Proof of a general noncommutative central limit theorem using a Feynman-Kac representation

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Abstract. The generalized noncommutative limit theorem conjectured in a preceding paper is proved using a Feynman-Kac representation analogous to that in an earlier paper by the author.

1. The Feynman-Kac representation.

In a preceding paper [1] the author and K. Neammanee conjectured a new noncommutative central-limit-type theorem which is a generalization of the second theorem proved in [2] to the case of higher spin values. Here I outline a proof of this conjecture along the lines of [2] indicating the necessary changes. It involves a Feynman-Kac representation like the one first derived by Griffiths [3], [4]. Given a set of $r+1$ creation and annihilation operators $a_0^*, \dots, a_r^*$ respectively $a_0, \dots, a_r$ satisfying

$$[a_i, a_j^*] = \delta_{i,j} \quad (i, j = 0, 1, \dots, r), \tag{1.1}$$

we introduce a new set $c_k, c_k^* \quad (k = 0, 1, \dots, r)$ defined by

$$c_k = \sum_{i=0}^{r} O_{k,i} a_i, \quad (k = 0, 1, \dots, r) \tag{1.2}$$

where O is an orthogonal matrix with the property that

$$O_{0,j} = (r+1)^{-1/2} \equiv \gamma_r \quad (j = 0, 1, \dots, r). \tag{1.3}$$

As in [2], the operators a_j and a_j^* can be represented on the n-particle space $\mathcal{F}_n$ by

$$a_j\big|_{\mathcal{F}_n} = \frac{1}{\sqrt{n}} \sum_{p=1}^{n} a_{p,j}\big|_{\mathcal{F}_n} \text{ and } a_j^*\big|_{\mathcal{F}_n} = \frac{1}{\sqrt{n+1}} \sum_{p=1}^{n+1} a_{p,j}^*\big|_{\mathcal{F}_n}, \tag{1.4}$$

where

$$a^*_{p,j}\,|x_1,\dots,x_n\rangle = |x_1,\dots,x_{p-1},j,x_p,\dots,x_n\rangle$$
$$a_{p,j}\,|x_1,\dots,x_n\rangle = \delta_{x_p,j}\,|x_1,\dots,\hat{x}_p,\dots,x_n\rangle. \tag{1.5a}$$

In this representation the number operator $c^*_k c_k$ is given by

$$c^*_k c_k = \sum_{p=1}^{n}\left(\sum_{i=0}^{r} O_{k,i}a^*_{p,i}\right)\left(\sum_{j=0}^{r} O_{k,j}a_{p,j}\right). \tag{1.6}$$

It follows that the operator $\sum_{k=1}^{r} c^*_k c_k$ appearing in the exponent of formula (1.16) in [1] can be written as

$$\sum_{k=1}^{r} c^*_k c_k = \sum_{p=1}^{n} G_p, \tag{1.7}$$

where the operator G_p acts on the p-th particle space according to

$$\begin{aligned}\langle x'|\,G_p\,|x\rangle &= \sum_{k=1}^{r}\sum_{i=0}^{r}\sum_{j=0}^{r} O_{k,i}O_{k,j}\langle x'|\,a^*_{p,i}a_{p,j}\,|x\rangle \\ &= \sum_{k=1}^{r} O_{k,x'}O_{k,x} = \delta_{x',x} - \frac{1}{r+1},\end{aligned} \tag{1.8}$$

independently of O. As in [2], the operator G_p is the generator of a simple jump process with transition probabilities given by

$$p_t(x,x') = \langle x'|\,\exp[-tG_p]\,|x\rangle = e^{-t}\delta_{x',x} + (1-e^{-t})\frac{1}{r+1}. \tag{1.9}$$

The corresponding infinitesimal transition probabilities are

$$\mathbf{P}[\xi_p(t+\delta t) = x' \mid \xi_p(t) = x] = \begin{cases} 1 - \frac{r}{r+1}\delta t & \text{if } x' = x, \\ \frac{1}{r+1}\delta t & \text{if } x' \neq x. \end{cases} \tag{1.10}$$

The operators Δ_k can be written as

$$\Delta_k = \sum_{j=0}^{r} O_{k,j}a^*_j\,a_j = \sum_{p=1}^{n} \Gamma_{k,p}, \tag{1.11}$$

where

$$\Gamma_{k,p} = \sum_{j=0}^{r} O_{k,j}\,a^*_{p,j}\,a_{p,j}. \tag{1.12}$$

These are diagonal in this representation and we have the following analogue of (2.15) in [2]:

$$\begin{aligned}&\langle x_1',\dots,x_n'|\ \exp\left[-\beta\sum_{p=1}^{n}G_p+f(\{\Gamma_{k,p}\}_{k,p})\right]\ |x_1,\dots,x_n\rangle\\&=\mathbf{E}^{(n)}_{x_1,\dots,x_n}\left[\exp\left[\int_0^\beta f\left(\left\{O_{k,\xi_p(t)}\right\}_{k,p}\right)dt\right]\prod_{i=1}^{n}1_{\{\xi_i(1)=x_i'\}}\right].\end{aligned}\tag{1.13}$$

Here f is any function of the nr arguments $\Gamma_{k,p}$ with $k=1,\dots,r$ and $p=1,\dots,n$ and and $\mathbf{E}^{(n)}_{x_1,\dots,x_n}$ is the expectation with respect to the n-fold product measure $\prod_{i=1}^n \mathbf{P}_{x_i}$.

Next we define the following measure on r-dimensional paths $(z_1(t),\dots,z_r(t))\in L^2([0,\beta],\mathbf{R}^r)$, with $z_i(t)=\sum_{p=1}^n O_{i,\xi_p(t)}$:

$$\begin{aligned}\mathbf{K}^{(n)}_\beta[F]=\frac{1}{Z_n(\beta)}&\sum_{n_0,\dots,n_r\geq 0:\sum n_i=n}\frac{n_0!\dots n_r!}{n!}\sum_{x_1,\dots,x_n}^{*}\sum_{x_1',\dots,x_n'}^{*}\\&\mathbf{E}^{(n)}_{x_1,\dots,x_n}\left\{F\left(\sum_{p=1}^n O_{1,\xi_p(t)},\dots,\sum_{p=1}^n O_{r,\xi_p(t)}\right)\prod_{i=1}^n 1_{\{\xi:\,\xi_i(\beta)=x_i'\}}\right\},\end{aligned}\tag{1.14}$$

for an arbitrary continuous function $F\left(\{z_i(\cdot)\}_{i=1}^r\right)$ of r paths, where the normalization factor $Z_n(\beta)$ is given by

$$Z_n(\beta)=\sum_{n_0,\dots,n_r:\sum_{i=0}^r n_i=n}\exp\left[-\beta\sum_{k=1}^r n_k\right]=\sum_{m=0}^n\binom{m+r-1}{r-1}e^{-\beta m},\tag{1.15}$$

and where the asterisk above the sum indicates that the sum is taken over all sequences $(x_1,\dots,x_n)$ satisfying $\#\{i:\,x_i=k\}=n_k$ for all $k=0,\dots,r$.

Using (1.4), (1.7) and (1.11) we now have

$$\frac{\mathrm{Trace}_n\exp\left\{-\beta\sum_{k=1}^r c_k^*c_k+f(\Delta_1,\dots,\Delta_r)\right\}}{\mathrm{Trace}_n\exp\left\{-\beta\sum_{k=1}^r c_k^*c_k\right\}}=\mathbf{K}^{(n)}_\beta[\tilde{f}_\beta],\tag{1.16}$$

where

$$\tilde{f}_\beta(z_1(\cdot),\dots,z_r(\cdot))=\exp\left[\int_0^\beta f(z_1(t),\dots,z_r(t))dt\right].\tag{1.17}$$

As in [2], we now want to compute the characteristic functional for the measures $\mathbf{K}^{(n)}_\beta$. We take a more operator-theoretic approach and express these in terms of *expansionals* as introduced by Fujiwara [5] and Araki [6,7]. If $A(t)$ ($t \in [0,\beta]$) is a continuous one-parameter family of operators then the (right) expansional of $A(t)$ is defined as

$$\mathrm{Exp} \int_0^\beta ; A(t)\, dt = \sum_{m=0}^\infty \int_0^\beta dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{m-1}} dt_m A(t_m) \ldots A(t_1). \tag{1.18}$$

Lemma 1. *The characteristic functional of the measure $\mathbf{K}_\beta^{(n)}$ is given by*

$$\mathbf{K}_\beta^{(n)} \left[\exp\left\{ i \int_0^\beta \sum_{k=1}^r u_k(t) z_k(t) dt \right\} \right] = \frac{1}{Z_n(\beta)} \sum_{n_0,\ldots,n_r : \sum_{i=0}^r n_i = n} \lambda_0^{n_0} \ldots \lambda_r^{n_r}, \tag{1.19}$$

where $\lambda_0, \ldots, \lambda_r$ are the eigenvalues of the matrix $U(\beta)$ given by

$$U(\beta) = \left(\mathrm{Exp} \int_0^\beta ; A(t) dt \right) V(\beta), \tag{1.20}$$

where the $(r+1) \times (r+1)$-matrices $A(t)$ are given by

$$A_{m,m'}(t) = \begin{cases} i \sum_{k=1}^r u_k(t) \sum_{x=0}^r O_{m,x} O_{k,x} O_{m',x} & \text{if } m, m' \neq 0, \\ i\gamma_r u_m(t) e^{-t} & \text{if } m' = 0, m \neq 0, \\ i\gamma_r u_{m'}(t) e^t & \text{if } m = 0, m' \neq 0. \end{cases} \tag{1.21}$$

and where $V(\beta)$ is a diagonal matrix with $V_{0,0}(\beta) = 1$ and $V_{m,m}(\beta) = e^{-\beta}$ if $m \neq 0$.

Proof. First define

$$\tilde{U}_{x,x'}(\beta) = \mathbf{E}_x \left\{ \exp\left[i \int_0^\beta \sum_{k=1}^r u_k(t) O_{k,\xi_p(t)} dt \right] 1_{\{\xi_p : \xi_p(\beta) = x'\}} \right\}. \tag{1.22}$$

Using (1.10) we have

$$\begin{aligned} \frac{d}{d\beta} \tilde{U}_{x,x'}(\beta) &= \left(i \sum_{k=1}^r u_k(\beta) O_{k,x'} \right) \tilde{U}_{x,x'}(\beta) \\ &\quad + \lim_{\delta\beta \to 0} \frac{1}{\delta\beta} \Bigg(\mathbf{E}_x \left\{ \exp\left[i \int_0^\beta \sum_{k=1}^r u_k(t) O_{k,\xi_p(t)} dt \right] 1_{\{\xi_p : \xi_p(\beta+\delta\beta) = x'\}} \right\} \\ &\qquad - \mathbf{E}_x \left\{ \exp\left[i \int_0^\beta \sum_{k=1}^r u_k(t) O_{k,\xi_p(t)} dt \right] 1_{\{\xi_p : \xi_p(\beta) = x'\}} \right\} \Bigg) \\ &= \left(i \sum_{k=1}^r u_k(\beta) O_{k,x'} - \frac{r}{r+1} \right) \tilde{U}_{x,x'}(\beta) + \frac{1}{r+1} \sum_{x'' \neq x'} \tilde{U}_{x,x''}(\beta). \end{aligned} \tag{1.23}$$

This can be written as

$$\frac{d}{d\beta}\tilde{U}(\beta) = \tilde{U}(\beta)\tilde{B}(\beta), \tag{1.24}$$

where

$$\tilde{B}_{x,x'}(\beta) = \left(i\sum_{k=1}^{r} u_k(\beta) O_{k,x'} - 1 \right) \delta_{x,x'} + \frac{1}{r+1}. \tag{1.25}$$

Next we transform to the c-representation and define

$$U(\beta) = O\tilde{U}(\beta)O^T. \tag{1.26}$$

Then

$$\frac{d}{d\beta}U(\beta) = O\frac{d}{d\beta}\tilde{U}(\beta)O^T = U(\beta)O\tilde{B}(\beta)O^T = U(\beta)B(\beta). \tag{1.27}$$

The matrix $B(\beta)$ is given by

$$\begin{aligned} B_{m,m'}(\beta) &= \sum_{x,x'} O_{m,x}\tilde{B}_{x,x'}(\beta)O_{m',x'} \\ &= i\sum_{k=1}^{r} u_k(\beta) \sum_x O_{m,x}O_{k,x}O_{m',x} - \sum_x O_{m,x}O_{m',x} \\ &\qquad\qquad + \frac{1}{r+1}\sum_{x,x'} O_{m,x}O_{m',x'} \\ &= i\sum_{k=1}^{r} u_k(\beta) \sum_x O_{m,x}O_{k,x}O_{m',x} - \delta_{m,m'} + \delta_{m,0}\delta_{m',0}. \end{aligned} \tag{1.28}$$

Using the simple fact that

$$\frac{d}{d\beta}V(\beta)^{-1} = (1-P_0)V(\beta)^{-1},$$

where $(P_0)_{m,m'} = \delta_{m,0}\delta_{m',0}$, we now find that

$$\frac{d}{d\beta}U(\beta)V(\beta)^{-1} = U(\beta)V(\beta)^{-1}A(\beta), \tag{1.29}$$

where $A(\beta) = (1-P_0) + V(\beta)B(\beta)V(\beta)^{-1}$ is given by (1.21). By a general result of Araki's [6] we then have

$$U(\beta)V(\beta)^{-1} = \mathrm{Exp}\int_0^\beta ; A(t)dt$$

as stated. QED

2. The Main Theorem

We now state the main theorem. The general noncommutative central limit theorem of [1] follows from this in the same way as in [2].

Theorem. *The rescaled measures given by* $\tilde{\mathbf{K}}_\beta^{(n)}(F) = \mathbf{K}_\beta^{(n)}(F_n)$, *where* $F_n(z_1, \ldots, z_r) = F(z_1/\sqrt{n}, \ldots, z_r/\sqrt{n})$, *converge weakly to the product measure* $\prod_{i=1}^r \mathbf{K}_\beta$ *given by*

$$\mathbf{K}_\beta[F] = \frac{1}{Z(\beta)} \sum_{m=0}^{\infty} \mathbf{E}_0 \left[\psi_m(q(0))\Omega_0(q(0))^{-1} F(\gamma_r q(\cdot))\psi_m(q(\beta))\Omega_0(q(\beta))^{-1}\right], \tag{2.1}$$

where $Z(\beta) = (1 - e^{-\beta})^{-1}$, *and where* $\mathbf{E}_0$ *is the expectation with respect to the Ornstein-Uhlenbeck process on* $\mathcal{C}[0, \beta]$ *starting at 0, and* Ω_0 *and* ψ_m *are the ground state and excited states of the harmonic oscillator* $H_0 = a^* a$ *in the position representation.*

Proof. We first prove that the corresponding characteristic functions converge. The largest eigenvalue λ_0 of $U(\beta)$ tends to 1 as $n \to \infty$. We show that the order-$1/\sqrt{n}$ term is zero. Notice first that to this order only the term $\int_0^\beta A(t)dt$ of the expansional is significant. Thus if we put $\lambda_0 = 1 + \frac{1}{\sqrt{n}}\alpha_1 + \mathcal{O}(n^{-1})$ and set the eigenvector equal to $v_0 + \frac{1}{\sqrt{n}} w$ where $w \perp v_0$ then

$$(1 + \frac{1}{\sqrt{n}} \int_0^\beta A(t)dt)(v_0 + e^{-\beta}\frac{1}{\sqrt{n}} w) = (1 + \frac{1}{\sqrt{n}}\alpha_1 + \mathcal{O}(n^{-1}))(v_0 + \frac{1}{\sqrt{n}} w).$$

Taking the scalar product with v_0 we find

$$\alpha_1 = \langle v_0 \,|\, \int_0^\beta A(t)dt\, v_0\rangle = 0$$

since $(v_0)_i = \delta_{0,i}$ is the eigenvector of $V(\beta)$ with eigenvalue 1. Denoting similarly $(v_m)_i = \delta_{m,i}$, the scalar product with v_m yields

$$\frac{1}{\sqrt{n}} \left(e^{-\beta} w_m + \int_0^\beta A_{m,0}(t)dt \right) + \mathcal{O}(n^{-1}) = \frac{1}{\sqrt{n}} w_m.$$

Thus the eigenvector corresponding to λ_0 is given by

$$v_0 + \frac{1}{\sqrt{n}} \frac{1}{1 - e^{-\beta}} \sum_{m=1}^{r} \int_0^\beta A_{m,0}(t)dt\, v_m + \mathcal{O}(n^{-1}). \tag{2.2}$$

We can do a similar calculation for the next term of λ_0. We now put $\lambda_0 = 1 + \alpha_2 n^{-1} + \mathcal{O}(n^{-3/2})$ and insert the just computed eigenvector:

$$\left(1 + \frac{1}{\sqrt{n}}\int_0^\beta A(t)dt + \frac{1}{n}\int_0^\beta dt_1 \int_0^{t_1} dt_2 A(t_2)A(t_1)\right)$$
$$\times \left(v_0 + \frac{1}{\sqrt{n}}\frac{e^{-\beta}}{1-e^{-\beta}}\sum_{m=1}^r \int_0^\beta A_{m,0}(t)dt\, v_m + \mathcal{O}(n^{-1})\right)$$
$$= (1 + \frac{1}{n}\alpha_2 + \mathcal{O}(n^{-3/2}))\left(v_0 + \frac{1}{\sqrt{n}}\frac{1}{1-e^{-\beta}}\sum_{m=1}^r \int_0^\beta A_{m,0}(t)dt\, v_m + \mathcal{O}(n^{-1})\right).$$

Taking again the scalar product with v_0 we obtain

$$\begin{aligned}\alpha_2 &= \frac{e^{-\beta}}{1-e^{-\beta}}\sum_{m=1}^r \int_0^\beta A_{0,m}(t)dt \int_0^\beta A_{m,0}(t)dt \\ &\quad + \sum_{m=1}^r \int_0^\beta dt_1 \int_0^{t_1} dt_2 A_{0,m}(t_2)A_{m,0}(t_1) \\ &= -\sum_{m=1}^r \frac{\gamma_r^2}{1-e^{-\beta}}\int_0^\beta dt_1 \int_0^{t_1} dt_2 u_m(t_1)u_m(t_2)(e^{-t_1+t_2} + e^{-\beta+t_1-t_2}).\end{aligned} \tag{2.3}$$

The other eigenvalues are all of the form $\lambda_k = e^{-\beta} + \mathcal{O}(n^{-1/2})$.

It follows that $\lim_{n\to\infty}\lambda_0^n = \lim_{n\to\infty}\left(1+\alpha_2 n^{-1} + \mathcal{O}(n^{-3/2})\right)^n = \exp(\alpha_2)$ and hence

$$\begin{aligned}&\lim_{n\to\infty}\int \exp\left[i\sum_{k=1}^r \int_0^\beta u_k(t)z_k(t)dt\right]\tilde{\mathbf{K}}_\beta^{(n)}(dz_1(\cdot),\ldots,dz_r(\cdot)) = \\ &\qquad = \lim_{n\to\infty}\frac{1}{Z_n(\beta)}\sum_{n_0,\ldots,n_r:\sum n_k = n}\lambda_0^{n_0}\lambda_1^{n_1}\ldots\lambda_r^{n_r} \\ &\qquad = (1-e^{-\beta})^r \lim_{n\to\infty}\lambda_0^n \prod_{k=1}^r\sum_{n_k=0}^\infty\left(\frac{\lambda_k}{\lambda_0}\right)^{n_k} \\ &\qquad = (1-e^{-\beta})^r e^{\alpha_2}\lim_{n\to\infty}\prod_{k=1}^r\sum_{n_k=0}^\infty (e^{-\beta}+\mathcal{O}(n^{-1/2}))^{n_k} = e^{\alpha_2}.\end{aligned} \tag{2.4}$$

This is just the characteristic function of the product measure $\prod_{i=1}^r \mathbf{K}_\beta$ as was proved in [2] (with a different normalization).

In order to complete the proof we need, as in [2], to find a compact set which contains all the measure up to ϵ uniformly in n (Prokhorov's theorem). This can be done in a similar way as in [2], but the technical details are more complicated. We need norm bounds on all terms of the series (1.20). As in [2] we will assume that $u \in \pi^{(m)}(L^2([0,\beta],\mathbf{R}^r))$ where $\pi^{(m)}$is the projection onto the space spanned by the scaled Haar functions h_q with $2^m \le q \le 2^{m+1}-1$ defined by

$$h_q(t) = \begin{cases} \beta^{-1/2}2^{-m/2} & \text{if } (2^{-m}q-1)\beta \le t < (2^{-m}(q+\frac{1}{2})-1)\beta, \\ -\beta^{-1/2}2^{-m/2} & \text{if } (2^{-m}(q+\frac{1}{2})-1)\beta \le t < (2^{-m}(q+1)-1)\beta, \\ 0 & \text{otherwise.} \end{cases} \tag{2.5}$$

We then have

$$u_k(t) = \sum_{q=2^m}^{2^{m+1}-1} u_{k,q} h_q(t). \tag{2.6}$$

We also write

$$A_{k'',k'}(t) = i\sum_{k=1}^{r} u_k(t) a_{k'',k,k'}(t), \tag{2.7}$$

where

$$a_{k'',k,k'}(t) = \begin{cases} \tilde{a}_{k'',k,k'} & \text{if } k', k'' \neq 0, \\ \tilde{a}_{k'',k,k'}e^{-t} & \text{if } k'' \neq 0,\ k'=0, \\ \tilde{a}_{k'',k,k'}e^{t} & \text{if } k' \neq 0,\ k''=0, \\ \tilde{a}_{k'',k,k'} \equiv 0 & \text{if } k'=k''=0. \end{cases} \tag{2.8}$$

Here

$$\tilde{a}_{k'',k,k'} = \sum_{x=0}^{r} O_{k'',x}O_{k,x}O_{k',x}. \tag{2.9}$$

In particular, if $k'=0$ or $k''=0$,

$$\tilde{a}_{0,k,k'} = \gamma_r \delta_{k,k'} \text{ and } \tilde{a}_{k'',k,0} = \gamma_r \delta_{k'',k}. \tag{2.10}$$

Inserting these expressions into the p-th term of (1.20) we have

$$\begin{aligned} &\left[\int_0^\beta dt_1 \ldots \int_0^{t_{p-1}} dt_p A(t_p)\ldots A(t_1)\right]_{k'',k'} = \\ &= i^p \sum_{k_1,\ldots,k_p=1}^{r} \sum_{q_1,\ldots,q_p=2^m}^{2^{m+1}-1} u_{k_1,q_1}\ldots u_{k_p,q_p} \\ &\times \sum_{k'_1,\ldots,k'_{p-1}=0}^{r} \int_0^\beta dt_1 \ldots \int_0^{t_{p-1}} dt_p\, h_{q_1}(t_1)\ldots h_{q_p}(t_p) \\ &\times a_{k'_p,k_p,k'_{p-1}}(t_p)\, a_{k'_{p-1},k_{p-1},k'_{p-2}}(t_{p-1})\ldots a_{k'_1,k_1,k'_0}(t_1), \end{aligned} \tag{2.11}$$

where $k'_p = k''$ and $k'_0 = k'$. We now remark that since $t_i > t_{i+1}$ for all i in this formula, and $h_{q_i}(t_i) = 0$ if $t_i \notin [q_i 2^{-m}-1, (q_i+1)2^{-m}-1]$, we must have $q_{i+1} \le q_i$.

Moreover, if $q_i > q_{i+1}$ then the integral $\int_0^{t_i} dt_{i+1}$ can be extended to $\int_0^\beta dt_{i+1}$ and the multiple integral splits into a product. Thus

$$\begin{aligned}
&\left[\int_0^\beta dt_1 \dots \int_0^{t_{p-1}} dt_p A(t_p)\dots A(t_1)\right]_{k'',k'} = \\
&= i^p \sum_{l=1}^{p\wedge 2^m} \sum_{p_1,\dots,p_l\geq 1:\sum p_i=p} \sum_{2^m\leq q_l<\dots<q_1\leq 2^{m+1}-1} \\
&\quad\times \sum_{k_1,\dots,k_p=1}^{r} \prod_{j=1}^{l} \left(\prod_{i=n_{j-1}+1}^{n_j} u_{k_i,q_j}\right) \\
&\quad\times \sum_{k'_1,\dots,k'_{p-1}=0}^{r} \prod_{j=1}^{l} I^{(p_j,q_j)}_{(k_{n_{j-1}+1},\dots,k_{n_j});(k'_{n_{j-1}},\dots,k'_{n_j})},
\end{aligned} \tag{2.12}$$

where $n_j = \sum_{i=1}^j p_i$ and

$$\begin{aligned}
&I^{(p_j,q_j)}_{(k_{n_{j-1}+1},\dots,k_{n_j});(k'_{n_{j-1}},\dots,k'_{n_j})} \\
&\quad= \int_0^\beta dt_1 \dots \int_0^{t_{p_j-1}} dt_{p_j}\, h_{q_j}(t_1)\dots h_{q_j}(t_{p_j}) \\
&\qquad\times a_{k'_{n_j},k_{n_j},k'_{n_j-1}}(t_{p_j})\dots a_{k'_{n_{j-1}+1},k_{n_{j-1}+1},k'_{n_{j-1}}}(t_1).
\end{aligned} \tag{2.13}$$

In operator form we can write

$$\begin{aligned}
&\int_0^\beta dt_1 \dots \int_0^{t_{p-1}} dt_p A(t_p)\dots A(t_1) = \\
&= i^p \sum_{l=1}^{p\wedge 2^m} \sum_{p_1,\dots,p_l\geq 1:\sum p_i=p} \sum_{2^m\leq q_l<\dots<q_1\leq 2^{m+1}-1} \\
&\quad\times \sum_{k_1,\dots,k_p=1}^{r} \prod_{j=1}^{l} \left(J^{(p_j,q_j)}_{k_{n_{j-1}+1},\dots,k_{n_j}} \prod_{i=n_{j-1}+1}^{n_j} u_{k_i,q_j}\right),
\end{aligned} \tag{2.14}$$

where $J^{(p_j,q_j)}_{k_{n_{j-1}+1},\dots,k_{n_j}}$ is a matrix with entries

$$\begin{aligned}
&\left(J^{(p_j,q_j)}_{k_{n_{j-1}+1},\dots,k_{n_j}}\right)_{k'',k'} = \\
&= \sum_{k'_1,\dots,k'_{p_j-1}=0}^{r} \int_0^\beta dt_1 \dots \int_0^{t_{p_j-1}} dt_{p_j}\, h_{q_j}(t_1)\dots h_{q_j}(t_{p_j}) \\
&\quad\times a_{k'',k_{n_j},k'_{p_j-1}}(t_{p_j}) a_{k'_{p_j-1},k_{n_j-1},k'_{p_j-2}}(t_{p_j-1})\dots a_{k'_1,k_{n_{j-1}+1},k'}(t_1).
\end{aligned} \tag{2.15}$$

We now construct a bound on the Hilbert-Schmidt norm of J. For simplicity we write p for p_j; q for q_j and $k_1, \ldots, k_p$ for $k_{n_{j-1}+1}, \ldots, k_{n_j}$ in the following. We distinguish the cases $k_i' = 0$ and $k_i' \neq 0$. Notice that, in formula (2.15), if $k_i' = 0$ then $k_{i+1}' \neq 0$ and $k_{i-1}' \neq 0$ because of (2.8) (last case). Writing $k'' = k_p'$ and $k' = k_0'$ we therefore have a maximum of $\left[\frac{p+1}{2}\right]$ possible i with $k_i' = 0$. Moreover, $\#\{i = 0, \ldots, p : k_i' = 0\}$ cannot be zero. Let

$$\nu = \#\{i = 1, \ldots, p-1 : k_i' = 0\}.$$

If $k_i' = 0$ then $k_{i+1}' = k_{i+1}$ and $k_{i-1}' = k_i$. The remaining number of variables to be summed over is at least $p - 1 - 3\nu$ and at most $p - 1 - 2\nu$. Next we estimate

$$\left| \int_0^\beta dt_1 \ldots \int_0^{t_{p-1}} dt_p \, h_q(t_1) \ldots h_q(t_p) e^{\pm(t_{i_1} - t_{i_2} + \ldots \pm t_{i_\nu})} \right|.$$

For ν even,

$$\begin{aligned} &\left| \int_0^\beta dt_1 \ldots \int_0^{t_{p-1}} dt_p \, h_q(t_1) \ldots h_q(t_p) e^{t_{i_1} - t_{i_2} + \ldots - t_{i_\nu}} \right| \\ &\quad \leq \frac{1}{p!} \beta_m^{p/2} (e^{\beta_m} - 1) \leq \frac{1}{p!} \beta_m^{p/2+1} e^{\beta_m} \end{aligned} \tag{2.16}$$

and

$$\begin{aligned} &\left| \int_0^\beta dt_1 \ldots \int_0^{t_{p-1}} dt_p \, h_q(t_1) \ldots h_q(t_p) e^{-\beta(t_{i_1} - t_{i_2} + \ldots - t_{i_\nu})} \right| \\ &\quad \leq \frac{1}{p!} \beta_m^{p/2} (1 - e^{-\beta_m}) \leq \frac{1}{p!} \beta_m^{p/2+1}, \end{aligned} \tag{2.17}$$

where $\beta_m = \beta 2^{-m}$. For ν odd,

$$\begin{aligned} &\left| \int_0^1 dt_1 \ldots \int_0^{t_{p-1}} dt_p \, h_q(t_1) \ldots h_q(t_p) e^{\beta(t_{i_1} - t_{i_2} + \ldots + t_{i_\nu})} \right| \\ &\quad \leq \frac{1}{p!} \beta_m^{p/2} \left(e^{\beta(2^{-m}(q+1) - 1)} - e^{\beta(2^{-m} q - 1)} \right) \\ &\quad \leq \frac{1}{p!} \beta_m^{1+p/2} e^{\beta(2^{-m}(q+1) - 1)}. \end{aligned} \tag{2.18}$$

and

$$\begin{aligned} &\left| \int_0^\beta dt_1 \ldots \int_0^{t_{p-1}} dt_p \, h_q(t_1) \ldots h_q(t_p) e^{-\beta(t_{i_1} - t_{i_2} + \ldots + t_{i_\nu})} \right| \\ &\quad \leq \frac{1}{p!} \beta_m^{p/2} \left(e^{-\beta(2^{-m} q - 1)} - e^{-\beta(2^{-m}(q+1) - 1)} \right) \\ &\quad \leq \frac{1}{p!} \beta_m^{1+p/2} e^{-\beta(2^{-m} q - 1)}. \end{aligned} \tag{2.19}$$

Now, if $k'_i = 0$ then by (2.8) and (2.10) the corresponding factors $a_{k'_i,k_i,k'_{i-1}}(t_i)$ and $a_{k'_{i+1},k_{i+1},k'_i}(t_{i+1})$ yield a factor $\gamma_r^2 e^{t_i - t_{i+1}}$. In the case $k'_0 \neq 0$ and $k'_p \neq 0$ the number ν of variables $k'_i = 0$ is at most $[p/2]$. Using (2.9) and summing over the remaining free variables k'_i we therefore get

$$
\begin{aligned}
\left| \left(J^{(p,q)}_{k_1,\dots,k_p} \right)_{k'_p,k'_0} \right| \leq \sum_{\nu=1}^{[p/2]} \gamma_r^{2\nu} & \sum_{1 \leq i_1 < \ldots < i_\nu \leq p-1} \\
& \Bigg| \sum_{k'_1,\dots,k'_{i_1-2}} \sum_{k'_{i_1+2},\dots,k'_{i_2-2}} \cdots \sum_{k'_{i_\nu+2},\dots,k'_{p-1}} \\
& \quad \times \tilde{a}_{k'_1,k_1,k'_0} \ldots \tilde{a}_{k'_{i_1-2},k_{i_1-2},k'_{i_1-3}} \tilde{a}_{k_{i_1},k_{i_1-1},k'_{i_1-2}} \\
& \quad \times \tilde{a}_{k'_{i_1+2},k_{i_1+2},k_{i_1+1}} \ldots \tilde{a}_{k'_{i_2-2},k_{i_2-2},k'_{i_2-3}} \tilde{a}_{k_{i_2},k_{i_2-1},k'_{i_2-2}} \\
& \qquad . \\
& \qquad . \\
& \quad \times \tilde{a}_{k'_{i_\nu+2},k_{i_\nu+2},k_{i_\nu+1}} \ldots \tilde{a}_{k'_{p-1},k_{p-1},k'_{p-2}} \tilde{a}_{k'_p,k_p,k'_{p-1}} \Bigg| \\
& \times \frac{1}{p!} \beta_m^{1+p/2} e^{\beta_m} \\
& = \frac{1}{p!} \beta_m^{1+p/2} e^{\beta_m} \sum_{\nu=1}^{[p/2]} \gamma_r^{2\nu} \sum_{1 \leq i_1 < \ldots < i_\nu \leq p-1} \\
& \quad \times \left| \sum_{x_0=0}^{r} O_{k'_0,x_0} O_{k_1,x_0} \ldots O_{k_{i_1},x_0} \right| \left| \sum_{x_1=0}^{r} O_{k_{i_1+1},x_1} \ldots O_{k_{i_2},x_1} \right| \\
& \quad \times \ldots \times \left| \sum_{x_\nu=0}^{r} O_{k_{i_\nu+1},x_\nu} \ldots O_{k_p,x_\nu} O_{k'_p,x_\nu} \right| .
\end{aligned}
\tag{2.20}
$$

Squaring this and summing over the variables $k'_0, k_1, \dots, k_p, k'_p$ we obtain sums of the form

$$
\begin{aligned}
& \sum_{k'_0,k_1,\dots,k_{i_1}=1}^{r} \left| \sum_{x_0=0}^{r} O_{k'_0,x_0} O_{k_1,x_0} \ldots O_{k_{i_1},x_0} \right|^2 \\
& \leq \sum_{x_0,x'_0=0}^{r} \sum_{k'_0,k_1,\dots,k_{i_1}=0}^{r} O_{k'_0,x_0} O_{k'_0,x'_0} \ldots O_{k_{i_1},x_0} O_{k_{i_1},x'_0} \\
& = \sum_{x_0,x'_0=0}^{r} \delta_{x_0,x'_0} = r+1 = \gamma_r^{-2}.
\end{aligned}
$$

Moreover, if for example $i_1 < i'_1$ then the sums overlap and $x_0 = x'_0 = x_1$. We can therefore first sum over the indices which occur in both factors. The remaining indices i and i' can be chosen from the spaces in between. This latter sum can be bounded by $3^{p-1-\nu}$.

It now follows that

$$
\begin{aligned}
&\sum_{k'_0,k_1,\dots,k_p,k'_p=1}^{r} \left| \left(J^{(p,q)}_{k_1,\dots,k_p} \right)_{k'_p,k'_0} \right|^2 \\
&\qquad \leq \frac{1}{(p!)^2} \beta_m^{2+p} e^{2\beta_m} \sum_{\nu=1}^{[p/2]} \gamma_r^{4\nu} \sum_{1 \leq i_1 < \dots < i_\nu \leq p-1} 3^{p-1-\nu} \gamma_r^{-2(\nu+1)} \\
&\qquad \leq \frac{1}{(p!)^2} \beta_m^{2+p} e^{2\beta_m} \gamma_r^{-2} 3^{p-1} \sum_{\nu=1}^{[p/2]} \binom{p-\nu}{\nu} \left(\frac{\gamma_r^2}{3} \right)^{\nu} \\
&\qquad \leq \frac{1}{(p!)^2} \beta_m^{2+p} e^{2\beta_m} \gamma_r^{-2} 3^{p-1} \left(\left(1 + \frac{\gamma_r^2}{3} \right)^{p-1} - 1 \right) \\
&\qquad \leq \frac{1}{(p!)^2} \beta_m^{2+p} e^{2\beta_m} 3^{p-2} p \left(1 + \frac{\gamma_r}{3} \right)^{p-2} \\
&\qquad \leq \frac{1}{(p!)^2} \beta_m^{2+p} e^{2\beta_m} p\, 4^{p-2}.
\end{aligned}
\tag{2.21}
$$

Similarly, if $k'_p = 0$ but $k'_0 \neq 0$,

$$\left| \left(J^{(p,q)}_{k_1,\ldots,k_p} \right)_{0,k'_0} \right| \leq \sum_{\nu=1}^{[(p-1)/2]} \gamma_r^{2\nu+1} \sum_{1 \leq i_1 < \ldots < i_\nu \leq p-2}$$

$$\times \left| \sum_{k'_1,\ldots,k'_{i_1-2}} \sum_{k'_{i_1+2},\ldots,k'_{i_2-2}} \cdots \sum_{k_{i_\nu+2},\ldots,k'_{p-2}} \right.$$

$$\times \tilde{a}_{k_p,k_{p-1},k'_{p-2}} \ldots \tilde{a}_{k'_{i_\nu+2},k_{i_\nu+2},k_{i_\nu+1}}$$

$$\vdots$$

$$\left. \times \tilde{a}_{k_{i_1},k_{i_1-1},k'_{i_1-2}} \ldots \tilde{a}_{k'_1,k_1,k'_0} \right| \times \frac{1}{p!} \beta_m^{1+p/2} e^{\beta(2^{-m}(q+1)-1)}$$

$$= \frac{1}{p!} \beta_m^{1+p/2} e^{\beta(2^{-m}(q+1)-1)} \sum_{\nu=1}^{[(p-1)/2]} \gamma_r^{2\nu+1} \sum_{1 \leq i_1 < \ldots < i_\nu \leq p-2}$$

$$\times \left| \sum_{x_0=0}^{r} O_{k'_0,x_0} O_{k_1,x_0} \ldots O_{k_{i_1},x_0} \right| \left| \sum_{x_1=0}^{r} O_{k_{i_1+1},x_1} \ldots O_{k_{i_2},x_1} \right|$$

$$\times \ldots \times \left| \sum_{x_{\nu-1}=0}^{r} O_{k_{i_{\nu-1}+1},x_{\nu-1}} \ldots O_{k_{i_\nu},x_{\nu-1}} \right| \left| \sum_{x_\nu=0}^{r} O_{k_{i_\nu+1},x_\nu} \ldots O_{k_p,x_\nu} \right|. \tag{2.22}$$

Hence

$$\sum_{k'_0,k_1,\ldots,k_p=1}^{r} \left| \left(J^{(p,q)}_{k_1,\ldots,k_p} \right)_{0,k'_0} \right|^2 \leq$$

$$\frac{1}{(p!)^2} \beta_m^{2+p} e^{2\beta(2^{-m}(q+1)-1)}$$

$$\sum_{\mu=0}^{[(p-1)/2]} \gamma_r^{4\mu+2} \sum_{1 \leq i_1 < \ldots < i_\mu \leq p-2} 3^{p-2-\mu} \gamma_r^{-2(\mu+1)} \tag{2.23}$$

$$\leq \frac{1}{(p!)^2} \beta_m^{2+p} e^{2\beta(2^{-m}(q+1)-1)} 3^{p-2} \sum_{\mu=0}^{[(p-1)/2]} \binom{p-2}{\mu} \left(\frac{\gamma_r^2}{3} \right)^{\mu}$$

$$\leq \frac{1}{(p!)^2} \beta_m^{2+p} e^{2\beta(2^{-m}(q+1)-1)} 3^{p-2} \left(1 + \frac{\gamma_r^2}{3} \right)^{p-2}$$

$$\leq \frac{1}{(p!)^2} \beta_m^{2+p} p \, 4^{p-2} e^{2\beta(2^{-m}(q+1)-1)}.$$

In the same way we find

$$\sum_{k',k_1,\dots,k_p=1}^{r}\left|\left(J^{(p,q)}_{k_1,\dots,k_p}\right)_{k',0}\right|^2 \le \frac{1}{(p!)^2}\beta_m^{2+p}\,p\,4^{p-2}e^{-2\beta(2^{-m}q-1)} \tag{2.24}$$

and

$$\sum_{k_1,\dots,k_p=1}^{r}\left|\left(J^{(p,q)}_{k_1,\dots,k_p}\right)_{0,0}\right|^2 \le \frac{1}{(p!)^2}\beta_m^{2+p}\,p\,4^{p-2}. \tag{2.25}$$

We next insert these bounds into (2.14). We first notice that after multiplying by $V(\beta)$ on the right-hand side, all exponential factors can be bounded by 1. For example, consider the case $l=4$ with $k'_0\ne 0$, $k'_{n_1}=0$, $k'_{n_2}\ne 0$, $k'_{n_3}\ne 0$ and $k'_{n_4}=k'_p=0$. In that case the exponential factors are:

$$e^{\beta(2^{-m}(q_4+1)-1)}e^{\beta_m}e^{-\beta(2^{-m}q_2-1)}e^{\beta(2^{-m}(q_1+1)-1)}e^{-\beta},$$

the last factor being due to $V(\beta)$. The product of the first three factors is ≤ 1 because $q_4+2\le q_2$. The entire product is thus ≤ 1 since $2^{-m}(q_1+1)-1\le 1$.

We now have

$$\begin{aligned}
&\left|\left|\int_0^\beta dt_1\dots\int_0^{t_{p-1}}dt_p\,A(t_p)\dots A(t_1)V(\beta)\right|\right|\\
&\le \sum_{l=1}^{p\wedge 2^m}\ \sum_{p_1,\dots,p_l\ge 1:\sum_{i=1}^l p_i=p}\ \sum_{2^m\le q_l<\dots<q_1\le 2^{m+1}-1}\\
&\quad\times\prod_{i=1}^{l}\left|\left|\sum_{k_1,\dots,k_{p_i}=1}^{r}\tilde J^{(p_i,q_i)}_{(k_1,\dots,k_{p_i})}\prod_{j=1}^{p_i}u_{k_j,q_i}\right|\right|_{HS}\\
&\le \sum_{l=1}^{p\wedge 2^m}\ \sum_{p_1,\dots,p_l\ge 1:\sum_{i=1}^l p_i=p}\ \sum_{2^m\le q_l<\dots<q_1\le 2^{m+1}-1}\\
&\prod_{i=1}^{l}\left(\sum_{k_1,\dots,k_{p_i}=1}^{r}||J^{(p_i,q_i)}_{(k_1,\dots,k_{p_i})}||^2_{HS}\right)^{1/2}\left(\sum_{k_1,\dots,k_{p_i}=1}^{r}\prod_{j=1}^{p_i}|u_{k_j,q_i}|^2\right)^{1/2}
\end{aligned}$$

and inserting the bounds:

$$
\begin{aligned}
&\left|\left| \int_0^\beta dt_1 \ldots \int_0^{t_{p-1}} dt_p\, A(t_p)\ldots A(t_1)V(\beta)\right|\right| \\
&\quad \leq \sum_{l=1}^{p\wedge 2^m} \sum_{p_1,\ldots,p_l\geq 1:\, \sum_{i=1}^l p_i=p}\ \sum_{2^m\leq q_l<\ldots<q_1\leq 2^{m+1}-1} \\
&\qquad \times \prod_{i=1}^{l}\left(\frac{1}{p_i!}\beta_m^{1+p_i/2}\sqrt{p_i}2^{p_i-2}\left(\sum_{k=1}^r u_{k,q_i}^2\right)^{p_i/2}\right) \\
&\quad \leq \sum_{l=1}^{p}\frac{1}{l!}\beta_m^{l+p/2}2^{p-2l}2^{ml/2} \\
&\qquad \times \sum_{p_1,\ldots,p_l\geq 1:\, \sum_{i=1}^l p_i=p}\ \prod_{i=1}^{l}\frac{1}{(p_i-1)!}\left(\sum_{q=2^m}^{2^{m+1}-1}\sum_{k=1}^{r}u_{k,q}^2\right)^{p/2} \\
&\quad \leq \beta_m^{p/2}4^p\left(\sum_{l=1}^{p}\frac{2^{-3l+ml/2}\beta_m^l}{l!(p-l)!}\right)||\vec{u}||^p \\
&\quad \leq \beta_m^{p/2}4^p\frac{1}{p!}\left[\left(1+\beta_m 2^{\frac{m}{2}-3}\right)^p-1\right]||\vec{u}||^p.
\end{aligned}
\tag{2.26}
$$

Finally, summing over p and putting in the factors $1/\sqrt{n}$, we obtain

$$
\begin{aligned}
&||U(\beta)-V(\beta)|| \\
&\quad \leq \sum_{p=1}^{\infty}\frac{1}{n^{p/2}}\beta_m^{p/2}4^p\frac{1}{p!}||\vec{u}||^p\left[\left(1+\beta_m 2^{\frac{m}{2}-3}\right)^p-1\right] \\
&\quad = \exp\left\{\frac{4}{\sqrt{n}}\beta_m^{1/2}\left(1+\beta_m 2^{m/2-3}\right)||\vec{u}||\right\}-\exp\left\{\frac{4}{\sqrt{n}}\beta_m^{1/2}||\vec{u}||\right\} \\
&\quad \leq \frac{1}{2\sqrt{n}}\beta_m^{3/2}2^{m/2}||\vec{u}||\exp\left\{\frac{4}{\sqrt{n}}\beta_m^{1/2}\left(1+\beta_m 2^{m/2-3}\right)||\vec{u}||\right\}.
\end{aligned}
\tag{2.27}
$$

This shows that $||U(\beta)-V(\beta)||=\mathcal{O}(2^{-m}n^{-1/2})$ which suffices for the eigenvalues λ_i ($i\neq 0$), but not for λ_0. However, it is not difficult to improve on (2.27) by splitting off the $1/\sqrt{n}$ term:

$$
\begin{aligned}
&||U(\beta) - \left(1 + \frac{1}{\sqrt{n}} \int_0^\beta A(t)dt\right) V(\beta)|| \\
&\quad \le \sum_{p=2}^{\infty} \frac{1}{n^{p/2}} \beta_m^{p/2} 4^p \frac{1}{p!} ||\vec{u}||^p \left[\left(1 + \beta_m 2^{\frac{m}{2}-3}\right)^p - 1 \right] \\
&\quad = \exp\left\{ \frac{4}{\sqrt{n}} \beta_m^{1/2} \left(1 + \beta_m 2^{m/2-3}\right) ||\vec{u}|| \right\} - \exp\left\{ \frac{4}{\sqrt{n}} \beta_m^{1/2} ||\vec{u}|| \right\} \qquad (2.28) \\
&\qquad - \frac{1}{2\sqrt{n}} \beta_m^{3/2} 2^{m/2} ||\vec{u}|| \\
&\quad \le \frac{1}{8n} \beta_m^3 2^m ||\vec{u}||^2 \exp\left\{ \frac{4}{\sqrt{n}} \beta_m^{1/2} \left(1 + \beta_m 2^{m/2-3}\right) ||\vec{u}|| \right\}.
\end{aligned}
$$

The eigenvalues of $U^{(1)}(\beta) = \left(1 + \frac{1}{\sqrt{n}} \int_0^\beta A(t)dt\right) V(\beta)$ are easily computed since $\int_0^\beta A_{k,k'}(t)dt = 0$ for $k, k' \neq 0$ if $u \in \pi^{(m)}(L^2)$ with $m \geq 0$, i.e. $(u, h_0) = 0$. The eigenvalues $\lambda_i^{(1)} = e^{-\beta}$ $(i = 2, \ldots, r)$ are unchanged but

$$
\lambda_{0,1}^{(1)} = \frac{1}{2} \left(1 + e^{-\beta} \pm \sqrt{(1 - e^{-\beta})^2 - \frac{4}{n} \gamma_r^2 \sum_{k=1}^{r} \int_0^\beta u_k(t) e^{-t} dt \int_0^\beta u_k(t) e^{-\beta + t} dt} \right), \qquad (2.29)
$$

so that $\lambda_0^{(1)} = 1 + \mathcal{O}(2^{-2m} n^{-1} ||u||^2)$ and $\lambda_1^{(1)} = e^{-\beta} + \mathcal{O}(2^{-2m} n^{-1} ||u||^2)$. Combining this with the bound (2.28) we have

$$
\lambda_0 = 1 + \mathcal{O}(2^{-2m} n^{-1} ||u||^2) \qquad (2.30)
$$

and

$$
\lambda_i = e^{-\beta} + \mathcal{O}(2^{-2m} n^{-1} ||u||^2) \qquad (2.31)
$$

for $i = 1, \ldots, r$. We can then proceed as in [2] and assume that $||u||^2 \leq \delta 2^{3m/2}$. In that case the exponential factor in (2.28) is indeed bounded and we have $\lambda_0 = 1 + \mathcal{O}(2^{-m/2} n^{-1})$ and $\lambda_i = e^{-\beta} + \mathcal{O}(2^{-m/2} n^{-1})$. This implies that the characteristic function of the measure projected on the space spanned by h_q with $2^m \leq q \leq 2^{m+1} - 1$ satisfies

$$
\psi_m^{(n)}[u] = \pi^{(m)} \left(\mathbf{K}_\beta^{(n)} \right) [\exp[i(\xi, u)]] = 1 + \mathcal{O}(\delta 2^{-m}). \qquad (2.32)
$$

To see this we write

$$
\begin{aligned}
|\psi_m^{(n)}[u]-1| &= \frac{\sum\limits_{n_0,\dots,n_r:\sum_{i=0}^r n_i=n} \left(\lambda_0^{n_0}\lambda_1^{n_1}\dots\lambda_r^{n_r} - e^{-\beta(n_1+\dots+n_r)}\right)}{\sum\limits_{n_0,\dots,n_r:\sum_{i=0}^r n_i=n} e^{-\beta(n_1+\dots+n_r)}} \\
&\le \max_{n_0,\dots,n_r:\sum_{i=0}^r n_i=n} \left| \frac{\lambda_0^{n_0}\lambda_1^{n_1}\dots\lambda_r^{n_r} - e^{-\beta(n_1+\dots+n_r)}}{e^{-\beta(n_1+\dots+n_r)}} \right| \\
&\le \max_{n_0,\dots,n_r:\sum_{i=0}^r n_i=n} \Bigg\{ |\lambda_0^{n_0}-1||c_1^{n_1}\dots c_r^{n_r}| \qquad (2.33)\\
&\qquad + \sum_{k=1}^r |c_k^{n_k}-1|\left|c_{k+1}^{n_{k+1}}\dots c_r^{n_r}\right| \Bigg\} \\
&\le \frac{1}{n}C\delta_m \max_{n_0,\dots,n_r:\sum_{i=0}^r n_i=n} \sum_{k=1}^r \left(1+\frac{1}{n}C\delta_m\right)^{n_k+\dots+n_r} \\
&\le \frac{r}{n}C\delta_m \exp[C\delta_m].
\end{aligned}
$$

where $\delta_m = \delta^2 2^{-m/2}$ and $c_k = \lambda_k e^{\beta}$, and we have used $|\lambda_0 - 1| \le \frac{1}{n}C\delta_m$ and $|c_k - 1| \le \frac{1}{n}C\delta_m$.

We also need an analysis of the space spanned by h_0. Thus, let $\vec{u}(t) = \vec{u}_0 h_0(t)$ where $h_0(t) = \beta^{-1/2}$ is the constant function with norm 1. We can estimate the norm of $A(t)$ in general as follows. We first compute $A(t)^*A(t)$ using (1.21):

$$
[A(t)^*A(t)]_{k,k'} = \begin{cases}
\gamma_r^2 e^{-2t}\sum_{k''=1}^r u_{k''}(t)^2 & \text{if } k=k'=0, \\
\gamma_r e^{-t}\sum_{x=0}^r O_{k',x}\left(\sum_{k''=1}^r O_{k'',x}u_{k''}(t)\right)^2 & \text{if } k=0, k'\neq 0, \\
\gamma_r e^{-t}\sum_{x=0}^r O_{k,x}\left(\sum_{k''=1}^r O_{k'',x}u_{k''}(t)\right)^2 & \text{if } k\neq 0, k'=0, \\
\sum_{x=0}^r O_{k,x}O_{k',x}\left(\sum_{k''=1}^r O_{k'',x}u_{k''}(t)\right)^2 + \gamma_r^2 u_k(t)u_{k'}(t) & \\
 & \text{if } k,k'\neq 0.
\end{cases} \tag{2.34}
$$

The trace then yields the Hilbert-Schmidt norm:

$$
\begin{aligned}
||A(t)||_{HS}^2 &= \gamma_r^2(e^{-2t}+1)\sum_{k=1}^r u_k(t)^2 + \sum_{x=0}^r\sum_{k=1}^r O_{k,x}^2\left(\sum_{k'=1}^r O_{k',x}u_{k'}(t)\right)^2 \\
&= (1+\gamma_r^2 e^{-2t})\sum_{k=1}^r u_k(t)^2 \le (1+\gamma_r^2)||\vec{u}(t)||^2.
\end{aligned} \tag{2.35}
$$

Inserting this into the expansional we obtain

$$||U(\beta) - U^{(1)}(\beta)|| \le \frac{1}{n}(1+\gamma_r^2)||u||^2 \exp\left((1+\gamma_r^2)^{1/2}\frac{||u||}{\sqrt{n}}\right). \tag{2.36}$$

If $(u, h_0) \neq 0$ we cannot compute the eigenvalues of $U^{(1)}(\beta)$ explicitly, however. To first order in $1/\sqrt{n}$ we have $\lambda_0^{(1)} = 1 + \mathcal{O}(n^{-1})$ (see the beginning of the proof), whereas $\lambda_k^{(1)}$ with $k \ge 1$ are given by the eigenvalues of the matrix

$$U_{k,k'}^{(1)} = e^{-\beta}\left(\delta_{k,k'} + i\sqrt{\frac{\beta}{n}}\sum_{k''=1}^{r} u_{0,k''}\sum_{x=0}^{r} O_{k,x}O_{k',x}O_{k'',x}\right) \qquad (k, k' = 1, \dots, r). \tag{2.37}$$

Clearly, the order-$1/\sqrt{n}$ term is not zero. However, notice that $\mathrm{Trace}\, U^{(1)}(\beta) = re^{-\beta}$, that is, the sum of the order-$1/\sqrt{n}$ terms is zero. In any case, we can use bounds on the perturbation expansion (see Kato [7], Chapter II) to estimate the higher-order terms. We use the representation (2.29) of [7,Chapter II]:

$$\lambda_0^{(1)} - 1 = \frac{1}{2\pi i}\mathrm{Trace}\int_\Gamma \sum_{p=1}^{\infty}\frac{1}{pn^{p/2}}\left(-\int_0^\beta A(t)dt\, V(\beta)R_\beta(\zeta)\right)^p d\zeta. \tag{2.38}$$

Here $R_\beta(\zeta) = (V(\beta)-\zeta)^{-1}$ is the unperturbed resolvent, and Γ is a contour around $\zeta = 1$. In our case the $p = 1$ term vanishes because the order-$1/\sqrt{n}$ term of $\lambda_0^{(1)}$ is zero. For the higher terms we use the following bound, which follows from (2.35),

$$||\int_0^\beta A(t)dtV(\beta)R_\beta(\zeta)|| \le \sqrt{\beta}||\vec{u}_0||\rho^{-1}, \tag{2.39}$$

where ρ is the radius of the contour Γ, which can be taken as $(1-e^{-\beta})/2$. Together with the fact that for any operator, $|\mathrm{Trace}\, B^p| \le ||B||_{HS}^p$ we see that, if $||\vec{u}_0||^2 \le \delta$ then $\lambda_0 = 1 + \mathcal{O}(\delta n^{-1})$. The argument (2.33) now has to be modified since $\lambda_k - e^{-\beta} = \mathcal{O}(1/\sqrt{n})$. Instead we can write

$$
\begin{aligned}
|\psi_m^{(n)}[u]-1| &= \left| \frac{\sum_{n_0,\ldots,n_r:\,\sum_{i=0}^r n_i=n} \left(\lambda_0^{n_0}\lambda_1^{n_1}\ldots\lambda_r^{n_r} - e^{-\beta(n_1+\ldots+n_r)}\right)}{\sum_{n_0,\ldots,n_r:\,\sum_{i=0}^r n_i=n} e^{-\beta(n_1+\ldots+n_r)}} \right| \\
&\le \max_{n_0\le n} |\lambda_0^{n_0}-1| + \left| \frac{\sum_{n_0,\ldots,n_r:\,\sum_{i=0}^r n_i=n} \lambda_0^{n_0}\left(\lambda_1^{n_1}\ldots\lambda_r^{n_r} - e^{-\beta(n_1+\ldots+n_r)}\right)}{\sum_{n_0,\ldots,n_r:\,\sum_{i=0}^r n_i=n} e^{-\beta(n_1+\ldots+n_r)}} \right| \\
&\le C\delta\exp[C\delta] + \frac{\sum_{k=0}^n |\lambda_0|^{n-k} e^{-\beta k}\left|\sum_{n_1,\ldots,n_r:\,\sum_{i=1}^r n_i=k}(c_1^{n_1}\ldots c_r^{n_r}-1)\right|}{\sum_{k=0}^n \binom{k+r-1}{k} e^{-\beta k}} \\
&\le \exp[C\delta]\left\{ C\delta + \frac{\sum_{k=0}^n \binom{k+r-1}{k} e^{-\beta k} C\sqrt{\frac{\delta}{n}}k\left(1+C\sqrt{\delta/n}\right)^k}{\sum_{k=0}^n \binom{k+r-1}{k} e^{-\beta k}} \right\} \\
&\le \exp[C\delta]\left\{ C\delta + \frac{\sum_{k=0}^\infty \binom{k+r-1}{k} e^{-\beta k} C\sqrt{\frac{\delta}{n}}k\left(1+C\sqrt{\delta/n}\right)^k}{\sum_{k=0}^\infty \binom{k+r-1}{k} e^{-\beta k}} \right\} \\
&\le \exp[C\delta]C\sqrt{\delta}\left\{1 - \frac{1}{\sqrt{n}}(1-e^{-\beta})^r \frac{d}{d\beta}(1-e^{-\beta+C\sqrt{\delta/n}})^{-r}\right\} \\
&\le \exp[C\delta]C\sqrt{\delta}\left\{1 + \frac{r}{\sqrt{n}(1-e^{-\beta})}\exp\left[-\beta+C\sqrt{\delta/n}\right]\right\} = \mathcal{O}(\sqrt{\delta}).
\end{aligned}
\tag{2.40}
$$

Although adequate for the proof of the main theorem, this bound is in fact not entirely satisfactory. Namely, whereas the bound (2.33) for $u \in \pi^{(m)}(L^2)$ is uniform in r, this is not the case for (2.40). This is due to the fact that $\lambda_i - e^{-\beta}$ is of order $1/\sqrt{n}$ instead of $1/n$. Nevertheless, it is possible to derive a more satisfactory bound which, although not uniform in r, is uniform in the density $\rho = n/r$, that is, the number of particles per degree of freedom. This will be done in an appendix. It was outlined in [9] how the noncommutative central limit theorem might be used for proving Bose-Einstein condensation in certain lattice models of a boson gas. In that case r plays the role of the volume and the uniformity in the volume is essential. The idea is that in the limit of large n the Bogoliubov approximation holds. (This was in fact proved by Ginibre in [10] by entirely different means and, it seems, in a less useful form.) This application will be worked out in a subsequent publication.

As in [2], we can now define a compact set K_δ by

$$K_\delta = \left\{ \xi \in L^2\left([0,\beta], \mathbf{R}^r\right) \mid \sum_{q=2^m}^{2^{m+1}-1} ||(\xi, h_p)||^2 \le \eta_m \,\forall m \ge 0 \text{ and } ||(\xi, h_0)|| \le M \right\}. \tag{2.41}$$

Then

$$\mathbf{K}_\beta^{(n)}[K_\delta^c] \le \mathbf{K}_\beta^{(n)}[||(\xi, h_0)|| > M] + \sum_{m=0}^{\infty} \mathbf{K}_\beta^{(n)}[\{\xi| \sum_{q=2^m}^{2^{m+1}-1} ||(\xi, h_p)||^2 > \eta_m\}]. \tag{2.42}$$

But, by (2.32) and Lemma 4.1 of [2],

$$\mathbf{K}_\beta^{(n)}[\xi|\, ||\pi^{(m)}(\xi)||_2 > \eta_m] \le \delta \mathcal{O}(2^{-m/2}) \tag{2.43}$$

if we take $\eta_m = \delta^{-1} 2^{-m/2}$, and similarly

$$\mathbf{K}_\beta^{(n)}[||(\xi, h_0)|| > M] \le \mathcal{O}(\sqrt{\delta}) \tag{2.44}$$

if $M = \delta^{-3/4}$. It follows that for δ small enough,

$$\mathbf{K}_\beta^{(n)}[K_\delta^c] < \epsilon. \tag{2.45}$$

This completes the proof of the main theorem. Q.E.D.

The generalised noncommutative central limit theorem follows from the main theorem in the same way as in [2] because of the fact that

$$\int_0^\beta f(\xi(t))dt$$

is continuous as a function of $\xi \in L^2\left([0,\beta], \mathbf{R}^r\right)$.

Appendix.

In this appendix we derive an improved bound on $\psi^{(n)}[\vec{u}]$ in the case that $\vec{u} \in L^2([0,\beta], \mathbf{R}^r)$ is a multiple of the constant function h_0: $u_i(t) = u_{0,i}\beta^{-1/2}$, with $\vec{u}_0 \in \mathbf{R}^r$. It depends on the following lemmas:

Lemma 1. *To order $1/\sqrt{n}$ the eigenvalues $\lambda_i^{(1)}$ $(i = 1, \ldots, r)$ satisfy*

$$\sum_{i=1}^r (\lambda_i^{(1)} - e^{-\beta}) = 0 \text{ and } \sum_{i=1}^r |\lambda_i^{(1)} - e^{-\beta}|^2 = \frac{\beta}{n}(1 - \gamma_r^2)^2 ||\tilde{u}||^2. \tag{A.1}$$

Proof. The first identity has already been proved. The second follows from the fact that to order $1/\sqrt{n}$, $\lambda_i^{(1)}$ $(i = 1, \ldots, r)$ are the eigenvalues of the matrix $U^{(1)}$ given by

$$U_{k,k'}^{(1)} = e^{-\beta} \left(\delta_{k,k'} + i\sqrt{\frac{\beta}{n}} \sum_{k''=1}^{r} \tilde{u}_{k''} \sum_{x=0}^{r} O_{k,x} O_{k',x} O_{k'',x} \right) \quad (k, k' = 1, \ldots, r). \tag{A.2}$$

Therefore

$$\begin{aligned} \sum_{i=1}^{r} |\lambda_i^{(1)} - e^{-\beta}|^2 &= \text{Trace}(U^{(1)} - e^{-\beta})^* (U^{(1)} - e^{-\beta}) \\ &= \frac{\beta}{n} \sum_{k_1,k_2=1}^{r} \sum_{k,k'=1}^{r} \sum_{x_1,x_2=0}^{r} \tilde{u}_{k_1} \tilde{u}_{k_2} O_{k,x_1} O_{k,x_2} O_{k',x_1} O_{k',x_2} O_{k_1,x_1} O_{k_2,x_2} \\ &= \frac{\beta}{n} (1 - \gamma_r^2)^2 ||\tilde{u}||^2 \end{aligned} \tag{A.3}$$

since $\sum_{k=1}^{r} O_{k,x_1} O_{k,x_2} = (1 - \gamma_r^2) \delta_{x_1,x_2}$. Q.E.D.

Lemma 2. *Let $k_1, \ldots, k_r$ be non-negative integers; $k = \sum_{i=1}^{r} k_i \geq 2$, and let $s_p = \#\{i|\, k_i = p\}$ for $p = 0, 1$. Then, for any set of complex numbers $z_1, \ldots, z_r$ with $\sum_{i=1}^{r} z_i = 0$,*

$$\left| \sum_{\sigma \in S_r} z_{\sigma(1)}^{k_1} \cdots z_{\sigma(r)}^{k_r} \right| \leq s_0!\, s_1!\, ||z||^k \tag{A.4}$$

where $||z||$ denotes the 2-norm: $||z||^2 = \sum_{i=1}^{r} |z_i|^2$.

Proof. We first prove the inequality (A.4) in the case that $s_k = k$, i.e. all $k_i = 0, 1$. In that case it can be written in the form

$$\left| \sum_{1 \leq i_1 < \ldots < i_k \leq r} z_{i_1} \ldots z_{i_k} \right| \leq ||z||^k. \tag{A.5}$$

We do induction on k. If $k = 2$ then in fact $\sum_{1 \leq i < j \leq r} z_i z_j = -\frac{1}{2} \sum_{i=1}^{r} z_i^2$. Next suppose (A.5) holds for all k' with $2 \leq k' \leq k$. Then we can write

$$\begin{aligned} 0 &= \left(\sum_{i=1}^{r} z_i \right) \left(\sum_{1 \leq i_1 < \ldots < i_k \leq r} z_{i_1} \ldots z_{i_k} \right) \\ &= (k+1) \sum_{1 \leq i_1 < \ldots < i_{k+1} \leq r} z_{i_1} \ldots z_{i_{k+1}} + \sum_{1 \leq i_1 < \ldots < i_k \leq r} \sum_{\nu=1}^{k} z_{i_\nu} z_{i_1} \ldots z_{i_k}. \end{aligned} \tag{A.6}$$

To compute the latter term we can write similarly

$$\left(\sum_{i=1}^{r} z_i^2\right)\left(\sum_{1\le i_1<\ldots<i_{k-1}\le r} z_{i_1}\ldots z_{i_{k-1}}\right) = \sum_{1\le i_1<\ldots<i_k\le r}\sum_{\nu=1}^{k} z_{i_\nu} z_{i_1}\ldots z_{i_k} + \sum_{1\le i_1<\ldots<i_{k-1}\le r}\sum_{\nu=1}^{k-1} z_{i_\nu}^2 z_{i_1}\ldots z_{i_{k-1}}. \tag{A.7}$$

Repeating this process we obtain

$$\begin{aligned} &\sum_{1\le i_1<\ldots<i_k\le r}\sum_{\nu=1}^{k} z_{i_\nu} z_{i_1}\ldots z_{i_k} \\ &\quad = \left(\sum_{i=1}^{r} z_i^2\right)\left(\sum_{1\le i_1<\ldots<i_{k-1}\le r} z_{i_1}\ldots z_{i_{k-1}}\right) \\ &\quad\quad - \left(\sum_{i=1}^{r} z_i^3\right)\left(\sum_{1\le i_1<\ldots<i_{k-2}\le r} z_{i_1}\ldots z_{i_{k-2}}\right) \\ &\quad\quad + \ldots\ldots + \\ &\quad\quad + (-1)^{k-1}\left(\sum_{i=1}^{r} z_i^{k-1}\right)\left(\sum_{1\le i_1<i_2\le r} z_{i_1} z_{i_2}\right) + (-1)^{k-1}\sum_{i=1}^{r} z_i^{k+1}. \end{aligned} \tag{A.8}$$

We thus obtain

$$\begin{aligned} &\left|\sum_{1\le i_1<\ldots<i_{k+1}\le r} z_{i_1}\ldots z_{i_{k+1}}\right| \\ &\quad\le \frac{1}{k+1}\left\{\sum_{p=2}^{k-1}\left|\sum_{i=1}^{r} z_i^p\right|\left|\sum_{1\le i_1<\ldots<i_{k-p+1}\le r} z_{i_1}\ldots z_{i_{k-p+1}}\right| + \left|\sum_{i=1}^{r} z_i^{k+1}\right|\right\} \\ &\quad\le \frac{1}{k+1}(k-1)||z||^{k+1}. \end{aligned} \tag{A.9}$$

Next we use induction on the number of $k_i \ge 2$. By symmetry we may then assume that $k_r \ge 2$ and write

$$\begin{aligned} \left|\sum_{\sigma\in S_r} z_{\sigma(1)}^{k_1}\cdots z_{\sigma(r)}^{k_r}\right| &\le \sum_{i=1}^{r} |z_i|^{k_r}\left|\sum_{\sigma\in S_{r-1}} \tilde{z}_{\sigma(1)}^{k_1}\cdots\tilde{z}_{\sigma(r-1)}^{k_{r-1}}\right| \\ &\le s_0!s_1!\sum_{i=1}^{r}|z_i|^{k_r}||\tilde{z}||^{k-k_r} \le s_0!s_1!||z||^k \end{aligned}$$

provided $k_1 + \ldots + k_{r-1} \geq 2$. Here $\tilde{z}_j = z_j$ if $j < i$ and $\tilde{z}_j = z_{j+1}$ if $j \geq i$. In case $k_1 + \ldots + k_{r-1} = 1$ the inequality (A.4) has the form

$$\left| \sum_{\sigma \in S_r} z_{\sigma(1)} z_{\sigma(2)}^{k-1} \right| \leq (r-2)! \, ||z||^k$$

and can also be written as

$$\left| \sum_{1 \leq i < j \leq r} (z_i z_j^{k-1} + z_i^{k-1} z_j) \right| \leq ||z||^k.$$

This follows immediately upon multiplying out $0 = (\sum_{i=1}^r z_i) \left(\sum_{j=1}^r z_j^{k-1} \right)$. If $k_i = 0$ for all $i \leq r-1$ then the inequality reads $\left| \sum_\sigma z_{\sigma(1)}^k \right| \leq (r-1)! ||z||^k$ which is trivial. Q.E.D.

Lemma 3. *Let k and p be positive integers, $k \geq p \geq 2$. Then, for any set of complex numbers $z_1, \ldots, z_r$ with $\sum_{i=1}^r z_i = 0$,*

$$\begin{aligned} &\left| \sum_{n_1, \ldots, n_r \geq 0 : \sum_{i=1}^r n_i = k} \; \sum_{k_i \leq n_i : \sum_{i=1}^r k_i = p} \; \prod_{i=1}^r \binom{n_i}{k_i} z_i^{k_i} \right| \\ &\quad \leq \binom{k+r-1}{k} k^p \, ||z||^p \begin{cases} \frac{(r-p)!}{r!} \vee \frac{1}{p! r} & \text{if } p \leq r, \\ \frac{1}{p! r} & \text{if } p > r. \end{cases} \end{aligned} \tag{A.10}$$

Proof. It is clear that it suffices to prove that

$$\frac{1}{r!} \left| \sum_{\sigma \in S_r} \; \sum_{k_i \leq n_i : \sum_{i=1}^r k_i = p} \; \prod_{i=1}^r \binom{n_i}{k_i} z_{\sigma(i)}^{k_i} \right| \leq k^p ||z||^p \begin{cases} \frac{(r-p)!}{r!} \vee \frac{1}{p! r} & \text{if } p \leq r, \\ \frac{k^p}{p! r} ||z||^p & \text{if } p > r. \end{cases} \tag{A.11}$$

Inserting Lemma A.2 we get

$$\begin{aligned} &\frac{1}{r!} \left| \sum_{\sigma \in S_r} \; \sum_{k_i \leq n_i : \sum_{i=1}^r k_i = p} \; \prod_{i=1}^r \binom{n_i}{k_i} z_{\sigma(i)}^{k_i} \right| \\ &\quad \leq \frac{1}{r!} \sum_{k_i \leq n_i : \sum_{i=1}^r k_i = p} \binom{n_1}{k_1} \cdots \binom{n_r}{k_r} s_0! \, s_1! \, ||z||^p \end{aligned}$$

and, using

$$k^p = p! \sum_{k_1,\ldots,k_r:\sum_{i=1}^r k_i=p} \prod_{i=1}^r \frac{n_i^{k_i}}{k_i!}$$

it remains to prove that

$$s_0!\, s_1! \leq \begin{cases} (r-1)! & \text{if } r < p, \\ p!(r-p)! \vee (r-1)! & \text{if } r \geq p. \end{cases} .$$

This is easily checked. Q.E.D.

We next insert this estimate into (2.43) to prove the following improved estimate:

Proposition. *If $\vec{u} \in L^2([0,\beta], \mathbf{R}^r)$ is a multiple of the constant function h_0, i.e. $u_i(t) = u_{0,i}h_0(t)$ $(i = 1, \ldots, r)$, and, moreover, $||\vec{u}_0||^2 \leq \delta$ then there exists a constant $C(\beta)$ depending on β but not on r or n such that*

$$|\psi^{(n)}[u] - 1| \leq C(\beta)\frac{r}{n}\delta. \tag{A.12}$$

Proof. We write $c_i = \lambda_i e^\beta$ and $z_i = c_i - 1$ for $i = 1, \ldots, r$. We now estimate the expression in the numerator of the third line of (2.40) by means of Lemmas 2 and 3:

$$\begin{aligned} &\left| \sum_{n_1,\ldots,n_r:\sum_{i=1}^r n_i=k} (c_1^{n_1} \ldots c_r^{n_r} - 1) \right| \\ &\quad = \left| \sum_{n_1,\ldots,n_r:\sum_{i=1}^r n_i=k} \sum_{p=1}^k \sum_{k_i \leq n_i:\sum_{i=1}^r k_i=p} \prod_{i=1}^r \binom{n_i}{k_i} z_i^{k_i} \right|. \end{aligned}$$

The term $p = 1$ vanishes because it equals

$$\sum_{n_1,\ldots,n_r:\sum_{i=1}^r n_i=k} \sum_{i=1}^r n_i z_i = \sum_{n_1,\ldots,n_r:\sum_{i=1}^r n_i=k} \frac{k}{r} \left(\sum_{i=1}^r z_i \right) = 0.$$

Thus, using Lemma 3 and the fact that $r(r-1)\dots(r-p+1) \le r(p-1)!$ if $r \ge p$,

$$\begin{aligned}
&\left| \sum_{n_1,\dots,n_r:\,\sum_{i=1}^r n_i = k} (c_1^{n_1}\dots c_r^{n_r} - 1) \right| \\
&\quad \le \sum_{p=2}^{k} \left| \sum_{n_1,\dots,n_r:\,\sum_{i=1}^r n_i = k} \;\; \sum_{k_i \le n_i:\,\sum_{i=1}^r k_i = p} \; \prod_{i=1}^{r} \binom{n_i}{k_i} z_i^{k_i} \right| \\
&\quad \le \sum_{p=2}^{k} \binom{k+r-1}{k} \frac{k^p ||z||^p}{r(p-1)!} \\
&\le \frac{k^2}{r} ||z||^2 \exp\left[k\,||z||\right].
\end{aligned} \tag{A.13}$$

Inserting into (2.40) we obtain

$$\begin{aligned}
|\psi^{(n)}[u] - 1| &\le e^{C\delta} \left\{ C\delta + \frac{\sum_{k=0}^{n} \binom{k+r-1}{k} e^{-\beta k} \frac{k^2}{r} ||z||^2 e^{k\,||z||}}{\sum_{k=0}^{n} \binom{k+r-1}{k} e^{-\beta k}} \right\} \\
&\le e^{C\delta} \left\{ C\delta + \frac{\sum_{k=0}^{\infty} \binom{k+r-1}{k} e^{-\beta k} \frac{k^2}{r} ||z||^2 e^{k\,||z||}}{\sum_{k=0}^{\infty} \binom{k+r-1}{k} e^{-\beta k}} \right\} \\
&= e^{C\delta} \left\{ C\delta + \frac{1}{r} ||z||^2 (1-e^{-\beta})^r \frac{d^2}{d\beta^2} (1 - e^{-\beta + ||z||})^{-r} \right\} \\
&= e^{C\delta} \left\{ C\delta + ||z||^2 \frac{e^{-\beta+||z||}(1 + r e^{-\beta+||z||})(1-e^{-\beta})^r}{(1-e^{-\beta+||z||})^{r+2}} \right\} \\
&\le e^{C\delta} \left\{ C\delta + C' r ||z||^2 \right\} \\
&\le e^{C\delta} \left\{ C\delta + C'' \frac{r}{n} \delta \right\} \\
&\le C(\beta)(1 + \frac{r}{n})\delta.
\end{aligned} \tag{A.14}$$

Here we used the fact that $||z||^2 \le \frac{\beta}{n} e^{2\beta} ||\vec{u}_0||^2$ by Lemma 1. Q.E.D.

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