalenko T.V., *Dobrushin’s uniqueness for quantum lattice systems with nonlocal interaction*, Commun. Math. Phys. **189**(2), 621–630 (1997)
- Alber1 [5] Alberverio S., Kondratiev Y., Kozitzky Y., Röckner M., *Euclidean Gibbs states of quantum lattice systems*, Rev. Math. Phys. **14** 1335 (2002)
- Alber2 [6] Alberverio S., Kondratiev Y., Pasurek T., Röckner M., *Euclidean Gibbs measures on loop lattices: existence and a priori estimates*, The Annals of Probability **32**(1A), 153–190 (2004)
- BK [7] Bellissard J., Høegh-Krohn R., *Compactness and the maximal Gibbs state for random Gibbs fields on a lattice*, Commun. Math. Phys. **84**, 297–327 (1982)
- Billi [8] Billingsley P., *Convergence of Probability Measures*, Second Edition, John Wiley & Sons, Inc., New York, 1999
- Bo1 [9] Bolley F., *Separability and completeness for the Wasserstein distance*, Séminaire de Probabilités XLI, Lectures Notes in Mathematics, Springer, 371–377 (2008)
- Bou [10] Bourbaki N., *Intégration Chapitre IX*, Éléments de mathématique, Hermann, Paris, 1969
- BR2 [11] Bratelli O., Robinson D.W., *Operator Algebras and Quantum Statistical Mechanics 2*, Second Edition, Springer-Verlag Berlin Heidelberg, 1996
- COPP [12] Cassandro M., Olivieri E., Pellegrinotti A., Presutti E., *Existence and uniqueness of DLR measures for unbounded spin systems*, Z. Wahrsch. verw. Gebiete **41**, 313–334 (1978)
- D0 [13] Dobrushin R.L., *Existence of a phase transition in two- and three-dimensional Ising models*, Teor. Veroyatn. Primen. **10**, 209–230 (1965) (In Russian)
- D1 [14] Dobrushin R.L., *The description of a random field by means of conditional probabilities and conditions of its regularity*, Teor. Veroyatnost. i Primenen. **13**(2), 201–229 (1968) (In Russian); Theory Probab. Appl. **13**(2), 197–224 (1968)
- D2 [15] Dobrushin R.L., *Gibbsian random fields for lattice systems with pairwise interaction*, Funkts. Anal. i Prilozhen. **2**(4), 31–43 (1968) (In Russian); Funct. Anal. Appl. **2**(4), 292–301 (1968)

- [D3] [16] Dobrushin R.L., *The problem of uniqueness of a Gibbsian random field and the problem of phase transition*, Funkts. Anal. i Prilozhen. **2**(4), 44–57 (1968) (In Russian); Funct. Anal. Appl. **2**(4), 302–312 (1968)
- [D4] [17] Dobrushin R.L., *Gibbsian random fields. The general case*, Funkts. Anal. i Prilozhen. **3**(1), 27–35 (1969) (In Russian); Funct. Anal. Appl. **3**(1), 22–28 (1969)
- [D5] [18] Dobrushin R.L., *Prescribing a system of random variables by conditional distributions*, Teor. Veroyatnost. i Primenen. **15**(3) 469–497 (1970) (In Russian); Theory Probab. Appl. **15**(3), 458–486 (1970)
- [D6] [19] Dobrushin R.L., Perchersky E.A., *A criterion of the uniqueness of Gibbs fields in the non-compact case*, In Probability Theory and Mathematical Statistics (Tbilisi, 1982), Lecture Notes in Math. 1021, Springer-Verlag, Berlin, 97–110 (1983)
- [DS] [20] Dobrushin R.L., Shlosman S.B., *Constructive criterion for the uniqueness of Gibbs field*, Statistical Physics and Dynamical Systems: Rigorous Results, Progress in Physics Vol.10, pp. 347–370, Birkhäuser, Boston-Basel-Stuttgart, 1985
- [Ed] [21] Edwards D.A., *On the Kantorovich-Rubinstein theorem*, Exposit. Mathematicae **29**, 387–398 (2011)
- [FV] [22] Friedli S., Velenik Y., *Statistical mechanics of lattice systems: A concrete mathematical introduction*, Cambridge University Press, Cambridge, 2017
- [Ge] [23] Georgii H.O., *Gibbs Measures and Phase Transitions*, de Gruyter Studies in Mathematics 9, Berlin, New York, 1988
- [Gro] [24] Gross L., *Decay of correlations in classical lattice models at high temperature*, Comm. Math. Phys. **68**(1), 9–27 (1979)
- [GRS] [25] Guerra F., Rosen L., Simon B., *The $P(\varphi)_2$ quantum field theory as classical statistical mechanics*, Ann. Math. **101**, 111–189 (1975)
- [HK] [26] Høegh-Krohn R., *Relativistic quantum statistical mechanics in two-dimensional space-time*, Comm. Math. Phys. **38**, 195–224 (1974)
- [Hug] [27] Hugenholtz N.M., *C^* -Algebras and Statistical Mechanics*, Proc. Symp. Pure Math., Vol. **38** Part 2, Ed. R. V. Kadison, (Providence, RI: American Mathematical Society), pp. 407–465, 1982
- [Isr] [28] Israel R.B., *Convexity in the theory of lattice gases*, Princeton Series in Physics, Princeton University Press, Princeton, New Jersey, 1979
- [KL] [29] Klein P., *Dobrushin’s uniqueness theorem and the decay of correlations in continuous statistical mechanics*, Comm. Math. Phys. **86**(2), 227–246 (1982)
- [KP] [30] Kozitsky Y., Pasarek T., *Euclidean Gibbs measures of quantum anharmonic oscillators*, J. Stat. Phys. **127**, 985–1047 (2007); *Addendum and Corrigendum*, J. Stat. Phys. **132**, 755–757 (2008)
- [Ku] [31] Künsch H., *Decay of correlations under Dobrushin’s uniqueness condition and its applications*, Comm. Math. Phys. **84**, 207–222 (1982)
- [LR] [32] Lanford O.E., Ruelle D., *Observables at infinity and states with short-range correlations in statistical mechanics*, Comm. Math. Phys. **13**, 194–215 (1969)
- [LP] [33] Lebowitz J., Presutti E., *Statistical mechanics of systems of unbounded spins*, Comm. Math. Phys. **50**, 195–218 (1976)
- [M1] [34] Minlos R.A., *Limiting Gibbs’ distribution*, Funkts. Anal. i Prilozhen. **1**(2), 60–73 (1967) (In Russian); Funct. Anal. Appl. **1**(2), 140–150 (1967)
- [M2] [35] Minlos R. A., *Regularity of the Gibbs limit distribution*, Funkts. Anal. i Prilozhen. **1**(3), 40–53 (1967) (In Russian); Funct. Anal. Appl. **1**(3), 206–217 (1967)
- [Mu] [36] Münster A., *Statistical thermodynamics*, Vol. 1 (Springer/Academic Press), 1969
- [Pa2] [37] Park Y.M., *The cluster expansion of classical and quantum lattice systems*, J. Stat. Phys. **27**, 553–576 (1982)
- [Pa] [38] Park Y.M., *Quantum statistical mechanics of unbounded continuous spin systems*, J. Korean Math. Soc. **22**, 43–74 (1985)
- [PaYo1] [39] Park Y.M., Yoo H.J., *A characterization of Gibbs states of lattice boson systems*, J. Stat. Phys. **75**, 215–239 (1994)
- [PaYo2] [40] Park Y.M., Yoo H.J., *Uniqueness and clustering properties of Gibbs states for classical and quantum unbounded spin systems*, J. Stat. Phys. **80**, 223–271 (1995)

- Ru2 [41] Ruelle D., *Statistical Mechanics: Rigorous Results*, W.A. Benjamin, New York, Amsterdam, 1969
- Ru0 [42] Ruelle D., *Superstable interactions in classical statistical mechanics*, Comm. Math. Phys. **18**(2), 127–159 (1970)
- Ru1 [43] Ruelle D., *Probability estimates for continuous spin systems*, Comm. Math. Phys. **50**, 189–194 (1976)
- Ru3 [44] Ruelle D., *Thermodynamic formalism: The mathematical structures of equilibrium statistical mechanics*, Second Edition, Cambridge University Press, Cambridge, 2004
- Sim0 [45] Simon B., *Functional integration and quantum Physics*, Academic Press, New York London, 1979
- Sim1 [46] Simon B., *The $P(\varphi)_2$ Euclidean field theory*, Princeton University Press, Princeton, 1974
- Sim2 [47] Simon B., *The statistical mechanics of lattice gases, Volume I*, Princeton University Press, Princeton, New Jersey, 1993
- Si [48] Sinai Ya.G., *Theory of phase transitions: Rigorous results*, Pergamon Press, First Edition, 1982
- Wass [49] Wasserstein L.N., *Markov processes on a countable product space describing large systems of automata*, Problemy Peredachi Informatsii, **5**(3), 64–73 (1969) (In Russian)
- Wig [50] Wightman A.S., *Convexity and the notion of equilibrium state in thermodynamics and statistical mechanics*, Introduction to Israel (1979)