Finite-Time Current Probabilities in the Asymmetric Exclusion Process on a Ring

T. C. Dorlas
Dublin Institute for Advanced Studies
School of Theoretical Physics
10 Burlington Road, Dublin 4, Ireland.

V. B. Priezzhev
Joint Institute for Nuclear Research
Laboratory of Theoretical Physics
Dubna, Russian Federation

February 13, 2007

Abstract

We calculate the time-dependent probability distribution of current through a selected bond in the totally asymmetric exclusion process with periodic boundary conditions. We derive a general formula for the probability that the integrated current exceeds a given value N at the moment of time t. The formula is written in a form of a contour integral of a determinant of a Toeplitz matrix. Transforming the determinant expression, we obtain a generalization of the known formula derived by Johansson for the infinite one-dimensional lattice. To check the general formula, we consider the specific case corresponding to the probability of a minimal non-zero current. For this case we get an explicit analytical expression and analyze its asymptotics.

1 Introduction

The study of space-time correlations in stochastic models of interacting particles is a central subject of the non-equilibrium statistical mechanics [1]. Among a variety of correlations functions, the current characteristics are the most natural and important ones for physical applications. During the past decade, there has been considerable progress in the study of current fluctuations in the totally asymmetric exclusion process (TASEP) which is a paradigm for non-equilibrium many-particle systems [2, 3, 4, 5].

Two main quantities are used for the description of current, depending on the geometry of system. For the ring geometry and the fully asymmetric process, an adequate quantity is the total distance Y_t covered by all of the particles between time 0 and t [6, 7, 8]. For the infinite chain, the time-integrated current can be measured by the number of particles Q_t which have crossed a particular bond up to time t [9]. For the finite chain which is in contact at its ends with two reservoirs, Q_t is the number of particles which have moved from the left reservoir into the system during time t [10].

Most of the known results obtained so far concern the limiting case of large time when the generating functions $\langle e^{\alpha Y_t}\rangle$ and $\langle e^{\alpha Q_t}\rangle$ increase exponentially with t,

$$\langle e^{\alpha Y_t}\rangle \sim e^{\lambda(\alpha)t}$$

and

$$\langle e^{\alpha Q_t}\rangle \sim e^{\mu(\alpha)t}$$

where $\lambda(\alpha)$ and $\mu(\alpha)$ are the largest eigenvalues of the properly defined generator matrices [6].

At the same time, much less is known about the finite-time behavior of Y_t and Q_t. The first exact result for the probability $P(x_1,...,x_P;t|a_1,...,a_P;0)$ of finding P particles on lattice sites $x_1,...,x_P$ at time t given that they were on sites $x_1^0,...,x_P^0$ at time 0, has been obtained in [11] (see also [12]) for the TASEP on the infinite chain. Based on this result, it became possible to find the probability distribution of the current $Q_t(x)$, i.e. the number of particles that have crossed the lattice bond $(x-1,x)$ up to time t for a specific boundary condition of the half filled infinite chain, when the sites from $-\infty$ to 0 are occupied and the right half is empty at $t=0$ [14].

The knowledge of $P(x_1,...,x_P;t|a_1,...,a_P;0)$ enables calculation of many other current properties for arbitrary time intervals. However, the infinite geometry is not sufficient for complete description of the relaxation phenomena because, in the case of an infinite lattice and a finite number of particles, the stationary state corresponds to zero density, so that the particles are non-interacting.

The probability $P(x_1,...,x_P;t|a_1,...,a_P;0)$ for the TASEP with P particles on a ring has been derived in [15]. This opens the prospect for studies of finite-time current probabilities during the whole process of relaxation from an initial configuration to a non-trivial steady state.

At the same time, much less is known about the finite-time behavior of Y_t and Q_t. The first exact result for the probability $P(x_1, ..., x_P; t|a_1, ..., a_P; 0)$ of finding P particles on lattice sites $x_1, ..., x_P$ at time t given that they were on sites $x_1^0, ..., x_P^0$ at time 0, has been obtained in [11] (see also [12]) for the TASEP on the infinite chain. Based on this result, it became possible to find the probability distribution of the current $Q_t(x)$, i.e. the number of particles that have crossed the lattice bond $(x-1, x)$ up to time t for a specific boundary condition of the half filled infinite chain, when the sites from $-\infty$ to 0 are occupied and the right half is empty at $t = 0$ [13], [14]. A remarkable formula for this case had been obtained earlier by Johansson [16] using a representation in terms of the symmetric group.

The knowledge of $P(x_1, ..., x_P; t|a_1, ..., a_P; 0)$ enables calculation of many other current properties for arbitrary time intervals. However, the infinite geometry is not sufficient for complete description of the relaxation phenomena because, in the case of an infinite lattice and a finite number of particles, the stationary state corresponds to zero density, so that the particles are non-interacting.

The probability $P(x_1, ..., x_P; t|a_1, ..., a_P; 0)$ for the TASEP with P particles on a ring has been derived in [15]. This opens the prospect for studies of finite-time current probabilities during the whole process of relaxation from an initial configuration to a non-trivial steady state.

In this paper, we consider the current $Q_t(0)$ on the ring of L sites which is the number of particles that have crossed the bond $(L-1, 0)$ up to time t. Our goal will be to compute the probability $Prob[Q_t(0) > N]$ that at least $N+1$ particles have crossed the bond $(L-1, 0)$ up to time t. In Section 2, we obtain a general expression for this probability assuming arbitrary initial positions of P particles on the ring. This result still contains a contour integral of a determinant of $P \times P$ matrix. In Section 3 we consider the particular initial conditions $a_1 = 0, a_2 = 1, \ldots, a_P = P-1$ and evaluate the determinant expression to get $Prob[Q_t(0) > N]$ in a form which is close to Johansson's formula for the infinite lattice [16]. In Section 4 we consider the simplest case $N = 0$ corresponding to the minimal current probability among all initial conditions. We derive an explicit analytical expression for $Prob[Q_t(0) > 0]$ and compare it with the result obtained by straightforward probabilistic calculations. Section 5 contains an analysis of the asymptotic behaviour of the resulting expression.

2 Current probabilities

Let C be a configuration of P particles on a ring of L sites, where the positions of particles are $0 \leq x_1 < x_2 < ... < x_P < L$. The TASEP is defined by the master equation for the probability $P_t(C)$ of finding the system in configuration C at time t,

$$\partial_t P_t(C) = \sum_{\{C'\}} [M_0(C,C') + M_1(C,C')]P_t(C'), \tag{2.1}$$

with the initial condition that the system is in configuration C_0 at time t. Here $M_1(C,C')$ is the probability of going from configuration C' to C during a time interval dt, and $M_0(C,C')$ is a diagonal matrix with diagonal elements

$$M_0(C,C) = -\sum_{\{C' \neq C\}} M_1(C',C). \tag{2.2}$$

The matrix elements of $M_1(C,C')$ obey the exclusion rule that, during dt, each particle jumps with probability dt to its right provided that the target site is empty. Given the initial positions of particles $0 \leq a_1 < a_2 < ... < a_P < L$ at the moment $t = 0$, $P_t(C)$ is the conditional probability $P(x_1, ..., x_P; t|a_1, ..., a_P; 0)$ of finding P particles on the sites $0 \leq x_1 < ... < x_P < L$ at time t.

The solution of (2.1) is [15]:

$$P_t(C) = \sum_{n_1=-\infty}^{\infty} ... \sum_{n_P=-\infty}^{\infty} (-1)^{(P-1)\sum_{i=1}^{P} n_i} \det \mathbf{M}. \tag{2.3}$$

Elements of the $P \times P$ matrix $\mathbf{M}$ are

$$M_{ij} = F_{s_{ij}}(a_i, x_j + n_i L|t), \tag{2.4}$$

where

$$s_{ij} = Pn_i - \sum_{k=1}^{P} n_k + j - i, \tag{2.5}$$

and $F_m(a, x|t)$ are functions introduced by Schütz [11]:

$$F_m(a, x|t) = \sum_{k=0}^{\infty} \begin{pmatrix} k+m-1 \\ m-1 \end{pmatrix} F_0(a-k, x|t), \tag{2.6}$$

if integer $m > 0$, and

$$F_m(a, x|t) = \sum_{k=0}^{-m} (-1)^k \begin{pmatrix} -m \\ k \end{pmatrix} F_0(a-k, x|t), \tag{2.7}$$

if integer $m < 0$. For $m = 0$ and $x \geq a$,

$$F_0(a, x|t) = \frac{e^{-t}t^K}{K!}, \tag{2.8}$$

where $K = x - a$. For $m = 0$ and $x < a$

$$F_0(a, x|t) = 0. \tag{2.9}$$

The derivation of (2.3) in ref.[15] contains, as an intermediate step, the evaluation of probabilities $\psi_n(C; t|C_0; 0)$ to reach configuration C from C_0 for time t after making n visits of the origin $0 \equiv L$ of the ring by particles in turn, starting with the last. Thus, the probability $P_t(C)$ is the sum

$$P_t(C) = \sum_{n=0}^{\infty} \psi_n(C; t|C_0; 0) = \sum_{n=0}^{\infty} \sum_{\{n_i\}_n} (-1)^{(P-1)n} \det \mathbf{M}. \tag{2.10}$$

where summation over $n_i, i = 1, 2, \ldots, P$ is restricted by the condition $n_1 + n_2 + \cdots + n_P = n$.

To find $Prob[Q_t(0) > N]$, we have to take the sum over all final configurations C which can be reached from C_0 after at least $N + 1$ visits of the origin,

$$Prob[Q_t(0) > N] = \sum_{n=N+1}^{\infty} \sum_{C} \psi_n(C; t|C_0; 0) = \tag{2.11}$$

$$\sum_{n=N+1}^{\infty} \sum_{0 \leq x_1 < x_2 < \cdots < x_P < L} \sum_{\{n_i\}_n} (-1)^{(P-1)n} \det \mathbf{M}, \tag{2.12}$$

or, in the explicit form,

$$Prob[Q_t(0) > N] = \sum_{n=N+1}^{\infty} \sum_{x_P = P-1}^{L-1} \sum_{x_{P-1} = P-2}^{x_P - 1} \cdots \sum_{x_2 = 1}^{x_3 - 1} \sum_{x_1 = 0}^{x_2 - 1} \sum_{\{n_i\}_n} (-1)^{(P-1)n} \times$$

$$\begin{vmatrix} F_{s_{11}}(a_1, x_1 + n_1 L) & F_{s_{12}}(a_1, x_2 + n_1 L) & \cdots & F_{s_{1P}}(a_1, x_P + n_1 L) \\ F_{s_{21}}(a_2, x_1 + n_2 L) & F_{s_{22}}(a_2, x_2 + n_2 L) & \cdots & F_{s_{2P}}(a_2, x_P + n_2 L) \\ \vdots & \vdots & & \vdots \\ F_{s_{P1}}(a_P, x_1 + n_P L) & F_{s_{P2}}(a_P, x_2 + n_P L) & \cdots & F_{s_{PP}}(a_P, x_P + n_P L) \end{vmatrix} \tag{2.13}$$

To evaluate these sums we proceed as in [17]. Using the identity

$$\sum_{x=x_1}^{x_2} F_s(a, x) = F_{s+1}(a, x_1) - F_{s+1}(a, x_2 + 1), \tag{2.14}$$

the first column of the determinant becomes, after summation over x_1,

$$\begin{gathered}
F_{s_{11}+1}(a_1, n_1 L) - F_{s_{11}+1}(a_1, x_2 + n_1 L) \\
F_{s_{21}+1}(a_2, n_2 L) - F_{s_{21}+1}(a_2, x_2 + n_2 L) \\
\vdots \\
F_{s_{P1}+1}(a_P, n_P L) - F_{s_{P1}+1}(a_P, x_2 + n_P L).
\end{gathered}$$

It follows from (2.5) that

$$s_{i1} + 1 = s_{i2} \tag{2.15}$$

for all $i = 1, 2, ..., P$, and we can reduce the first column by adding the second to it. Continuing this process up to the sum over x_P, we get the first $P-1$ columns in the form

$$\begin{array}{c}
F_{s_{1k}+1}(a_1, k-1+n_1 L) \\
F_{s_{2k}+1}(a_2, k-1+n_2 L) \\
\vdots \\
F_{s_{Pk}+1}(a_P, k-1+n_P L)
\end{array} \tag{2.16}$$

for $k = 1, \dots, P-1$, and only the last column remains nonreduced,

$$\begin{gathered}
F_{s_{1P}+1}(a_1, P-1+n_1 L) - F_{s_{1P}+1}(a_1, L+n_1 L) \\
F_{s_{2P}+1}(a_2, P-1+n_2 L) - F_{s_{2P}+1}(a_2, L+n_2 L) \\
\vdots \\
F_{s_{PP}+1}(a_P, P-1+n_P L) - F_{s_{PP}+1}(a_P, L+n_P L).
\end{gathered} \tag{2.17}$$

Thus, the resulting determinant splits into two determinants D_1 and D_2 corresponding to two summands in the last column (2.17). The first determinant D_1 has a convenient form

$$\left| \begin{array}{cccc}
F_{s_{11}+1}(a_1, n_1 L) & F_{s_{12}+1}(a_1, 1+n_1 L) & \cdots & F_{s_{1P}+1}(a_1, P-1+n_1 L) \\
F_{s_{21}+1}(a_2, n_2 L) & F_{s_{22}+1}(a_2, 1+n_2 L) & \cdots & F_{s_{2P}+1}(a_2, P-1+n_2 L) \\
\vdots & \vdots & & \vdots \\
F_{s_{P1}+1}(a_P, n_P L) & F_{s_{P2}+1}(a_P, 1+n_P L) & \cdots & F_{s_{PP}+1}(a_P, P-1+n_P L)
\end{array} \right|. \tag{2.18}$$

Consider the determinant D_2. Using the property

$$F_m(a, x) = \sum_{k=0}^{\infty} F_{m-1}(a, x+k) \tag{2.19}$$

we can write the i-th element of the first column as

$$F_{s_{i1}+1}(a_i, n_i L) = F_{s_{i1}}(a_i, n_i L) + F_{s_{i1}+1}(a_i, 1 + n_i L) \tag{2.20}$$

for all $i = 1, 2, ..., P$. We now prove that the contribution from the first term of Eq.(2.20) into the sum

$$\sum_{n=N+1}^{\infty} \sum_{\{n_i\}_n} (-1)^{(P-1)n} \det D_2 \tag{2.21}$$

vanishes.

Expanding the determinant in (2.21), we select among terms containing the first summand in (2.20) those which contain the j-th element of the last column: $F_{s_{i1}}(a_i, n_i L) \times F_{s_{jP}+1}(a_j, L + n_j L)$. Consider the unique "mirror" terms which coincide with the selected terms except two factors, one from the j-th element of the first column and the second from the i-th element of the last column: $F_{s_{j1}}(a_j, n_j^{'} L) \times F_{s_{iP}+1}(a_i, L + n_i^{'} L)$, where $n_i^{'} = n_i - 1$ and $n_j^{'} = n_i + 1$. The indices $s_{jk} = s_{jk}(\mathbf{n})$ are functions of the vector $\mathbf{n} = (n_1, n_2, \ldots, n_P)$. We denote by $\mathbf{n}^{'}$ the vector obtained from $\mathbf{n}$ by replacement n_i and n_j by $n_i^{'}$ and $n_j^{'}$ Taking into account that

$$s_{j1}(\mathbf{n}^{'}) = Pn_j^{'} - \sum_{k=1}^{P} n_k^{'} + 1 - j = Pn_j - \sum_{k=1}^{P} n_k + P - j + 1 = s_{jP}(\mathbf{n}) + 1 \tag{2.22}$$

and

$$s_{iP}(\mathbf{n}^{'}) = Pn_i^{'} - \sum_{k=1}^{P} n_k^{'} + P - i = Pn_i - \sum_{k=1}^{P} n_k - i = s_{i1}(\mathbf{n}) - 1, \tag{2.23}$$

we see that the two selected terms are equal and enter into (2.21) with opposite signs because the sum $\sum n_k = \sum n_k^{'}$ and the sign of the permutation of indices 1 and P is always negative. Thus, the set of terms containing the first summand in (2.20) splits into two subsets cancelling one another.

As the contribution from the first term of (2.20) vanishes, we obtain instead of D_2 a determinant where the first two columns have the same arguments:

$$\begin{vmatrix} F_{s_{11}+1}(a_1, 1+n_1L) & F_{s_{12}+1}(a_1, 1+n_1L) & \cdots & F_{s_{1,P-1}+1}(a_1, P-2+n_1L) & F_{s_{1P}+1}(a_1, L+n_1L) \\ F_{s_{21}+1}(a_2, 1+n_2L) & F_{s_{22}+1}(a_2, 1+n_2L) & \cdots & F_{s_{2,P-1}+1}(a_2, P-2+n_2L) & F_{s_{2P}+1}(a_2, L+n_2L) \\ \vdots & \vdots & & \vdots & \vdots \\ F_{s_{P1}+1}(a_P, 1+n_PL) & F_{s_{P2}+1}(a_P, 1+n_PL) & \cdots & F_{s_{P,P-1}+1}(a_P, P-2+n_PL) & F_{s_{PP}+1}(a_P, L+n_PL) \end{vmatrix}. \tag{2.24}$$

Again, using (2.19)

$$F_{s_{i2}+1}(a_i, n_iL) = F_{s_{i2}}(a_i, n_iL) + F_{s_{i2}+1}(a_i, 1+n_iL) \tag{2.25}$$

for all $i = 1, 2, ..., P$, we obtain the sum of determinants

$$\begin{vmatrix} F_{s_{11}+1}(a_1, 1+n_1L) & F_{s_{12}}(a_1, 1+n_1L) & F_{s_{13}+1}(a_1, 2+n_1L) & \cdots \\ F_{s_{21}+1}(a_2, 1+n_2L) & F_{s_{22}}(a_2, 1+n_2L) & F_{s_{23}+1}(a_2, 2+n_2L) & \cdots \\ \vdots & \vdots & \vdots & \\ F_{s_{P1}+1}(a_P, 1+n_PL) & F_{s_{P2}}(a_P, 1+n_PL) & F_{s_{P3}+1}(a_P, 2+n_PL) & \cdots \end{vmatrix} \tag{2.26}$$

and

$$\begin{vmatrix} F_{s_{11}+1}(a_1, 1+n_1L) & F_{s_{12}+1}(a_1, 2+n_1L) & F_{s_{13}+1}(a_1, 2+n_1L) & \cdots \\ F_{s_{21}+1}(a_2, 1+n_2L) & F_{s_{22}+1}(a_2, 2+n_2L) & F_{s_{23}+1}(a_2, 2+n_2L) & \cdots \\ \vdots & \vdots & \vdots & \\ F_{s_{P1}+1}(a_P, 1+n_PL) & F_{s_{P2}+1}(a_P, 2+n_PL) & F_{s_{P3}+1}(a_P, 2+n_PL) & \cdots \end{vmatrix}. \tag{2.27}$$

Two columns in the first determinant coincide because $s_{i1} + 1 = s_{i2}$ for all $i = 1, 2, ..., P$ and D_2 gets reduced to the determinant (2.27) with equal arguments in the second and third columns. Continuing this procedure, we obtain finally

$$\begin{vmatrix} F_{s_{11}+1}(a_1, 1+n_1L) & \cdots & F_{s_{1(P-1)}+1}(a_1, P-1+n_1L) & F_{s_{1P}+1}(a_1, L+n_1L) \\ F_{s_{21}+1}(a_2, 1+n_2L) & \cdots & F_{s_{2(P-1)}+1}(a_2, P-1+n_2L) & F_{s_{2P}+1}(a_2, L+n_2L) \\ \vdots & & \vdots & \vdots \\ F_{s_{P1}+1}(a_P, 1+n_PL) & \cdots & F_{s_{P(P-1)}+1}(a_P, P-1+n_PL) & F_{s_{PP}+1}(a_P, L+n_PL) \end{vmatrix} \tag{2.28}$$

We expand the determinant (2.28) by the last column and consider the sum

$$\sum_{n=N+1}^{\infty} \sum_{\{n_i\}_n} (-1)^{(P-1)\sum n_k} \sum_{i=1}^{P} (-1)^{i+P} F_{s_{iP}+1}(a_i, (1+n_i)L) M_{iP}, \tag{2.29}$$

where M_{iP} is a minor of the matrix in Eq.(2.28). Given the i-th element of the sum Eq.(2.29), we introduce a vector $\mathbf{n}' = (n_1', n_2', \ldots, n_P')$ with $n_1' = n_1, \ldots, n_{i-1}' = n_{i-1}, n_i' = n_i + 1, n_{i+1}' = n_{i+1}, \ldots, n_P' = n_P$, so that

$$\sum_{i=1}^{P} n_i' = \sum_{i=1}^{P} n_i + 1. \tag{2.30}$$

We have

$$s_{iP}(\mathbf{n}) = Pn_i - \sum n_k + P - i = Pn_i' - \sum n_k' + 1 - i = s_{i1}(\mathbf{n}') \tag{2.31}$$

and

$$s_{jm}(\mathbf{n}) = Pn_j - \sum n_k + m - j = Pn_j' - \sum n_k' + 1 + m - j = s_{j(m+1)}(\mathbf{n}') \tag{2.32}$$

for $j \neq i$. Then the sum (2.29) becomes

$$\sum_{n=N+2}^{\infty} (-1)^{P-1} \sum_{i=1}^{P} \sum_{\{n_i'\}_n} (-1)^{(P-1)\sum n_k'} (-1)^{i+P} F_{s_{i1}+1}(a_i, (n_i')L) M_{iP} =$$

$$\sum_{n=N+2}^{\infty} (-1)^{(P-1)} \sum_{\{n_i'\}_n} (-1)^{(P-1)\sum n_k'} \times$$

$$\left| \begin{matrix} F_{s_{12}+1}(a_1, 1+n_1'L) & \cdots & F_{s_{1P}+1}(a_1, P-1+n_1'L) & F_{s_{11}+1}(a_1, n_1'L) \\ F_{s_{22}+1}(a_2, 1+n_2'L) & \cdots & F_{s_{2P}+1}(a_2, P-1+n_2'L) & F_{s_{21}+1}(a_2, n_2'L) \\ \vdots & & \vdots & \vdots \\ F_{s_{P2}+1}(a_P, 1+n_P'L) & \cdots & F_{s_{PP}+1}(a_P, P-1+n_P'L) & F_{s_{P1}+1}(a_P, n_P'L) \end{matrix} \right|. \tag{2.33}$$

Performing a cyclic permutation in Eq.(2.33), we see that the sum (2.33) is similar to the sum

$$\sum_{n=N+1}^{\infty} \sum_{\{n_i\}_n} (-1)^{(P-1)n} D_1, \tag{2.34}$$

where D_1 is given by Eq.(2.18). The only difference is in the ranges of summation over n. Remembering that D_1 and D_2 have opposite signs, we see that only terms obeying $\sum n_k = N+1$ remain and we obtain

$$Prob[Q_t(0) > N] = \sum_{n_1+\cdots+n_P=N+1} (-1)^{(P-1)(N+1)} \left| \begin{matrix} F_{s_{11}+1}(a_1, n_1L) & F_{s_{12}+1}(a_1, 1+n_1L) & \cdots & F_{s_{1P}+1}(a_1, P-1+n_1L) \\ F_{s_{21}+1}(a_2, n_2L) & F_{s_{22}+1}(a_2, 1+n_2L) & \cdots & F_{s_{2P}+1}(a_2, P-1+n_2L) \\ \vdots & \vdots & & \vdots \\ F_{s_{P1}+1}(a_P, n_PL) & F_{s_{P2}+1}(a_P, 1+n_PL) & \cdots & F_{s_{PP}+1}(a_P, P-1+n_PL) \end{matrix} \right|, \tag{2.35}$$

where $s_{ij} = Pn_i - N + j - i - 1$.

To proceed with (2.35), it is convenient to use the integral representation of functions $F_m(a,x)$ (see, e.g. [11]),

$$F_m(a,x) = \frac{1}{2\pi} \int_0^{2\pi} dq \frac{e^{iq(x-a)-\varepsilon_q t}}{(1-e^{iq})^m} \tag{2.36}$$

where $\varepsilon_q = 1 - e^{-iq}$ and the pole in the integrand is defined by $q \to q + i0$.

We introduce the generating functions

$$G_{ij}(z,t) = \sum_{n_i=-\infty}^{\infty} F_{s_{ij}+1}(a_i, j + n_iL - 1) z^{n_i} \tag{2.37}$$

and

$$g(z,q) = \sum_{n=-\infty}^{\infty} \frac{z^n e^{iqLn}}{(1-e^{iq})^{Pn}}. \tag{2.38}$$

Using (2.5),(2.36) and performing independent summations by $n_1, \dots, n_P$, we can write (2.37) and (2.35) as

$$G_{ij}(z,t) = \frac{1}{2\pi}\int_0^{2\pi} dq \frac{e^{iq(j-a_i-1)}e^{-(1-e^{-iq})t}}{(1-e^{iq})^{j-i-N}} g(z,q) \tag{2.39}$$

and

$$Prob[Q_t(0) > N] = \frac{1}{2\pi i}\oint \frac{dz}{z^{N+2}}(-1)^{(P-1)(N+1)} \det \mathbf{G} \tag{2.40}$$

where $\mathbf{G}$ is the matrix with elements (2.39).

For specific but commonly used initial conditions $a_1 = 0, a_2 = 1, \dots, a_P = P-1$, matrix $\mathbf{G}$ has the Toeplitz form $G_{ij} \equiv G(i-j)$. Using notations $\omega = e^{iq}$ and $\bar{\omega} = 1 - e^{iq}$, we obtain the elements of the Toeplitz matrix

$$G_{ij}(z,t) = \frac{1}{2\pi i}\oint \bar{\omega}^N g(z,\omega)\exp(\frac{\bar{\omega}}{\omega}t)(\frac{\bar{\omega}}{\omega})^{i-j}\frac{d\omega}{\omega} \tag{2.41}$$

where

$$g(z,\omega) = \sum_{n=-\infty}^{\infty}(\frac{z\omega^L}{\bar{\omega}^P})^n \tag{2.42}$$

and the integration contour is a small circle around 0.

The solution (2.40) with (2.41,2.42) has a fairly cumbersome form. To check it, we consider in Section 4 the simplest case $N = 0$ which can be computed independently by elementary probabilistic means. However, we first show that the solution can be brought into a form similar to that obtained by Johansson [16] (see also [14] and [13]) for the infinite lattice. In this way, we obtain a generalization of the known result of [16] to the case of finite periodic lattice which can be used for evaluation of finite-size effects.

3 Generalization of Johansson's formula

To obtain a generalization of the formula found by Johansson [16], we recall the particular case of the TASEP considered there. Consider P particles initially fixed at sites $a_1 = 0, a_2 = 1, ..., a_P = P-1$ of the infinite lattice. The problem is to find the probability $\mathbb{P}(M,P,t)$ that the particle initially at position $a_1 = 0$ has moved at least M steps in time t. Johansson's formula reads [16]:

$$\mathbb{P}(M,P,t) = \prod_{i=1}^{P}\frac{1}{i!(M-i)!}\int_{[0,t]^P} d^P\tau \prod_{i=1}^{P}\tau_i^{M-P}e^{-\tau_i}\prod_{1\le i<j\le P}(\tau_i-\tau_j)^2 \tag{3.1}$$

To get $\mathbb{P}(M,P,t)$ for the ring from (2.35), we change notations for the initial coordinates and put $a_1 = -\nu L, a_2 = -\nu L+1, ..., a_P = -\nu L + P - 1$, so that the minimal distance travelled by the first particle is $M = \nu L$ and the minimal number of particles crossing the bond $(L-1,0)$ is $N+1 = \nu P$. For the sake of simplicity, we take ν integer. For this choice of a_i, all n_i, $i = 1, ..., P$ are shifted by ν, $n_i \to n_i + \nu$ and formula (2.35) can be written as

$$\mathbb{P}(M,P,t) = \sum_{n_1+\cdots+n_P=0} (-1)^{(P-1)\nu P} \det |F_{s_{ij}+1}(a_i, n_i L + j - 1)| \tag{3.2}$$

with $s_{ij} = Pn_i + j - i$ or, equivalently,

$$\mathbb{P}(M,P,t) = \sum_{n_1+\cdots+n_P=0} (-1)^{(P-1)\nu P} \begin{vmatrix} F_{Pn_1+1}(n_1L+M) & F_{Pn_2}(n_2L+M-1) & \cdots & F_{Pn_P-P+2}(n_PL+M-P+1) \\ F_{Pn_1+2}(n_1L+M+1) & F_{Pn_2+1}(n_2L+M) & \cdots & F_{Pn_P-P+3}(n_PL+M-P+2) \\ \vdots & \vdots & & \vdots \\ F_{Pn_1+P}(n_1L+M+P-1) & F_{Pn_2+P-1}(n_2L+M+P-2) & \cdots & F_{Pn_P+1}(n_PL+M) \end{vmatrix}, \tag{3.3}$$

where we transposed the matrix and put $F_m(a,x) \equiv F_m(x-a)$.

Using (2.38) we introduce a new function

$$\tilde{F}_m(x) = \frac{1}{2\pi}\int_0^{2\pi} dq \frac{e^{iqx-\varepsilon_q t}}{(1-e^{iq})^m} g(z,q) \tag{3.4}$$

The function $\tilde{F}_m(x)$ obeys several useful relations similar to those for $F_m(x)$

$$\tilde{F}_m(x|t) = \int_0^t \tilde{F}_{m-1}(x-1|\tau)d\tau \tag{3.5}$$

and

$$\tilde{F}_m(x|t) = \tilde{F}_{m+1}(x|t) - \tilde{F}_{m+1}(x+1|t) \tag{3.6}$$

or, more generally,

$$\tilde{F}_m(x|t) = \int_0^t d\tau_1 \ldots \int_0^{\tau_{n-1}} d\tau_n \tilde{F}_{m-n}(x-n|\tau_n) \tag{3.7}$$

and

$$\tilde{F}_m(x|t) = \sum_{k=0}^{n} (-1)^k \left(\begin{array}{c} n \\ k \end{array} \right) \tilde{F}_{m+n}(x+k|t) \tag{3.8}$$

Performing summation over n_i in each column $i = 1, 2, \ldots, P$, we can continue (3.3) as

$$\mathbb{P}(M,P,t) = \frac{1}{2\pi i} \oint \frac{dz}{z} \times$$

$$\left| \begin{array}{llll} \tilde{F}_1(M|t) & \tilde{F}_0(M-1|t) & \cdots & \tilde{F}_{-P+2}(M-P+1|t) \\ \tilde{F}_2(M+1|t) & \tilde{F}_1(M|t) & \cdots & \tilde{F}_{-P+1}(M-P+2|t) \\ \vdots & \vdots & & \vdots \\ \tilde{F}_P(M+P-1|t) & \tilde{F}_{P-1}(M+P-2|t) & \cdots & \tilde{F}_1(M|t) \end{array} \right| \tag{3.9}$$

The resulting determinant expression coincides with that in [14] where it is written for functions $F_m(x|t)$. Similarity between properties (3.7), (3.8) of $\tilde{F}_m(x|t)$ and $F_m(x|t)$ means that $\mathbb{P}(M,P,t)$ can be represented in integral form [13],[14]:

$$\mathbb{P}(M,P,t) = \frac{1}{2\pi i} \oint \frac{dz}{z} (-1)^{[\frac{P}{2}]} \prod_{i=1}^{P-1} \frac{1}{i!} \int_{[0,t]^P} d^P\tau \tau_1^0 \tau_2^1 \ldots \tau_P^{P-1} \times$$

$$\left| \begin{array}{llll} \tilde{F}_0(M-1|\tau_1) & \tilde{F}_0(M-2|\tau_1) & \cdots & \tilde{F}_0(M-P|\tau_1) \\ \tilde{F}_0(M-1|\tau_2) & \tilde{F}_0(M-2|\tau_2) & \cdots & \tilde{F}_0(M-P|\tau_2) \\ \vdots & \vdots & & \vdots \\ \tilde{F}_0(M-1|\tau_P) & \tilde{F}_0(M-2|\tau_P) & \cdots & \tilde{F}_0(M-P|\tau_P) \end{array} \right| \tag{3.10}$$

or, after anti-symmetrization of the product $\tau_1^0 \tau_2^1 \ldots \tau_P^{P-1}$,

$$\mathbb{P}(M,P,t) = \frac{1}{2\pi i} \oint \frac{dz}{z} \prod_{i=1}^{P} \frac{1}{i!} \int_{[0,t]^P} d^P\tau \prod_{1 \le i,j \le P} (\tau_i - \tau_j) \times$$

$$\left| \begin{array}{llll} \tilde{F}_0(M-1|\tau_1) & \tilde{F}_0(M-2|\tau_1) & \cdots & \tilde{F}_0(M-P|\tau_1) \\ \tilde{F}_0(M-1|\tau_2) & \tilde{F}_0(M-2|\tau_2) & \cdots & \tilde{F}_0(M-P|\tau_2) \\ \vdots & \vdots & & \vdots \\ \tilde{F}_0(M-1|\tau_P) & \tilde{F}_0(M-2|\tau_P) & \cdots & \tilde{F}_0(M-P|\tau_P) \end{array} \right| \tag{3.11}$$

Returning back from functions $\tilde{F}_m(x|t)$ to $F_m(x|t)$ we finally get

$$\mathbb{P}(M,P,t) = \frac{1}{2\pi i}\oint \frac{dz}{z}\int_{[0,t]^P} d^P\tau \prod_{1\le i,j\le P}(\tau_i-\tau_j)\prod_{i=1}^{P}\frac{1}{i!}\times$$

$$\sum_{n_i=-\infty}^{\infty} z^{n_i}\sum_{k_i=0}^{\infty}\left(\begin{array}{c} k_i+Pn_i-1 \\ Pn_i-1 \end{array}\right)\frac{\tau_i^{M-P}e^{-\tau_i}}{(M+Ln_i+k_i-i)!}\times$$

$$\left|\begin{array}{cccc} \tau_1^{P-1+k_1+Ln_1} & \tau_1^{P-2+k_2+Ln_2} & \cdots & \tau_1^{k_P+Ln_P} \\ \tau_2^{P-1+k_1+Ln_1} & \tau_2^{P-2+k_2+Ln_2} & \cdots & \tau_2^{k_P+Ln_P} \\ \vdots & \vdots & & \vdots \\ \tau_P^{P-1+k_1+Ln_1} & \tau_P^{P-2+k_1+Ln_2} & \cdots & \tau_P^{k_P+Ln_P} \end{array}\right| \qquad (3.12)$$

where the binomial coefficient is defined by the Γ-function. For the infinite lattice where $n_i = 0, k_i = 0, i = 1, 2, \ldots, P$, we obtain Johansson's formula (3.1) as the determinant then has the Vandermonde form.

4 Minimal current probability

The probability of the non-zero current through bond $(L-1, 0)$ depends on the initial configuration of particles. This probability is minimal for the ordered initial conditions $a_1 = 0, a_2 = 1, \ldots, a_P = P-1$ because the particle at site 0 has a maximal obstacle to clear this site and the first particle which can cross the bond $(L-1, 0)$ has a maximal distance to the target site $0 \equiv L$.

In the following, let $\mathcal{E}_t^{(P)}$ denote the event that before time t at least one particle crosses the bond $(L-1, 0)$ given the initial conditions $a_1 = 0, a_2 = 1, \ldots, a_P = P-1$. In this section we obtain an explicit expression for $Prob[\mathcal{E}_t(P)] = Prob[Q_t(0) > 0]$. This quantity serves as a testing example for the general theory because it can be obtained by direct probabilistic calculations. Indeed, the whole process of the motion from the initial state to the first crossing of the bond $(L-1, 0)$ can be divided into three stages.

The first stage is the step of P-th particle from the site $P-1$ to the site P with the exponentially distributed time of rest. The second stage is the motion of P-th particle from the site P to the site $L-1$ and the independent motion of the hole from the site $P-1$ to the site 0. If the P-th particle reaches the site $L-1$ first, it waits for the arrival of the hole to the site 0 and, vice versa, if the hole reaches the target site 0 first, it waits for the arrival of the P-th particle. Therefore, the distribution of time of the second stage is

$$f(t) = g_{L-P-1}(t) \int_0^t g_{P-1}(\tau) d\tau + g_{P-1}(t) \int_0^t g_{L-P-1}(\tau) d\tau \tag{4.1}$$

where

$$g_n(t) = \frac{t^{n-1}}{(n-1)!} e^{-t} = F_0(n-1, t) \tag{4.2}$$

is the distribution of the sum of n independent exponentially distributed times of rest preceding n consecutive steps.

The last stage is simply the step of the P-th particle from the site $L-1$ to the empty site 0. The distribution function of the whole process is

$$Prob(\mathcal{E}_t^{(P)}) = \int_0^t dt_1 \int_0^{t-t_1} dt_2 \int_0^{t-t_1-t_2} dt_3 e^{-t_1-t_3} f(t_2). \tag{4.3}$$

To simplify notations, we use the fact that functions $F_m(a,x)$ depend only on the difference of their arguments and write $F_m(a,x) \equiv F_m(x-a)$.

Below, we obtain $Prob(\mathcal{E}_t^{(P)})$ from the general formula (2.35) to see how the exact P-particle dynamics produces the correct probabilistic distribution. However, first, let us express the integrals in (4.3) in terms of functions $F_0(x,t)$ and $F_1(x,t)$. Notice that, since $\int_0^t F_0(x-1,t_1)dt_1 = F_1(x,t)$ and $g_n(t) = F_0(n-1,t)$, we have

$$\begin{aligned} f(t) &= F_0(P-2,t)F_1(L-P-1,t) + F_1(P-1,t)F_0(L-P-2,t) \\ &= \frac{d}{dt}[F_1(P-1,t)F_1(L-P-1,t)]. \end{aligned} \tag{4.4}$$

Inserting into (4.3) we get

$$\begin{aligned} Prob[\mathcal{E}_t^{(P)}] &= \int_0^t dt_1 e^{-t_1} F_1(P-1,t-t_1)F_1(L-P-1,t-t_1) \\ &\quad -e^{-t}\int_0^t dt_1 \int_0^{t-t_1} dt_2 e^{t_2} \frac{d}{dt_2}[F_1(P-1,t_2)F_1(L-P-1,t_2)] \\ &= e^{-t}\int_0^t dt_1 \int_0^{t-t_1} dt_2 e^{t_2} F_1(P-1,t_2)F_1(L-P-1,t_2) \\ &= e^{-t}\int_0^t dt_1 \int_0^{t_1} dt_2 e^{t_2} F_1(P-1,t_2)F_1(L-P-1,t_2), \end{aligned} \tag{4.5}$$

where we used integration by parts. Next we use the formula

$$\begin{aligned}
\int e^t F_1(x-1,t)dt &= e^t F_1(x-1,t) - \int e^t F_0(x-2,t)dt \\
&= e^t F_1(x-1,t) - \int \frac{t^{x-2}}{(x-2)!}dt \\
&= e^t F_1(x-1,t) - \frac{t^{x-1}}{(x-1)!} = e^t F_1(x,t) \qquad (4.6)
\end{aligned}$$

to rewrite this as

$$\begin{aligned}
Prob[Q_t(0)>0] &= e^{-t}\int_0^t dt_1 e^{t_1} F_1(P,t_1)F_1(L-P-1,t_1) \\
&\quad -e^{-t}\int_0^t dt_1 \frac{t_1^{L-P-1}}{(L-P-1)!}F_1(P,t_1) \\
&\quad +e^{-t}\int_0^t dt_1 \int_0^{t_1} dt_2 \frac{t_2^{L-P-1}}{(L-P-1)!}F_0(P-1,t_2) \\
&= e^{-t}\int_0^t dt_1 e^{t_1} F_1(P,t_1)F_1(L-P-1,t_1) \\
&\quad -e^{-t}\frac{t^{L-P}}{(L-P)!}F_1(P,t) \\
&\quad +e^{-t}\int_0^t dt_1 \frac{t_1^{L-P}}{(L-P)!}F_0(P-1,t_1) \\
&\quad +e^{-t}\binom{L-2}{P-1}\int_0^t dt_1 F_1(L-1,t_1) \\
&= e^{-t}\int_0^t dt_1 e^{t_1} F_1(P,t_1)F_1(L-P-1,t_1) \\
&\quad -e^{-t}\frac{t^{L-P}}{(L-P)!}F_1(P,t) + \binom{L-1}{P-1}e^{-t}F_1(L,t) \\
&\quad +\binom{L-2}{P-1}e^{-t}F_2(L,t). \qquad (4.7)
\end{aligned}$$

Using (4.6) again, now with $x = L - P$, we have for the first term:

$$
\begin{aligned}
&e^{-t}\int_0^t dt_1 e^{-t_1} F_1(P,t_1)F_1(L-P-1,t_1) = \\
&= F_1(P,t)F_1(L-P,t) - e^{-t}\int_0^t dt_1 e^{t_1} F_0(P-1,t_1)F_1(L-P,t_1) \\
&= F_1(P,t)F_1(L-P,t) - e^{-t}\int_0^t dt_1 \frac{t_1^{P-1}}{(P-1)!} F_1(L-P,t_1) \\
&= F_1(P,t)F_1(L-P,t) - e^{-t}\frac{t^P}{P!} F_1(L-P,t) \\
&\quad + e^{-t}\int_0^t dt_1 \frac{t_1^P}{P!} F_0(L-P-1,t_1) \\
&= F_1(P,t)F_1(L-P,t) - F_0(P,t)F_1(L-P,t) + \binom{L-1}{P} e^{-t} F_1(L,t).
\end{aligned}
\tag{4.8}
$$

The final result is thus

$$
\begin{aligned}
Prob[\mathcal{E}_t^{(P)}] &= F_1(P,t)F_1(L-P,t) \\
&\quad - F_0(P,t)F_1(L-P,t) - F_1(P,t)F_0(L-P,t) \\
&\quad + \binom{L}{P} e^{-t} F_1(L,t) + \binom{L-2}{P-1} e^{-t} F_2(L,t).
\end{aligned}
\tag{4.9}
$$

Notice that this is manifestly invariant under particle-hole interchange.

To evaluate the same using the general formula (2.35), notice first of all that only the terms with $n_1 = n_2 = \cdots = n_{i-1} = n_{i+1} = \cdots = n_P = 0$, $n_i = 1$, $i = 1, \ldots, P$ do not vanish in (2.35). Indeed, assume that $n_i < 0$ for some i, $1 \le i \le P$. Then, the i-th row in (2.35)

$$
F_{s_{i1}+1}(a_i, n_i L), \ldots, F_{s_{iP}+1}(a_i, P-1+n_i L) \tag{4.10}
$$

vanishes owing to the condition $F_{-m}(a,x) = 0$ if $x - a < -m$, $m \ge 0$ and the inequalities $s_{ik} + 1 = Pn_i + k - i > n_i L + k - 1 - a_i$ and $a_i \ge i - 1$.

Inserting the initial conditions $a_1 = 0, a_2 = 1, \ldots, a_P = P-1$ and the possible values of $n_1, \ldots, n_P$ in (2.35) we obtain

$$Prob(\mathcal{E}_t^{(P)}) = \sum_{i=1}^{P} (-1)^{P-1} \times \begin{vmatrix} F_0(0) & F_1(1) & \cdots & F_{P-1}(P-1) \\ F_{-1}(-1) & F_0(0) & \cdots & F_{P-2}(P-2) \\ \vdots & \vdots & & \vdots \\ F_{P-i+1}(L-i+1) & F_{P-i+2}(L-i+2) & \cdots & F_{2P-i}(L+P-i) \\ \vdots & \vdots & & \vdots \\ F_{-P+1}(-P+1) & F_{-P+2}(-P+2) & \cdots & F_0(0) \end{vmatrix} \tag{4.11}$$

Using the fact that $F_{-p}(-p) = (-1)^p F_0(0)$ and performing simple column operations, we can write this as

$$Prob(\mathcal{E}_t^{(P)}) = \sum_{i=1}^{P} (-1)^{P-1} \Delta_P^{(i)}, \tag{4.12}$$

where

$$\Delta_P^{(i)} = \begin{vmatrix} F_1(0) & F_2(1) & \cdots & \cdots & F_{P-1}(P-2) & F_{P-1}(P-1) \\ 0 & F_1(0) & \cdots & \cdots & F_{P-2}(P-3) & F_{P-2}(P-2) \\ \vdots & \vdots & & & \vdots & \vdots \\ F_{P-i+2}(L-i+1) & F_{P-i+3}(L-i+2) & \cdots & \cdots & F_{2P-i}(L+P-1-i) & F_{2P-i}(L+P-i) \\ \vdots & \vdots & & & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & F_1(0) & F_1(1) \\ & & & \cdots & 0 & F_0(0) \end{vmatrix}. \tag{4.13}$$

This can be further simplified to

$$Prob(\mathcal{E}_t^{(P)}) = (-1)^{P-1} \left[e^{-t} \sum_{i=2}^{P-1} \Delta_i^* + \Delta_P^{(1)} + \Delta_P^{(P)} \right], \tag{4.14}$$

where

$$\Delta_P^{(1)} = e^{-t} F_{P+1}(L). \tag{4.15}$$

We now evaluate the determinant Δ_i^* using the fact that

$$F_{n+1}(n) = \frac{t^n}{n!}. \tag{4.16}$$

Writing $x_k = F_{P-i+k+1}(L-i+k)$ we have

$$\Delta_i^* = \begin{vmatrix} 1 & t & \cdots & \frac{1}{(i-2)!}t^{i-2} & \frac{1}{(i-1)!}t^{i-1} \\ 0 & 1 & \cdots & \frac{1}{(i-3)!}t^{i-3} & \frac{1}{(i-2)!}t^{i-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & t \\ x_1 & x_2 & \cdots & x_{i-1} & x_i \end{vmatrix}. \tag{4.17}$$

Applying row operations this can be reduced to

$$\Delta_i^* = \begin{vmatrix} 1 & 0 & \cdots & 0 & \frac{(-1)^i}{(i-1)!}t^{i-1} \\ 0 & 1 & \cdots & 0 & \frac{(-1)^{i-1}}{(i-2)!}t^{i-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & t \\ x_1 & x_2 & \cdots & x_{i-1} & x_i \end{vmatrix}. \tag{4.18}$$

Indeed, after the bottom rows from $i-k+1$ down to $i-1$ have been cleared, we subtract these rows $t^r/r!$ times from the $(i-k)$-th row $(r = 1, \ldots, k-1)$ to get

$$\frac{t^k}{k!} + \sum_{r=1}^{k-1} \frac{t^r}{r!} \frac{(-t)^{k-r}}{(k-r)!} = -\frac{(-t)^k}{k!} \tag{4.19}$$

in the last column. (The sum is the coefficient of t^k in the expansion of $e^t e^{-t}$ except for the terms $r = 0$ and $r = k$.) The determinant now easily evaluates to

$$\Delta_i^* = \sum_{k=0}^{i-1} \frac{(-t)^k}{k!} F_{P+1-k}(L-k). \tag{4.20}$$

This sums to

$$\sum_{i=2}^{P-1} \Delta_i^* = (P-2)F_{P+1}(L) + \sum_{k=1}^{P-2} \frac{(-t)^k}{k!}(P-k-1)F_{P-k+1}(L-k). \tag{4.21}$$

The determinant $\Delta_P^{(P)}$ can be treated similarly. It is given by

$$\Delta_P^{(P)} = \left| \begin{array}{ccccc} 1 & t & \cdots & \frac{t^{P-2}}{(P-2)!} & F_{P-1}(P-1) \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & t & F_2(2) \\ 0 & 0 & \cdots & 1 & F_1(1) \\ F_2(L-P+1) & \cdots & \cdots & F_P(L-1) & F_P(L) \end{array} \right| . \tag{4.22}$$

The entries in the last column are given by

$$F_{n+1}(n+1) = \sum_{k=0}^{n} (-1)^k \frac{t^{n-k}}{(n-k)!} + (-1)^{n+1} e^{-t}. \tag{4.23}$$

A similar row reduction as for Δ_i^* now yields

$$\Delta_P^{(P)} = \left| \begin{array}{ccccc} 1 & 0 & \cdots & 0 & F_1(P-1) \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -F_1(2) \\ 0 & 0 & \cdots & 1 & F_1(1) \\ F_2(L-P+1) & \cdots & \cdots & F_P(L-1) & F_P(L) \end{array} \right| . \tag{4.24}$$

Indeed, the reduction of the k-th row from the bottom leads to

$$F_k(k) + \sum_{r=1}^{k-1} (-1)^r \frac{t^{k-r}}{(k-r)!} F_1(r) = -(-1)^k F_1(k). \tag{4.25}$$

The result is

$$\Delta_P^{(P)} = F_P(L) + \sum_{k=1}^{P-1} (-1)^k F_1(k) F_{P-k+1}(L-k). \tag{4.26}$$

Using the relation

$$\sum_{k=r+1}^{P-1} (-1)^k F_{P-k+1}(L-k) = (-1)^{P-1} F_1(L-P+1) + (-1)^{r-1} F_{P-r}(L-r), \tag{4.27}$$

this can be written as

$$\begin{aligned} \Delta_P^{(P)} &= (-1)^{P-1} F_1(L-P+1) + \\ & \quad -e^{-t} \sum_{r=0}^{P-2} \frac{t^r}{r!} \left[(-1)^{P-1} F_1(L-P+1) + (-1)^{r-1} F_{P-r}(L-r) \right]. \end{aligned} \tag{4.28}$$

Inserting into (4.14) we obtain the following expression for the probability of $\mathcal{E}_t^{(P)}$:

$$Prob(\mathcal{E}_t^{(P)}) = F_1(P-1)F_1(L-P+1)$$
$$+(-1)^{P-1}e^{-t}\sum_{k=0}^{P-2}\frac{(-t)^k}{k!}\left[(P-k-1)F_{P-k+1}(L-k)+F_{P-k}(L-k)\right]. \tag{4.29}$$

Using the properties of functions $F_p(n)$ and several combinatoric identities (see Appendix), we obtain finally

$$\begin{aligned} Prob(\mathcal{E}_t^{(P)}) &= F_1(P-1)F_1(L-P+1) \\ &\quad -e^{-2t}\sum_{r=L}^{\infty}\frac{t^r}{r!}\left[(P-1)\binom{L-1}{P}-\binom{L-1}{P-1}\right. \\ &\quad\quad \left.-r\binom{L-2}{P-1}+\binom{r+1}{P}\right]. \end{aligned} \tag{4.30}$$

A equivalence of (4.30) and (4.9) is not entirely obvious. We elaborate on this in the following section.

5 Analysis of the minimal current probability

Figure 1 shows a plot of $Prob(\mathcal{E}_t^{(P)})$ for $P=2$ and a number of values of L. It is clear that the probability increases from 0 to 1 as t increases, as it should.

We can rewrite (4.30) in a more symmetric way as follows:

$$\begin{aligned} & F_1(P-1)F_1(L-P+1) = \qquad (5.1) \\ &= \left(F_1(P)+\frac{t^{P-1}}{(P-1)!}e^{-t}\right)\left(F_1(L-P)-\frac{t^{L-P}}{(L-P)!}e^{-t}\right) \\ &= F_1(P)F_1(L-P)+\frac{t^{P-1}}{(P-1)!}\sum_{k=L-P}^{\infty}\frac{t^k}{k!}e^{-2t} \\ &\quad -\frac{t^{L-P}}{(L-P)!}\sum_{k=P}^{\infty}\frac{t^k}{k!}e^{-2t} \\ &= F_1(P)F_1(L-P)+\sum_{r=L}^{\infty}\frac{t^r}{r!}\left[\binom{r}{P-1}-\binom{r}{L-P}\right]. \qquad (5.2) \end{aligned}$$

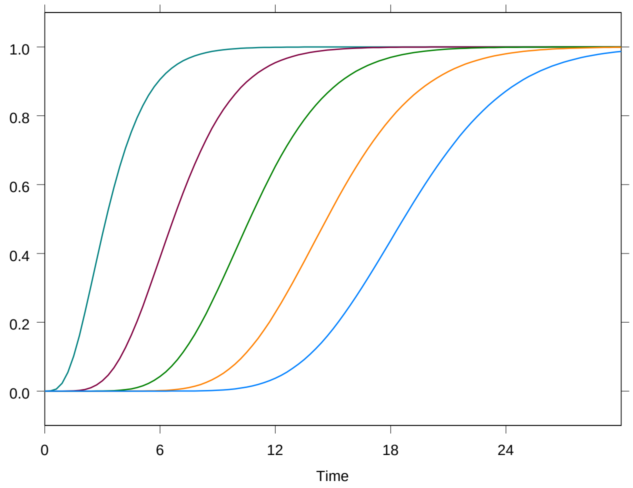

Figure 1: *The probability of at least one of two particles reaching the end of an interval of length* $L = 4, 6, 8, 10$ *and* 12*, as a function of time.*

Inserting this, we get

$$
\begin{aligned}
&Prob(\mathcal{E}_t^{(P)}) = &&(5.3)\\
&= \quad F_1(P)F_1(L-P) - e^{-2t}\sum_{r=L}^{\infty}\frac{t^r}{r!}\times \\
&\qquad \left[\left(\frac{P(L-P)}{L} - 1 - \frac{P(L-P)}{L(L-1)}r\right)\binom{L}{P} + \binom{r}{P} + \binom{r}{L-P}\right]. &&(5.4)
\end{aligned}
$$

This formula is manifestly symmetric under exchange of particles and holes, i.e. $P \leftrightarrow L-P$. As a particular case we have

$$Prob(\mathcal{E}_t^{(L-1)}) = Prob(\mathcal{E}_t^{(1)}) = F_1(L). \tag{5.5}$$

The formula (5.4) also reveals the equivalence with (4.9). Indeed,

$$e^{-t}\sum_{r=L}^{\infty}\frac{t^r}{r!}\binom{r}{P} = e^{-t}\frac{t^P}{P!}\sum_{k=L-P}^{\infty}\frac{t^k}{k!} = F_0(P,t)F_1(L-P,t), \tag{5.6}$$

and similarly for the last term in (5.4). The term

$$e^{-2t}\sum_{r=L}^{\infty}\frac{t^r}{r!}\binom{L}{P} = \binom{L}{P}e^{-t}F_1(L,t) \tag{5.7}$$

The remaining two terms can be written as

$$\binom{L-2}{P-1} e^{-2t} \sum_{r=L}^{\infty} (r-L+1)\frac{t^r}{r!} = \binom{L-2}{P-1} e^{-t} F_2(L,t). \tag{5.8}$$

It is clear from (4.30) that $Prob(\mathcal{E}_t^{(P)})$ is bounded by 1. In fact,

$$(P-1)\binom{L-1}{P} - \binom{L-1}{P-1} - r\binom{L-2}{P-1} + \binom{r}{P} + \binom{r}{L-P} > 0 \tag{5.9}$$

for $r \geq L$. This is easily seen by induction, as it is zero for $r = L-1$ and increases in r. The same relation is also useful to prove that $Prob(\mathcal{E}_t^{(P)})$ is increasing. Indeed, the derivative is given by

$$\begin{aligned}
& F_0(P-1)F_1(L-P) + F_1(P)F_0(L-P-1) \\
& + \sum_{r=L}^{\infty} e^{-2t}\left(2\frac{t^r}{r!} - \frac{t^{r-1}}{(r-1)!}\right)\left[(P-1)\binom{L-1}{P} - \binom{L-1}{P-1} - r\binom{L-2}{P-1}\right. \\
& \qquad\qquad \left. + \binom{r}{P} + \binom{r}{L-P}\right] \\
= \; & e^{-2t} \sum_{r=L-1}^{\infty} \frac{t^r}{r!}\left[\binom{r}{P-1} + \binom{r}{L-P-1}\right] \\
& + e^{-2t}\sum_{r=L}^{\infty}\frac{t^r}{r!}\left[(P-1)\binom{L-1}{P} - \binom{L-1}{P-1} - (r-1)\binom{L-2}{P-1}\right. \\
& \qquad\qquad \left. + \binom{r}{P} - \binom{r}{P-1} + \binom{r}{L-P} - \binom{r}{L-P-1}\right] \\
& - e^{-2t}\frac{t^{L-1}}{(L-1)!}\left[(P-1)\binom{L-1}{P} - \binom{L-1}{P-1}\right. \\
& \qquad\qquad \left. - L\binom{L-2}{P-1} + 2\binom{L}{P}\right] \\
= \; & e^{-2t}\sum_{r=L}^{\infty}\frac{t^r}{r!}\left[\binom{r}{P} + \binom{r}{L-P} - (r-L)\binom{L-2}{P-1}\right] \\
& + e^{-2t}\frac{t^{L-1}}{(L-1)!}\binom{L-2}{P-1}.
\end{aligned} \tag{5.10}$$

It is now clear that $Prob(\mathcal{E}_t^{(P)})$ must increase from 0 at $t = 0$ to 1 as $t \to \infty$.

It is natural to scale the time with L. It is not difficult to see that at constant P, $Prob(\mathcal{E}_{Lt}^{(P)})$ tends to a step function as $L \to \infty$. Indeed, the maximum term in

$$F_1(L-P+1, Lt) = \sum_{k=L-P+1}^{\infty} \frac{(Lt)^k}{k!} e^{-Lt} \tag{5.11}$$

is attained for $k = L-P+1$ if $t < 1$ and for $k \approx Lt$ for $t > 1$ so that

$$\lim_{L\to\infty} F_1(L-P+1, Lt) = \begin{cases} 0 & \text{if} \quad t < 1, \\ 1 & \text{if} \quad t > 1. \end{cases} \tag{5.12}$$

Moreover, $F_1(P-1, Lt) \to 1$ and the second term tends to zero.

A more interesting limit is the thermodynamic limit, where both t and P scale with L. This can be analysed roughly as follows. We write $t = L\tau$ and $P = \rho L$. Clearly, $F_1(P-1) \sim 1_{\{\tau>\rho\}}$ and $F_1(L-P+1) \sim 1_{\{\tau>1-\rho\}}$ so

$$F_1(P-1)F_1(L-P+1) \sim 1_{\{\tau>\rho\vee 1-\rho\}}. \tag{5.13}$$

In analysing the second term of (4.30), we may assume $L - P \geq P$. We have seen that the second term is positive and therefore bounded by

$$\begin{aligned} e^{-2t}\sum_{r=L}^{\infty}\frac{t^r}{r!}\binom{r+1}{P} &\sim e^{-2t}\left(\frac{t^P}{P!}+\frac{t^{P-1}}{(P-1)!}\right)\sum_{r=L-P}^{\infty}\frac{t^r}{r!} \\ &\sim e^{-t}\left(\frac{t^P}{P!}+\frac{t^{P-1}}{(P-1)!}\right) \to 0 \end{aligned} \tag{5.14}$$

if $\tau > 1-\rho$. Otherwise, the convergence is even faster.

The next interesting question is, what happens in the neighbourhood of $\tau = 1-\rho$ (assuming $\rho < \frac{1}{2}$). The correct scaling is then presumably with $\sqrt{L}$. The following figure shows graphs of $Prob(\mathcal{E}^{\rho L}_{(1-\rho)L+\sqrt{L}\tau})$ as a function of τ for $\rho = 1/3$ and a number of values of L.

It suggests that there exists a constant ξ (depending on ρ) such that

$$Prob(\mathcal{E}^{\rho L}_{(1-\rho)L+\sqrt{L}\tau}) \to \int_{-\infty}^{\tau} e^{-t^2/2\xi}\frac{dt}{\sqrt{2\pi\xi}}. \tag{5.15}$$

Assuming $\rho > \frac{1}{2}$, we insert $t = L\rho + \sqrt{L}\tau$ into $F_1(P-1)F_1(L-P+1)$. The second factor is very close to 1. The first factor can be approximated as

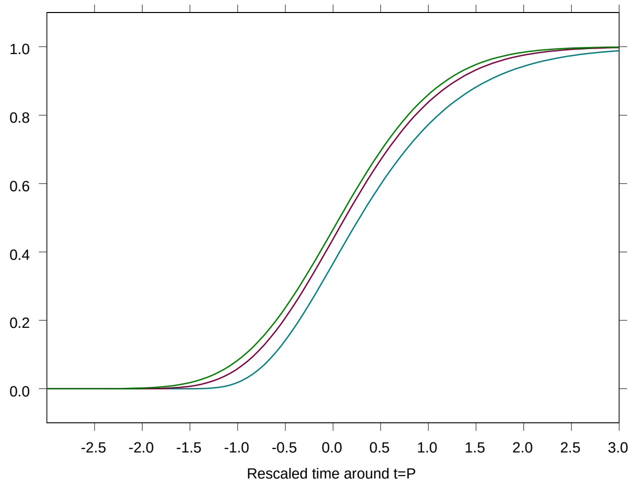

Figure 2: *The probability $Prob(\mathcal{E}_t)$ for $L = 6$ (right-most curve), 30 (middle curve) and 90 (left-most curve), as a function of τ where $t = L\rho + \sqrt{L}\tau$ and $P = L/3$.*

follows:

$$\begin{aligned}
e^{-t}\sum_{k=0}^{P}\frac{t^k}{k!} &\approx e^{-L\rho-\sqrt{L}\tau}\sum_{n=0}^{L\rho}\frac{(L\rho+\sqrt{L}\tau)^{L\rho-n}}{(L\rho-n)^{L\rho-n}e^{-L\rho+n}\sqrt{2\pi(L\rho-n)}} \\
&= \sum_{n=0}^{L\rho}\left(\frac{L\rho+\sqrt{L}\tau}{L\rho-n}\right)^{L\rho-n}\frac{e^{-n-\sqrt{L}\tau}}{\sqrt{2\pi L\rho}} \\
&\approx \sum_{n=0}^{L\rho}\exp\left[(L\rho-n)\left(\frac{\tau}{\rho\sqrt{L}}+\frac{n}{\rho L}-\frac{\tau^2}{2\rho^2 L}+\frac{n^2}{2\rho^2L^2}\right)\right]\frac{e^{-n-\sqrt{L}\tau}}{\sqrt{2\pi\rho L}} \\
&\approx \frac{1}{\sqrt{2\pi\rho L}}\sum_{n=0}^{\infty}\exp\left[-\frac{n\tau}{\sqrt{L}\rho}-\frac{n^2}{2L\rho}-\frac{\tau^2}{2\rho}\right] \\
&\approx \int_{\tau}^{\infty}e^{-x^2/2\rho}\frac{dx}{\sqrt{2\pi\rho}}. \qquad (5.16)
\end{aligned}$$

The second term in (4.30) still does not contribute in this limit, so (5.15) holds with $\xi = \rho$.

Notice that there is one exception: if $\rho = \frac{1}{2}$ the both factors behave like (5.16), so the result for $Prob(\mathcal{E}^{\rho L}_{(1-\rho)L+\sqrt{L}\tau})$ is the square of the error function.

6 Appendix

Using the general formula

$$
\begin{aligned}
F_p(n) &= \sum_{k=0}^{p-1}(-1)^{p-k+1}\frac{t^k}{k!}\binom{n-k-1}{p-k-1} \\
&\qquad +(-1)^p e^{-t}\sum_{k=0}^{n-p}\binom{n-k-1}{p-1}\frac{t^k}{k!},
\end{aligned}
\tag{6.1}
$$

valid for $n \geq p$, we can rewrite the second term in (4.29) in a more convenient form. We have

$$
\begin{aligned}
&(-1)^{P-1}\sum_{k=0}^{P-2}\frac{(-t)^k}{k!}(P-k-1)F_{P-k+1}(L-k) \\
&= -\sum_{k=0}^{P-2}\frac{t^k}{k!}(P-k-1)\sum_{l=0}^{P-k}\frac{(-t)^l}{l!}\binom{L-k-l-1}{P-k-l} \\
&\quad +e^{-t}\sum_{k=0}^{P-2}\frac{t^k}{k!}(P-k-1)\sum_{l=0}^{L-P-1}\frac{t^l}{l!}\binom{L-k-l-1}{P-k} \\
&= -\sum_{r=0}^{P}\frac{t^r}{r!}\binom{L-r-1}{P-r}\sum_{k=0}^{r\wedge(P-2)}(-1)^{r-k}(P-k-1)\binom{r}{k} \\
&\quad +e^{-t}\sum_{k=0}^{P}\frac{t^k}{k!}(P-k-1)\sum_{l=0}^{L-P-1}\frac{t^l}{l!}\binom{L-k-l-1}{P-k}+\frac{t^P}{P!}\sum_{l=0}^{L-P-1}\frac{t^l}{l!}.
\end{aligned}
\tag{6.2}
$$

In the first term we now use the simple identities

$$\sum_{k=0}^{r}(-1)^{r-k}\binom{r}{k}=0 \tag{6.3}$$

if $r > 0$, and

$$\sum_{k=0}^{r}(-1)^{r-k}k\binom{r}{k}=0 \tag{6.4}$$

if $r > 1$ to write (for $P \geq 2$)

$$\begin{aligned}
&(-1)^{P-1} \sum_{k=0}^{P-2} \frac{(-t)^k}{k!} (P-k-1) F_{P-k+1}(L-k) \\
&= -(P-1)\binom{L-1}{P} + \binom{L-2}{P-1} t - \frac{t^P}{P!} \\
&\quad + e^{-t}(P-1) \sum_{r=0}^{L-1} \frac{t^r}{r!} \sum_{k=0 \vee (P+r+1-L)}^{P \wedge r} \binom{r}{k} \binom{L-r-1}{P-k} \\
&\quad - e^{-t} \sum_{k=1}^{P} \frac{t^k}{(k-1)!} \sum_{l=0}^{L-P-1} \frac{t^l}{l!} \binom{L-k-l-1}{P-k} + \frac{t^P}{P!} \sum_{l=0}^{L-P-1} \frac{t^l}{l!}. \quad (6.5)
\end{aligned}$$

Rewriting the last but one term as

$$\sum_{r=0}^{L-2} \frac{t^{r+1}}{r!} \sum_{k'=0 \vee (P+r+1-L)}^{r \wedge (P-1)} \binom{r}{k'} \binom{L-r-2}{P-k'-1}, \quad (6.6)$$

and using the identity

$$\sum_{k=0 \vee (p+r-n)}^{r \wedge p} \binom{n-r}{p-k} \binom{r}{k} = \binom{n}{p} \quad (6.7)$$

we obtain

$$\begin{aligned}
&(-1)^{P-1} \sum_{k=0}^{P-2} \frac{(-t)^k}{k!} (P-k-1) F_{P-k+1}(L-k) \\
&= -(P-1)\binom{L-1}{P} + \binom{L-2}{P-1} t - \frac{t^P}{P!} \\
&\quad + e^{-t} \sum_{r=0}^{L-1} \frac{t^r}{r!} \left[(P-1)\binom{L-1}{P} - r\binom{L-2}{P-1} \right] + \frac{t^P}{P!} \sum_{l=0}^{L-P-1} \frac{t^l}{l!}.
\end{aligned} \quad (6.8)$$

A similar analysis yields

$$\begin{aligned}
(-1)^{P-1} \sum_{k=0}^{P-2} \frac{(-t)^k}{k!} F_{P-k}(L-k) &= \binom{L-1}{P-1} - \frac{t^{P-1}}{(P-1)!} \\
&\quad - e^{-t} \sum_{r=0}^{L-1} \frac{t^r}{r!} \binom{L-1}{P-1} + \frac{t^{P-1}}{(P-1)!} \sum_{l=0}^{L-P} \frac{t^l}{l!}.
\end{aligned} \quad (6.9)$$

The complete result for the second term of (4.29) is

$$
\begin{aligned}
&(-1)^{P-1}e^{-t}\sum_{k=0}^{P-2}\frac{(-t)^k}{k!}[(P-k-1)F_{P-k+1}(L-k)+F_{P-k}(L-k)] \\
&= -e^{-t}\left\{(P-1)\binom{L-1}{P}-\binom{L-1}{P-1}-\binom{L-2}{P-1}t+\frac{t^{P-1}}{(P-1)!}+\frac{t^P}{P!}\right\} \\
&\quad +e^{-2t}\sum_{r=0}^{L-1}\frac{t^r}{r!}\left[(P-1)\binom{L-1}{P}-r\binom{L-2}{P-1}-\binom{L-1}{P-1}\right] \\
&\quad +\frac{t^{P-1}}{(P-1)!}\sum_{l=0}^{L-P}\frac{t^l}{l!}+\frac{t^P}{P!}\sum_{l=0}^{L-P-1}\frac{t^l}{l!}.
\end{aligned} \tag{6.10}
$$

Next we expand the e^{-t} term:

$$
\begin{aligned}
&-e^{-t}\left\{(P-1)\binom{L-1}{P}-\binom{L-1}{P-1}-\binom{L-2}{P-1}t+\frac{t^{P-1}}{(P-1)!}+\frac{t^P}{P!}\right\} \\
&= -e^{-2t}\left(\sum_{r=0}^{L-1}\frac{t^r}{r!}+\sum_{r=L}^{\infty}\frac{t^r}{r!}\right)\left[(P-1)\binom{L-1}{P}-\binom{L-1}{P-1}\right] \\
&\quad +e^{-2t}\binom{L-2}{P-1}\left(\sum_{r=0}^{L-2}\frac{t^{r+1}}{r!}+\sum_{r=L}^{\infty}\frac{t^r}{(r-1)!}\right) \\
&\quad -e^{-2t}\frac{t^P}{P!}\left(\sum_{r=0}^{L-P-1}\frac{t^r}{r!}+\sum_{r=L-P}^{\infty}\frac{t^r}{r!}\right) \\
&\quad -e^{-2t}\frac{t^{P-1}}{(P-1)!}\left(\sum_{r=0}^{L-P}\frac{t^r}{r!}+\sum_{r=L-P+1}^{\infty}\frac{t^r}{r!}\right).
\end{aligned} \tag{6.11}
$$

It is clear that the terms up to order $L-1$ cancel the e^{-2t} contribution, and we find

$$
\begin{aligned}
Prob(\mathcal{E}_t^{(P)}) &= F_1(P-1)F_1(L-P+1) \\
&\quad -e^{-2t}\sum_{r=L}^{\infty}\frac{t^r}{r!}\left[(P-1)\binom{L-1}{P}-\binom{L-1}{P-1}\right. \\
&\quad\quad \left.-r\binom{L-2}{P-1}+\binom{r+1}{P}\right].
\end{aligned} \tag{6.12}
$$

7 Acknowledgements

This work was supported in part by the Russian Foundation for Basic Research under Grant No. 03-01-00780. V.B.P acknowledges the hospitality of DIAS, Dublin. The authors are indebted to referees for constructive remarks.

References

[1] T.M. Liggett, *Interacting Particle Systems* (Springer-Verlag, New York, 1985).

[2] L. H. Gwa and H. Spohn, Six-Vertex Model, Roughened Surfaces, and an Asymmetric Spin Hamiltonian, Phys. Rev. Lett. **68**:725 (1992).

[3] B. Derrida, An exactly soluble non-equilibrium system: The asymmetric simple exclusion process, Phys. Rep. **301**:65 (1998).

[4] H. Spohn, *Large Scale Dynamics of Interacting Particles* (Springer-Verlag, New York, 1991).

[5] G. M. Schütz, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and J. L. Lebowitz (Academic Press, London, 2001), Vol. 19.

[6] B. Derrida and J.L. Lebowitz, Exact Large Deviation Function in the Asymmetric Exclusion Process, 1998, Phys. Rev. Lett. **80**:209 (1998).

[7] B. Derrida and C.Appert, Universal Large-Deviation Function of the Kardar-Parisi-Zhang Equation in One Dimension, J. Stat. Phys. **94**:1 (1999).

[8] D.-S. Lee and D. Kim, Large deviation function of the partially asymmetric exclusion process, Phys. Rev. E **59**:6476 (1999).

[9] M. Prähofer and H. Spohn, Current fluctuations for the totally asymmetric simple exclusion process, in *In and out of equilibrium*, ed. V. Sidoravicius, Progress in Probability, Birkhäuser, 2002.

[10] B. Derrida, B. Doucot and P.-E. Roche, Current fluctuations in the one dimensional symmetric exclusion process with open boundaries, J. Stat. Phys. **115**:717 (2004).

[11] G. M. Schütz, Exact Solution for Asymmetric Exclusion Process, J. Stat. Phys. **88**:427 (1997).

[12] T. Sasamoto and M. Wadati, Exact results for one-dimensional totally asymmetric diffusion models, J. Phys. A: Math.Gen. **31**:6057 (1998).

[13] T. Nagao and T. Sasamoto, Asymmetric Simple Exclusion Process and Modified Random Matrix Ensembles, Nucl.Phys. B **699**:487 (2004).

[14] A. Rakos and G.M. Schütz, Current distribution and random matrix ensembles for an integrable asymmetric fragmentation process, cond-mat/0405464.

[15] V. B. Priezzhev, Exact Non-Stationary Probabilities in the Asymmetric Exclusion Process on a Ring, Phys. Rev. Lett. **91**:050601 (2003).

[16] K. Johansson, Shape fluctuations and Random Matrices, Commun. Math. Phys. **209**: 437 (2000).

[17] T. C. Dorlas and V. B. Priezzhev, A Normalization identity for the asymmetric exclusion process on a ring. J. Stat. Mech.: Theory and Exp., P11002 (2004).