Entanglement of a Gaussian state interacting with a heat bath

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The effect of a thermal reservoir is observed on a bipartite Gaussian state using continuous variables. We provide the pre-Lindblad master equation in the non-rotating wave approximation for the system. We show correlation between disentanglement and the distance between the particles. The master equation is then solved for a bipartite harmonic oscillator hamiltonian. We show that the entanglement reacts in very surprising ways.

Motivation

Entanglement is one of quantum mechanics most fascinating features. It has baffled many a physicist since it was first described in the 1935 Einstein, Podolsky and Rosen paper [1] and is widely used in quantum information. Moreover the fragility of entanglement when subjected to "outside" influence is still very much an open problem. In the current work, we wish to study a bipartite system with a Gaussian wavefunction. The state is prepared such that it is entangled, then shared between two parties who let their respective particle evolve naturally, resulting in each particle being allowed to interact freely with its own environment, so that the loss of entanglement between the particles may be studied. To do so, we derive the pre-lindblad non-rotating wave master equation, [2, 3], starting with the Quantum Langevin Equation as derived in [4] and using a rather simple perturbation method.

The loss of entanglement in such a system is of course quite well-known a phenomenon. It has been termed entanglement sudden-death (E.S.D.), to account for the very sharp loss of entanglement when compared to a decoherence time scale, and has been studied in various ways, see [5–10] to cite but a few. It has however been studied mainly in the context of qubits and Rotating Wave Approximation (R.W.A.), whereas this work presents a study of E.S.D. in a continuous variables setting and using the Non-Rotating Wave (N.R.W) approximation. This approach is somewhat more dangerous, since the master equation obatined in the N.R.W approximation is not of the well known Lindblad form, hence may have unphysical results. However, the unphysical behaviour is obtained for short times and one can easily check for the validity of the density matrix by checking its positive semi-definite nature. On the other hand, the master equation works well for systems strongly coupled to the environment.

In [11], Ficek and Tanás study a two qubits system coupled to a radiation field where they allow spontaneous decay of the atoms. They show that the entanglement vanishes but is revived twice. In [12], the authors study the emergence of entanglement between two initially non-entangled qubits due to spontaneous emission, provided both atoms are initially excited and in the asymmetric state. Their results suggest that allowing an interaction between two particles initially entangled will delay the vanishing of the entanglement and revive it, or create entanglement between two initially non-entangled particles. We show that when using a harmonic potential as the interaction, the entanglement revival depends on the damping, i.e. how strong the coupling is with respect to the oscillator's frequency. We show that if the damping is small enough, the entanglement oscillates towards a central value and does not in fact vanish.

Section I will recall the Langevin equation and the main steps in the derivation of the master equation. It will then establish the formalism used to describe and use Gaussian states and lay the way to the estimation of the degree of entanglement. Section II will illustrate E.S.D. while Section III shows the main result of this paper. Section IV concludes this article.

I. FRAMEWORK

To derive the Quantum Langevin Equation, let us consider two independent oscillator heat baths, that we can couple to a bipartite system, such that the baths are independent but the particles are entangled. The resulting Hamiltonian has the following form

$$\begin{aligned} H &= \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + V(x_1) + V(x_2) \\ &\quad + \frac{1}{2}\sum_j \left\{ \frac{p_j^2}{m_j} + m_j\omega_j^2 (q_j - x_1)^2 \right\} \\ &\quad + \frac{1}{2}\sum_n \left\{ \frac{p_n^2}{m_n} + m_n\omega_n^2 (q_n - x_2)^2 \right\} \end{aligned} \tag{1}$$

Solving the Heisenberg equations of motion yields

$$\begin{aligned} q_j(t) &= q_j^h(t) + x_1(t) - \int_{-\infty}^t \cos\left[\omega_j(t-t')\right] \dot{x}_1(t')\, dt' \\ q_n(t) &= q_n^h(t) + x_2(t) - \int_{-\infty}^t \cos\left[\omega_n(t-t')\right] \dot{x}_2(t')\, dt' \\ q_k^h(t) &= q_k \cos(\omega_k t) + \frac{p_k}{\omega_k m_k} \sin(\omega_k t) \end{aligned} \tag{2}$$

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Introducing

$$\begin{aligned}\mu_1(t) &= \sum_j m_j\omega_j^2\cos(\omega_j t)\Theta(t)\\ \xi_1(t) &= \sum_j m_j\omega_j^2 q_j^h(t)\end{aligned}\qquad(3)$$

and their counterparts for the other particle. $\Theta(t)$ is the Heaviside function. We can write the Quantum Langevin Equations

$$\begin{aligned}m\ddot{x}_1+\int_{-\infty}^{t}\mu_1(t-t')\dot{x}_1(t')\,dt'+V'(x_1) &= \xi_1(t)\\ m\ddot{x}_2+\int_{-\infty}^{t}\mu_2(t-t')\dot{x}_2(t')\,dt'+V'(x_2) &= \xi_2(t)\end{aligned}\qquad(4)$$

where the dot denote the derivative with respect to time and the prime that with respect to x. $\mu(t)$ and $\xi(t)$ describe the interaction with the system as respecively a memory function and an operator-valued random force. We can also introduce the spectral distribution

$$\begin{aligned}G(\omega) &= \mathcal{R}e\left[\tilde{\mu}(\omega+\imath 0^+)\right]\\ &= \frac{\pi}{2}\sum_k m_k\omega_k^2\left[\delta(\omega-\omega_k)+\delta(\omega+\omega_k)\right]\end{aligned}\qquad(5)$$

so that one can write the autocorrelation of $\xi(t)$ as

$$\frac{1}{2}\langle[\xi(t),\xi(t')]_+\rangle = \frac{1}{\pi}\int_0^{\infty}G(\omega)\hbar\omega\coth\left(\frac{\hbar\omega}{2kT}\right)\cos\left[\omega(t-t')\right]\,d\omega\qquad(6)$$

where the $[\,,]_+$ denote anticommutation. For a general observable Y, one can write

$$\begin{aligned}\dot{Y} &= \frac{\imath}{\hbar}\left[H_s,Y\right]-\frac{\imath}{2\hbar}\left[\left[x_1,Y\right],\xi_1(t)\right]_+-\frac{\imath}{2\hbar}\left[\left[x_2,Y\right],\xi_2(t)\right]_+\\ &+\frac{\imath}{2\hbar}\left[\left[x_1,Y\right],\int_{-\infty}^{t}\mu_1(t-t')\dot{x}_1(t')\,dt'\right]_+\\ &+\frac{\imath}{2\hbar}\left[\left[x_2,Y\right],\int_{-\infty}^{t}\mu_2(t-t')\dot{x}_2(t')\,dt'\right]_+\end{aligned}\qquad(7)$$

If we work with Ohmic heat baths, then

$$\int_{-\infty}^{t}\mu(t')\dot{x}(t')\,dt'\rightarrow\gamma\dot{x}(t)\text{and}G(\omega)=\gamma\qquad(8)$$

so that the Quantum Langevin equation for an arbitrary operator is

$$\begin{aligned}\dot{Y} &= \frac{\imath}{\hbar}\left[H_s,Y\right]-\frac{\imath}{2\hbar}\left[\left[x_1,Y\right],\xi_1(t)\right]_+-\frac{\imath}{2\hbar}\left[\left[x_2,Y\right],\xi_2(t)\right]_+\\ &+\frac{\imath}{2\hbar}\left[\left[x_1,Y\right],\gamma_1\dot{x}_1(t)\right]_++\frac{\imath}{2\hbar}\left[\left[x_2,Y\right],\gamma_2\dot{x}_2(t)\right]_+\end{aligned}\qquad(9)$$

The Langevin equation is an equation for the system operators, where as a master equation is an approximate equation acting on the density operator of the quantum system under study. The adjoint equation provides a link between the two formalisms. One can define

$$\mathrm{Tr}_s\left\{Y(t)\rho_s\right\}=\mathrm{Tr}_s\left\{Y\rho(t)\right\}\qquad(10)$$

where Tr_s is the trace over the system. Applying (10) onto (9) yields the adjoint equation

$$\begin{aligned}\dot{\rho}(t) &= -\frac{\imath}{\hbar}\left[H_s,\rho(t)\right]-\frac{\imath}{2\hbar}\left[\left[\xi_1(t),\rho(t)\right]_+,x_1\right]\\ &-\frac{\imath}{2\hbar}\left[\left[\xi_2(t),\rho(t)\right]_+,x_2\right]+\frac{\imath\gamma_1}{2\hbar}\left[\left[\dot{x}_1,\rho(t)\right]_+,x_1\right]\\ &+\frac{\imath\gamma_2}{2\hbar}\left[\left[\dot{x}_2,\rho(t)\right]_+,x_2\right]\end{aligned}\qquad(11)$$

To derive the master equation from the adjoint equation, we assume that the noise is small and that both bath follow the same dynamics so we can approximate $\xi(t)\rightarrow\epsilon\xi(t)$. We can then write $\rho(t)$ to second order in ϵ as

$$\rho(t)=\rho_0(t)+\epsilon\rho_1(t)+\epsilon^2\rho_2(t)$$

We also assume the baths and the system are initially decoupled so that $\rho_0(t)=\nu_0(t)\rho_B$. Then

$$\dot{\rho}(t)=\dot{\rho}_0(t)+\epsilon\dot{\rho}_1(t)+\epsilon^2\dot{\rho}_2(t)$$

We can expand the terms using (11), then study each term separately. Particularly, the first order term can be written as

$$\begin{aligned}\rho_1(t) = -\frac{\imath}{2\hbar}\int_{-\infty}^{t}e^{A_s(t-t')}\Big\{&\left[\nu_0(t'),x_1\right]\otimes\left[\xi_1(t'),\rho_B\right]_+\\ &+\left[\nu_0(t'),x_2\right]\otimes\left[\xi_2(t'),\rho_B\right]_+\Big\}\,dt'\end{aligned}$$

where A_s is a super operator applying on $\rho_1(t)$ as

$$\begin{aligned}A_s\rho_1(t) &= -\frac{\imath}{\hbar}\left[H_s,\rho_1(t)\right]\\ &+\frac{\imath\gamma_1}{2\hbar}\left[\left[\dot{x}_1,\rho_1(t)\right]_+,x_1\right]+\frac{\imath\gamma_1}{2\hbar}\left[\left[\dot{x}_2,\rho_1(t)\right]_+,x_2\right]\end{aligned}\qquad(12)$$

Finally, we recombine all the terms to get

$$\dot{\nu}(t) = -\frac{\imath}{\hbar}\left[H_s,\nu(t)\right] + \frac{\imath\gamma_1}{2\hbar}\left[\left[\dot{x}_1,\nu(t)\right]_+, x_1\right] + \frac{\imath\gamma_1}{2\hbar}\left[\left[\dot{x}_2,\nu(t)\right]_+, x_2\right]$$
$$-\frac{1}{\hbar^2}\int_{-\infty}^{t}\left[e^{A_s(t-t')}\left[\nu_0(t'),x_1\right],x_1\right]\times\frac{1}{2}\langle[\xi_1(t),\xi_1(t')]_+\rangle\,dt'$$
$$-\frac{1}{\hbar^2}\int_{-\infty}^{t}\left[e^{A_s(t-t')}\left[\nu_0(t'),x_2\right],x_2\right]\times\frac{1}{2}\langle[\xi_2(t),\xi_2(t')]_+\rangle\,dt' \qquad (13)$$

If we recall our assumption that the baths be Ohmic and have a closer look at the autocorrelation function, we can see that in the classical limit[4]

$$\frac{1}{2}\langle[\xi(t),\xi(t')]_+\rangle \rightarrow 2kT\gamma\delta(t-t') \qquad (14)$$

If we inset(14) into (13), we get the non-rotating wave master equation in the brownian motion limit

$$\dot{\nu}(t) = -\frac{\imath}{\hbar}\left[H_s,\nu(t)\right] + \frac{\imath\gamma_1}{2\hbar}\left[\left[\dot{x}_1,\nu(t)\right]_+, x_1\right] + \frac{\imath\gamma_1}{2\hbar}\left[\left[\dot{x}_2,\nu(t)\right]_+, x_2\right] - \frac{kT_1\gamma_1}{\hbar^2}\left[\left[\nu(t),x_1\right],x_1\right] - \frac{kT_2\gamma_2}{\hbar^2}\left[\left[\nu(t),x_2\right],x_2\right] \qquad (15)$$

where we assume that though the baths follow the same dynamics, they may not be at the same temperature nor have the same coupling constant γ at any one time.

To solve it, we can assume that both particles undergo free motion so that

$$H_s = \frac{p_1^2}{2m} + \frac{p_2^2}{2m}$$

with $p|x\rangle = -\imath\hbar\frac{\partial}{\partial x}|x\rangle$. Using a change of variables $x = u + \hbar z$, $x' = u - \hbar z$ and $\rho(\mathbf{x},\mathbf{x'},0) \rightarrow P(\mathbf{u},\mathbf{z},0)$, we can write

$$\partial t P(\mathbf{u},\mathbf{z},t) = \Bigg[\frac{\imath}{2m}\left(\frac{\partial^2}{\partial u_1\partial z_1} + \frac{\partial^2}{\partial u_2\partial z_2}\right) - \frac{\gamma_1}{m}z_1\frac{\partial}{\partial z_1} - \frac{\gamma_2}{m}z_2\frac{\partial}{\partial z_2} - 4\gamma_1 kT_1 z_1^2 - 4\gamma_2 kT_2 z_2^2\Bigg]P(\mathbf{u},\mathbf{z},t) \qquad (16)$$

Applying a Fourier transform with respect to $\mathbf{u}$ as

$$\tilde{P}(\mathbf{q},\mathbf{z},t) = \frac{1}{4\pi^2}\int P(\mathbf{u},\mathbf{z},t)e^{-\imath q_1 u_1 - \imath q_2 u_2}du_1\,du_2 \qquad (17)$$

we can derive the solution to the master equation to be

$$\tilde{P}(\mathbf{q},\mathbf{z},t) = \tilde{P}(\mathbf{q},\mathbf{z}_0,0)\exp\left[-\tau_2\,q_2^2 - \tau_1\,q_1^2\right]$$
$$\times\exp\Bigg[-\lambda_1\left(z_1+\frac{q_1}{2\gamma_1}\right)^2 - \lambda_2\left(z_2+\frac{q_2}{2\gamma_2}\right)^2 + \alpha_1\left(z_1+\frac{q_1}{2\gamma_1}\right) + \alpha_2\left(z_1+\frac{q_2}{2\gamma_2}\right)\Bigg] \qquad (18)$$

with

$$\begin{aligned}\lambda &= 2mkT(1-e^{-2\gamma t/m})\\ \alpha &= \frac{4mkT}{\gamma}(1-e^{-\gamma t/m})\\ \tau &= \frac{kT\,t}{\gamma}\\ z_0 &= z\,e^{-\gamma t/m} - \frac{q}{2\gamma}\left(1-e^{-\gamma t/m}\right)\end{aligned} \qquad (19)$$

We now have an expression giving the time dependency of the state of our choice. Since the state we will study is Gaussian, we will now present an easy way to write Gaussian states.

Gaussian states can be completely specified in terms of their first and second moments [13–15], described respectively by the displacement vector

$$d_j = \langle R_j\rangle_\rho = \mathrm{Tr}[R_j\rho]$$

and the covariance matrix

$$\gamma_{j,k} = 2\mathcal{R}e\ \mathrm{Tr}[\rho(R_j - \langle R_j\rangle_\rho)(R_k - \langle R_k\rangle_\rho)]$$

where R is the vector $R^T = (q_1,....,q_n;p_1,...p_n)$; q_n and p_n are the canonical variables of a system of n oscillators with the usual canonical relations written as $[R_j,R_k] = \imath\sigma_{jk}$ and σ a real skew-symmetric $2n\times 2n$ block matrix given by

$$\sigma = \begin{pmatrix} 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0\end{pmatrix}$$

We take $\hbar = 1$ here. It is important to know that the first moments represented by the displacement vector can always be made vanishing with local operations, rendering them irrelevant in the study of entanglement. We can then reduce the covariance matrix to

$$\gamma_{j,k} = 2\mathcal{R}e \ \mathrm{Tr}[\rho R_j R_k] \tag{20}$$

Any real symmetric positive definite matrix A can be brought to its Williamson normal form [16] via symplectic transformations, i.e. transformations that preserve the canonical commutation relations, $A_{WF} = SAS^T = \mathrm{diag}(a_1, a_1, ...a_n, a_n)$ where the a_i's are the symplectic eigenvalues of A. One can calculate them as the positive eigenvalues of $\imath\sigma A$ or more simply as the positive square root of the eigenvalues of $-\sigma A \sigma A$.

Chosen here because of the easy way to evaluate it with covariance matrices as

$$\mathcal{E}_{\mathcal{N}}(\rho) = -\sum_{i=1}^{n+m} \log_2 \left(\min \left(1\,, \mid \lambda_i \mid\right)\right) \tag{21}$$

the logarithmic negativity[13–15, 17] is a commonly used entanglement measure for mixed states and is perfectly well-adapted for the present work. It vanishes for a separable state, does not increase under LOCC and stays invariant under local unitary transformations. However it does not reduce to the Von Neumann entropy of entanglement for a pure state. To use it requires a partially transposed γ, obtained by reversing the time in all variables of one system. If we choose to transpose with respect to particle 1, we replace $x_1 \to x_1$ and $p_1 \to -p_1$. The λ_i's are then the symplectic eigenvalues of γ^{T_1}, obtained without explicit diagonalisation as the positive square root of the eigenvalues of $-\sigma\gamma^{T_1}\sigma\gamma^{T_1}$.

II. THE STATE AND ITS EVOLUTION

Using the framework established in the previous section, we can study the state in quite an easy way. Let us consider a initial bipartite state with the Gaussian wavefunction

$$\Psi(x_1, x_2) = \Omega^{1/2} e^{-\frac{(x_1-x_2)^2}{4\sigma^2}} e^{-\frac{(x_1+x_2)^2}{16d^2}} \tag{22}$$

The corresponding density matrix is then

$$\rho(t=0) = \Omega e^{-\epsilon_+ ({x_1}^2 + {x_2}^2 + {x'_1}^2 + {x'_2}^2) + 2\epsilon_- (x_1 x_2 + x'_1 x'_2)} \tag{23}$$

where $\Omega = \frac{1}{2\pi\sigma d}$, $\epsilon_+ = \frac{1}{4\sigma^2} + \frac{1}{16d^2}$ and $\epsilon_- = \frac{1}{4\sigma^2} - \frac{1}{16d^2}$.

It is easy to calculate the covariance matrix of this state and examine how the entanglement behaves as the distance σ between the two particles varies. Performing the partial transposition as previously described, we obtain

$$\gamma^{T_1} = \begin{pmatrix} \alpha & 0 & \beta & 0 \\ 0 & \delta & 0 & \epsilon \\ \beta & 0 & \alpha & 0 \\ 0 & \epsilon & 0 & \delta \end{pmatrix} \tag{24}$$

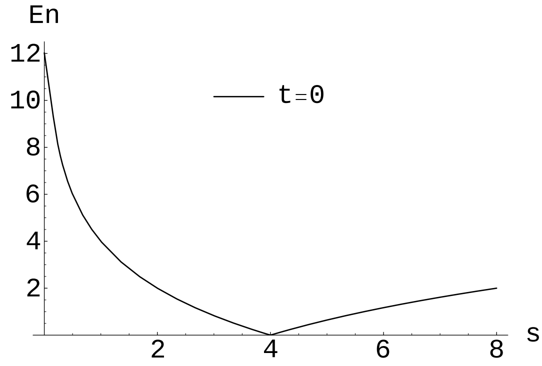

FIG. 1: Logarithmic negativity vs σ for $d = 2$

with $\alpha = 2d^2 + \frac{\sigma^2}{2}$, $\beta = 2d^2 - \frac{\sigma^2}{2}$, $\delta = \frac{1}{2\sigma^2} + \frac{1}{8d^2}$, $\epsilon = \frac{1}{2\sigma^2} - \frac{1}{8d^2}$.

The symplectic eigenvalues are then calculated to be

$$\lambda_{1,2} = \frac{4d^2}{\sigma^2} \qquad \lambda_{2,4} = \frac{\sigma^2}{4d^2} \tag{25}$$

and the logarithmic negativity becomes

$$\mathcal{E}_{\mathcal{N}}(\rho) = -2\log_2\left(\min(1, \frac{2d}{\sigma})\right) - 2\log_2\left(\min(1, \frac{\sigma}{2d})\right) \tag{26}$$

As can be seen on Figure 1, the state becomes separable when $\sigma = 2d$.

We next compute the time dependent state by applying (17) to (23)

$$\tilde{P}(\mathbf{q}, \mathbf{z}_0; 0) = \frac{\Omega}{4\pi^2}\sqrt{\frac{\pi}{\epsilon_+^2 - \epsilon_-^2}} \tag{27}$$

$$\times \exp\left[-\epsilon_+ \hbar^2 z_{01}^2 - \epsilon_+ \hbar^2 z_{02}^2 + 2\epsilon_- \hbar^2 z_{01} z_{02}\right]$$

$$\times \exp\left[-\frac{{q_2}^2}{4\epsilon_+} - \frac{(q_1 + \frac{\epsilon_- q_2}{\epsilon_+})^2}{4(\epsilon_+ - \frac{\epsilon_-^2}{\epsilon_+})}\right] \tag{28}$$

which is then substituted into (18) to get [18]

$$\rho(x_1, x_2; x'_1, x'_2; t) = \Omega' \times e^{-\mu_1 {x_1}^2 - \mu_2 {x_2}^2 - \mu'_1 {x'_1}^2 - \mu'_2 {x'_2}^2} \times e^{2\nu_1 x_1 x'_1 + 2\nu_2 x_2 x'_2 + \eta x_1 x_2 + \eta' x'_1 x'_2 + \zeta x_1 x'_2 + \zeta' x_2 x'_1} \tag{29}$$

where the coefficients are rather complicated expressions of the coefficients (31) to which they reduce when the covariance matix terms are calculated.

As for the unevolved case, we perform the previously described partial transposition. The covariance matrix is then

$$\gamma_t^{T_1} = \begin{pmatrix} 4A_1 & -C_{11} & E & -C_{21} \\ -C_{11} & B_1 & C_{12} & D \\ E & C_{12} & 4A_2 & C_{22} \\ -C_{21} & D & C_{22} & B_2 \end{pmatrix} \tag{30}$$

which is real and symmetric. It may be of interest to give

$$
\begin{aligned}
A_j &= \frac{\epsilon_+}{4(\epsilon_+^2-\epsilon_-^2)}+\tau_j-\frac{\alpha_j}{2\gamma_j}+\frac{\lambda_j}{4\gamma_j{}^2}+\frac{\hbar^2\epsilon_+}{4\gamma_j{}^2}(1-e^{-\frac{\gamma_j t}{m}})^2\\
B_j &= \hbar^2\epsilon_+\, e^{-\frac{2\gamma_j t}{m}}+\lambda_j\\
C_{jj} &= \frac{\hbar^2\epsilon_+}{\gamma_j}(e^{-\frac{\gamma_j t}{m}}-e^{-\frac{2\gamma_j t}{m}})-\frac{\lambda_j}{\gamma_j}+\alpha_j\\
D &= \hbar^2\epsilon_-\, e^{-\frac{\gamma_1 t}{m}}\, e^{-\frac{\gamma_2 t}{m}}\\
E &= \frac{\epsilon_-}{\epsilon_+^2-\epsilon_-^2}-\frac{\hbar^2\epsilon_-}{\gamma_1\gamma_2}(1-e^{-\frac{\gamma_1 t}{m}})(1-e^{-\frac{\gamma_2 t}{m}})\\
C_{jk} &= \frac{\hbar^2\epsilon_-}{\gamma_k}\, e^{-\frac{\gamma_j t}{m}}\,(1-e^{-\frac{\gamma_k t}{m}})
\end{aligned}
\tag{31}
$$

The symplectic eigenvalues are given by

$$
\begin{aligned}
\lambda^T_{1,2} =& \frac{\varepsilon_{11}+\varepsilon_{33}}{2}+\sqrt{(\varepsilon_{11}-\varepsilon_{33})^2+4\varepsilon_{13}\varepsilon_{24}-4\varepsilon_{14}\varepsilon_{23}}\\
\lambda^T_{3,4} =& \frac{\varepsilon_{11}+\varepsilon_{33}}{2}-\sqrt{(\varepsilon_{11}-\varepsilon_{33})^2+4\varepsilon_{13}\varepsilon_{24}-4\varepsilon_{14}\varepsilon_{23}}
\end{aligned}
\tag{32}
$$

where $\varepsilon_{12}=\varepsilon_{21}=\varepsilon_{34}=\varepsilon_{43}=0$ and

$$
\begin{aligned}
\varepsilon_{11}=\varepsilon_{22}=&4A_1B_1-DE+C_{12}C_{21}-C_{11}^2\\
\varepsilon_{33}=\varepsilon_{44}=&4A_2B_2-C_{22}^2-DE+C_{12}C_{21}\\
\varepsilon_{13}=\varepsilon_{42}=&2EB_1-2A_2D-C_{11}C_{12}+C_{12}C_{22}\\
\varepsilon_{14}=-\varepsilon_{32}=&-C_{12}B_2-C_{21}B_1+C_{11}D/2+C_{22}D/2\\
\varepsilon_{23}=-\varepsilon_{41}=&-2EC_{11}+4A_1C_{12}+4A_2C_{21}-2EC_{22}\\
\varepsilon_{24}=\varepsilon_{31}=&2EB_2-C_{22}C_{21}+C_{11}C_{21}-2A_1D
\end{aligned}
\tag{33}
$$

The logarithmic negativity then becomes

$$
\mathcal{E}_{\mathcal{N}}(\rho)=-2\left(\log_2\left(\min(1,|\lambda^T_{1,2}|)\right)+\log_2\left(\min(1,|\lambda^T_{3,4}|)\right)\right)
\tag{34}
$$

On Figure 2, we have plotted the logarithmic negativity vs σ for various t's. Figure 3 shows the time evolution of the logarithmic negativity for various σ.

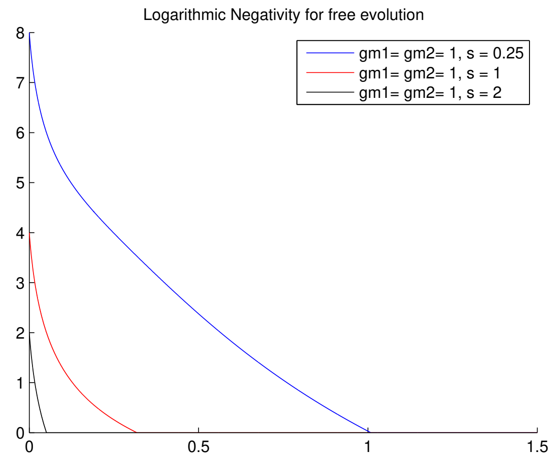

FIG. 2: Logarithmic negativity vs t for three values of s

The values of σ are : blue $\sigma=0.25$, red $\sigma=1$, black $\sigma=2$

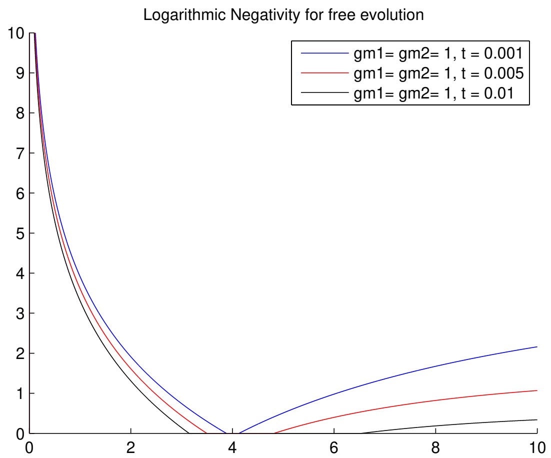

FIG. 3: Logarithmic negativity vs σ for three times

The plot are for : blue $t=0.001$, red $t=0.005$, black $t=0.01$

III. THE HARMONIC POTENTIAL

A bipartite harmonic potential Hamiltonian can be written as

$$
H_s=\frac{p_1^2}{2m}+\frac{p_2^2}{2m}+\frac{m\omega_0^2}{2}(x_1-x_2)^2
\tag{35}
$$

With position coupling, the master equation becomes

$$
\begin{aligned}
\dot\rho=-\frac{\imath}{\hbar}\left[H_s,\rho\right]&-\frac{\imath\gamma_1}{2\hbar}\left[x_1,[p_1,\rho]_+\right]-\frac{\gamma_1 kT_1}{\hbar^2}\left[x_1,[x_1,\rho]\right]\\
&-\frac{\imath\gamma_2}{2\hbar}\left[x_2,[p_2,\rho]_+\right]-\frac{\gamma_2 kT_2}{\hbar^2}\left[x_2,[x_2,\rho]\right]
\end{aligned}
\tag{36}
$$

Writing the density matrix in position representation, $\rho(x_1,x_2;y_1,y_2)$, we get

$$
\begin{aligned}
\frac{\partial\rho}{\partial t}=&\frac{\imath\hbar}{2m}\left(\frac{\partial^2}{\partial x_1^2}-\frac{\partial^2}{\partial y_1^2}+\frac{\partial^2}{\partial x_2^2}-\frac{\partial^2}{\partial y_2^2}\right)\rho\\
&-\frac{\imath m\omega_0^2}{2\hbar}\left((x_1-x_2)^2-(y_1-y_2)^2\right)\rho\\
&-\frac{\gamma_1}{2m}(x_1-y_1)\left(\frac{\partial}{\partial x_1}-\frac{\partial}{\partial y_1}\right)\rho-\frac{\gamma_1 kT_1}{\hbar^2}(x_1-y_1)^2\rho\\
&-\frac{\gamma_2}{2m}(x_2-y_2)\left(\frac{\partial}{\partial x_2}-\frac{\partial}{\partial y_2}\right)\rho-\frac{\gamma_2 kT_2}{\hbar^2}(x_2-y_2)^2\rho
\end{aligned}
\tag{37}
$$

After applying the change of variable used in Section I and (17), we obtain the differential equation

$$\frac{\partial \tilde{P}}{\partial t}(\mathbf{q},\mathbf{z},t) = \Bigg\{ -\frac{1}{2m}\left(q_1\frac{\partial}{\partial z_1} + q_2\frac{\partial}{\partial z_2}\right) - \frac{\gamma_1}{m}\,z_1\,\frac{\partial}{\partial z_1} - \frac{\gamma_2}{m}\,z_2\,\frac{\partial}{\partial z_2} + 2m\omega_0^2\left(\frac{\partial}{\partial q_1} - \frac{\partial}{\partial q_2}\right)(z_1 - z_2) - 4\gamma_1 kT_1\,z_1^2 - 4\gamma_2 kT_2\,z_2^2 \Bigg\}\tilde{P}(\mathbf{q},\mathbf{z},t) \tag{38}$$

That we can solve using the method of characteristics (taking $\gamma_1 = \gamma_2 = \gamma$ and $T_1 = T_2 = T$ for simplicity). After some unpleasant algebra, we can write

$$\begin{aligned}\tilde{P} =& \exp\big[-\mathcal{A}q_1^2 - \mathcal{A}q_2^2 - \mathcal{L}q_1q_2 - \mathcal{B}z_1^2 - \mathcal{B}z_2^2 - \mathcal{D}z_1z_2 \\ & - \mathcal{C}_1z_1q_1 - \mathcal{C}_1z_2q_2 - \mathcal{C}_2z_1q_2 - \mathcal{C}_2z_2q_1\big]\end{aligned} \tag{39}$$

The partial transpose is then

$$\tilde{\gamma}_t^T = \begin{bmatrix} 4\mathcal{A} & \mathcal{C}_1 & 2\mathcal{L} & -\mathcal{C}_2 \\ \mathcal{C}_1 & \mathcal{B} & \mathcal{C}_2 & -\mathcal{D}/2 \\ 2\mathcal{L} & \mathcal{C}_2 & 4\mathcal{A} & -\mathcal{C}_1 \\ -\mathcal{C}_2 & -\mathcal{D}/2 & -\mathcal{C}_1 & \mathcal{B} \end{bmatrix} \tag{40}$$

The symplectic eigenvalues can then be determined to be :

$$\begin{aligned}\tilde{\lambda}_{1,2}^T =& \tilde{\varepsilon}_{11} + \sqrt{\tilde{\varepsilon}_{13}^2 - \tilde{\varepsilon}_{14}\tilde{\varepsilon}_{23}} \\ \tilde{\lambda}_{3,4}^T =& \tilde{\varepsilon}_{11} - \sqrt{\tilde{\varepsilon}_{13}^2 - \tilde{\varepsilon}_{14}\tilde{\varepsilon}_{23}}\end{aligned} \tag{41}$$

where $\tilde{\varepsilon}_{12} = \tilde{\varepsilon}_{21} = \tilde{\varepsilon}_{34} = \tilde{\varepsilon}_{43} = 0$

$$\begin{aligned}\tilde{\varepsilon}_{11} = \tilde{\varepsilon}_{22} = \tilde{\varepsilon}_{33} = \tilde{\varepsilon}_{44} =& 4\mathcal{A}\mathcal{B} - \mathcal{D}\mathcal{L} + \mathcal{C}_2^2 - \mathcal{C}_1^2 \\ \tilde{\varepsilon}_{13} = \tilde{\varepsilon}_{24} = \tilde{\varepsilon}_{31} = \tilde{\varepsilon}_{42} =& 2\mathcal{L}\mathcal{B} - 2\mathcal{A}\mathcal{D} \\ \tilde{\varepsilon}_{14} = -\tilde{\varepsilon}_{32} =& \mathcal{C}_1\mathcal{D} - 2\mathcal{C}_2\mathcal{B} \\ \tilde{\varepsilon}_{23} = -\tilde{\varepsilon}_{41} =& 8\mathcal{A}\mathcal{C}_2 - 4\mathcal{L}\mathcal{C}_1\end{aligned}$$

The logarithmic negativity then becomes

$$\mathcal{E}_{\mathcal{N}}(\rho) = -2\left(\log_2\left(\min(1, |\tilde{\lambda}_{1,2}^T|)\right) + \log_2\left(\min(1, |\tilde{\lambda}_{3,4}^T|)\right)\right) \tag{42}$$

IV. CONCLUDING REMARKS

We can observe on Figure **??** that for small enough t's, the entanglement vanishes for $\sigma = 2d$, as one can expect from the form of the initial state. As t increases however, $\mathcal{E}_{\mathcal{N}}(\rho)$ vanishes for a range of σ. We can then deduce that for a value of d, there is a range of distances σ between the particles for which entanglement will exist, a range that becomes more and more restrained as the time increases.

On Figure **??**, we can observe the disentanglement between the particles for various σ and a varying time. The sharp decrease, trademark of E.S.D. is quite obvious.

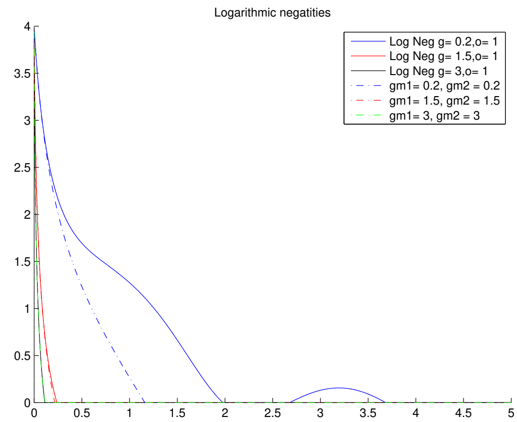

FIG. 4: Logarithmic Negativities with and without potential for $\omega_0 = 1$

The dashed lines represent $\mathcal{E}$ plotted without the potential. The plots are ($\gamma = \gamma_1 = \gamma_2$): green $\gamma = 3$, red $\gamma = 1.5$ and blue $\gamma = 0.2$

On Figure 4, one can identify two damping behaviours, namely over-damped if $\gamma \gg \omega_0$ and under-damped when $\gamma \ll \omega_0$. It can easily be seen that if the damping is strong, the entanglement vanishes just as quickly as it would without the potential. However, if the damping is small, the entanglement reappears several times. Figure 5 illustrates this phenomenon in more details. It can easily be seen indeed that as the damping decreases, the entanglement does not vanish, but tends towards a particular value.

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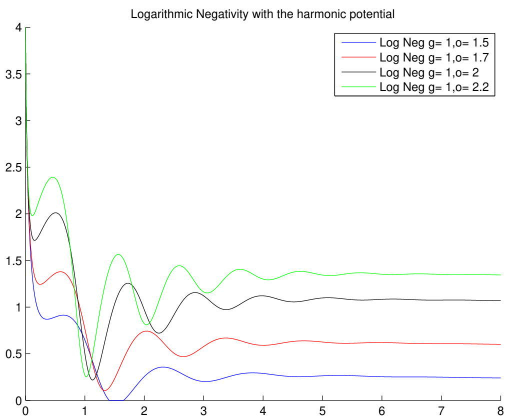

FIG. 5: $\mathcal{E}$ in the highly under-damped case

The plots are obtained with $\gamma = 1$ and : blue $\omega_0 = 1.5$, red $\omega_0 = 1.7$, black $\omega_0 = 2$ and green $\omega_0 = 2.2$

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[18] The symmetry of the original state was broken when performing the Fourier transform. It was chosen to perform all integrals with respect to the second variables first. This convention was also applied to the calculation of all the convariance matrix terms. This choice was very much arbitrary and similar results are expected, should one wish to perform the integrals with respect to the first variables.