

For

$$H_0 := -\Delta + \alpha\delta(x), \quad (0.1)$$

Green's function is, for  $k^2 \notin \mathbb{R}_+ \cup \{-\alpha^2/4\}$ ,

$$G_0(k^2; x; x') = \frac{i}{2k} e^{ik|x-x'|} + \frac{\alpha}{2k(2k + i\alpha)} e^{ik(|x|+|x'|)}. \quad (0.2)$$

Consider

$$H_1 := -\frac{1}{2}\Delta_{x_1} - \frac{1}{2}\Delta_{x_2} + \alpha\delta(x_1 - x_2). \quad (0.3)$$

Letting  $x_1 = (X + Y)/2$  and  $x_2 = (Y - X)/2$  this transforms to

$$H_2 := -\Delta_Y - \Delta_X + \alpha\delta(X) \quad (0.4)$$

with Green's function

$$G_2(z; X, Y; X', Y') = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(Y-Y')} G_0(z - k^2; X; X') dk \quad (0.5)$$

for  $z \notin [-\alpha^2/4, \infty)$ . In the original coordinates Green's function is

$$\begin{aligned} G_1(z; \mathbf{x}; \mathbf{x}') &= G_2(z; x_1 - x_2, x_1 + x_2; x'_1 - x'_2, x'_1 + x'_2) \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik\{(x_1 - x'_1) + (x_2 - x'_2)\}} G_0(z - k^2; x_1 - x_2; x'_1 - x'_2) dk. \end{aligned}$$

Let

$$\begin{aligned} K_{11}(z; s; s') &= G_2(z; 0, s; 0, s') = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(s-s')} G_0(z - k^2; -s; -s') dk, \\ K_{22}(z; s; s') &= G_2(z; s, 0; s', 0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(s-s')} G_0(z - k^2; s; s') dk, \\ K_{12}(z; s; s') &= G_2(z; 0, s; s', 0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(s-s')} G_0(z - k^2; -s; s') dk, \\ K_{21}(z; s; s') &= G_2(z; s, 0; 0, s') = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(s-s')} G_0(z - k^2; s; -s') dk. \end{aligned}$$

Then

$$K_{11}(z; s; s') = K_{22}(z; s; s') = K_{12}(z; s; -s') = K_{21}(z; s; -s'). \quad (0.6)$$

Let

$$H_3 := -\frac{1}{2}\Delta_{x_1} - \frac{1}{2}\Delta_{x_2} + \alpha\delta(x_1 - x_2) + a\delta(x_1) + b\delta(x_2). \quad (0.7)$$

Let  $A$  and  $B$  be the integral operators on  $L^2(\mathbb{R})$  with kernels  $K_{11} = K_{22}$  and  $K_{12} = K_{21}$  respectively. Then Green's function for  $H_3$  can be expressed in the form:

$$\begin{aligned} G_3(z; \mathbf{x}; \mathbf{x}') &= G_1(z; \mathbf{x}; \mathbf{x}') \\ &- \left( \begin{array}{cc} G_1(z; \mathbf{x}; 0, s) & G_1(z; \mathbf{x}; s, 0) \end{array} \right) \left( \begin{array}{cc} a^{-1} + A & B \\ B & b^{-1} + A \end{array} \right)^{-1} \left( \begin{array}{c} G_1(z; 0, s; \mathbf{x}') \\ G_1(z; s, 0; \mathbf{x}') \end{array} \right) \end{aligned}$$