Quantum Zeno and anti-Zeno Effects: An Exact Model

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Abstract

Recent studies suggest that both the quantum Zeno (increase of the natural lifetime of an unstable quantum state by repeated measurements) and anti-Zeno (decrease of the natural lifetime) effects can be made manifest in the same system by simply changing the dissipative decay rate associated with the environment. We present an exact calculation confirming this expectation.

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The quantum Zeno effect (QZE) predicts that the lifetime of an excited state increases by repeated measurements. It has been a subject of interest for many years [1, 2] and recent reviews appear in [3] and [4]. More recently, it has been pointed out that a decrease in lifetime, referred to as the Inverse or anti-Zeno effect (AZE) can also occur [5–7]. Whereas there are claims that the QZE has been observed, to our knowledge there has been no experimental verification yet of the AZE. On the other hand, the detailed calculations of Pascazio and Facchi [5] and Kofman and Kurizki [6] lend strong credence to the possible existence of the AZE. Since the calculations of [5] and [6] by their nature required various assumptions (such as the Weisskopf-Wigner approximation), we consider it desirable to present an <u>exact</u> calculation which should also delineate in a clear-cut manner the nature and magnitude of the external environment that is necessary to achieve the transition from QZE to AZE.

The system we analyze is the decay of a free particle that is placed initially in a Gaussian state:

$$\psi(x,0) = \left(2\pi\sigma^2\right)^{-1/4} \exp\left\{-\frac{x^2}{4\sigma^2}\right\}, \tag{1}$$

where σ^2 is the variance. The particle is regarded as part of a larger system of a particle coupled to a reservoir and the complete system is initially in equilibrium at temperature T. This was the scenario considered by Ford et al. [8] who used distribution functions defined in accordance with the quantum theory of measurement to obtain exact results for the spreading of the wave packet and for the probability at time t given by

$$P(x,t) = \frac{1}{\sqrt{2\pi w^2(t)}} \exp\left\{-\frac{x^2}{2w^2(t)}\right\}. \tag{2}$$

Here

$$\begin{aligned} w^2(t) &= \sigma^2 - \frac{[x(0),x(t)]^2}{4\sigma^2} + s(t) \\ &\equiv \sigma^2 + \sigma_q^2 + s(t), \end{aligned} \tag{3}$$

where σ^2 is the initial variance, $[x(0),x(t)]$ is the commutator,

$$s(t) = \left\langle \{x(t) - x(0)\}^2 \right\rangle, \tag{4}$$

is the mean square displacement and σ_q^2 is the contribution to the spreading due to temperature-independent quantum effects. A measure of the decay rate $R(t)$ is simply given by the ratio of the probabilities at times t and 0. However, this ratio is clearly dependent on x so, from henceforth, we take $x = 0$ (corresponding to the maximum of the wave-packet) and write

$$\begin{aligned} R(t) &= \frac{P(0,t)}{P(0,0)} \\ &= \left\{ \frac{\sigma^2}{w^2(t)} \right\}^{1/2} . \end{aligned} \tag{5}$$

Hence, our calculation reduces to an evaluation of the width of the wave-packet at time t.

The quantities appearing in (3) and (4) are evaluated by use of the quantum Langevin equation [9], which is a Heisenberg equation of motion for $x(t)$, the dynamical variable corresponding to the coordinate of a Brownian particle interacting with a linear passive heat bath. For the case of a free particle, this equation for the stationary process has the well known form,

$$m\ddot{x} + \int_{-\infty}^{t} dt' \mu(t-t')\dot{x}(t') = F(t), \tag{6}$$

where $\mu(t)$ is the memory function and $F(t)$ is a fluctuating operator force with mean zero. The solution of the quantum Langevin equation (6) can be written

$$x(t) = \int_{-\infty}^{t} dt' G(t-t') F(t'), \tag{7}$$

where $G(t)$, the Green function, can in turn be written

$$G(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \alpha(\omega + i0^+) e^{-i\omega t}, \tag{8}$$

in which $\alpha(z)$ (the Fourier transform of the Green function) is the response function. For the free particle the response function has the general form

$$\alpha(z) = \frac{1}{-mz^2 - iz\tilde{\mu}(z)}, \tag{9}$$

in which $\tilde{\mu}(z)$ is the Fourier transform of the memory function,

$$\tilde{\mu}(z) = \int_0^\infty dt \mu(t) e^{izt}, \qquad \mathrm{Im}\{z\} > 0. \tag{10}$$

Using these results, we find that [9, 10] the mean square displacement is given by the formula

$$s(t) = \frac{2\hbar}{\pi} \int_0^\infty d\omega \mathrm{Im}\left\{\alpha\left(\omega + i0^+\right)\right\} \coth \frac{\hbar\omega}{2kT}(1 - \cos\omega t), \tag{11}$$

while the commutator, which is temperature independent, is given by the formula

$$[x(0), x(t)] = \frac{2i\hbar}{\pi} \int_0^\infty d\omega \mathrm{Im}\left\{\alpha\left(\omega + i0^+\right)\right\} \sin\omega t. \tag{12}$$

These expressions are valid for arbitrary temperature and arbitrary dissipation. (Indeed, with the appropriate expression for the response function, they are valid in the presence of an external oscillator potential.) Here, we confine our attention to the case of zero temperature and Ohmic dissipation, where $\tilde{\mu}(z) = m\gamma$. It then follows that

$$s(t) = \frac{2\hbar\gamma}{\pi m} t^2 \mathrm{I}(\gamma t), \tag{13}$$

where

$$\mathrm{I}(\gamma t) = \int_0^\infty dy \frac{(1 - \cos y)}{y\left[y^2 + (\gamma t)^2\right]}. \tag{14}$$

In addition, the commutator is given by

$$[x(0), x(t)] = i\hbar G(t) = \frac{i\hbar}{m\gamma}\left(1 - e^{-\gamma t}\right), \tag{15}$$

so that

$$\sigma_q^2 = \frac{\hbar^2}{m^2\gamma^2} \; \frac{\left(1 - e^{-\gamma t}\right)^2}{4\sigma^2}. \tag{16}$$

Hence, we now have all the tools at our disposal in order to carry out an exact calculation of $P(x, t)$ and hence the rate $R(t, \gamma)$, where we have added the argument γ to R in order to emphasize the fact that this dependence will be the crucial element in our calculation. Combining the various results given above, we may write explicitly

$$R(t, \gamma) = \left\{\left[\sigma^2 + \left(\frac{\hbar^2}{4m^2\sigma^2}\right) t^2 \left\{\frac{\left(1 - e^{-\gamma t}\right)}{\gamma t}\right\}^2 + \frac{2\hbar t}{\pi m}(\gamma t)\mathrm{I}(\gamma t)\right] / \sigma^2\right\}^{-1/2}. \tag{17}$$

Our goal will be to calculate $R_n(t)$, the rate corresponding to n measurements on the system at time intervals $\tau = t/n$, and then compare it to $R(t,\gamma)$. As we shall see, the result depends crucially on the value of γt.

The only quantity left requiring explicit evaluation is I(γt) given by (14). In order to obtain more physical insight into the nature of the results obtained, we will first evaluate R analytically for both small and large values of γt, which we will demonstrate correspond to the QZE and AZE, respectively. However, in order to determine for what value of γt the transition between the two regimes occur, it will be necessary to carry out a numerical evaluation of I(γ). First, we turn to the analytic calculation.

(a) $\underline{\gamma t << 1}$

Then, from (16),

$$\sigma_q^2 \approx \frac{\hbar^2}{4m^2\sigma^2} t^2, \tag{18}$$

which corresponds to the usual dynamical wave packet spreading in the absence of a dissipative environment. In addition, (13) reduces to [11]

$$s(t) = \frac{\hbar\gamma}{\pi m} t^2 \left\{ -\log(\gamma t) + \frac{3}{2} - \gamma_E \right\}, \tag{19}$$

where $\gamma_E = 0.577$ is Euler's constant.

Using these results in (3) leads to

$$w^2(t) = \sigma^2 + \langle v^2 \rangle t^2, \tag{20}$$

where

$$\langle v^2 \rangle = \frac{\hbar^2}{4m^2\sigma^2} + \frac{\hbar\gamma}{\pi m} \left\{ -\log(\gamma t) + 0.92 \right\}. \tag{21}$$

Since for most reasonable scenarios $\langle v^2 \rangle t^2 << \sigma^2$, we may expand $w^2(t)$ in a power series in t^2 to get

$$\begin{aligned} R(t) &= \left\{ \frac{\sigma^2}{w^2(t)} \right\}^{1/2} \approx 1 - \frac{1}{2\sigma^2} \left\{ \sigma_q^2 + s(t) \right\} \\ &\approx \left(1 - \frac{\langle v^2 \rangle t^2}{2\sigma^2} + - - - \right). \end{aligned} \tag{22}$$

Apart from the weak $\log(\gamma t)$ dependence of $\langle v^2 \rangle$, as manifest in (21), we note the ubiquitous short-time t^2 behaviour of $R(t)$, which is characteristic of the QZE. To remind the reader, we consider n instantaneous measurements at intervals $\tau = t/n$. The corresponding rate can be obtained by computing the widths w_k at the intermediate times $k\tau$ by inserting the previous width w_{k-1} for σ in (3), i.e. $w_0 = \sigma$ and

$$w_k^2 = w_{k-1}^2 + \frac{\hbar^2}{4m^2 w_{k-1}^2} \left(\frac{t}{n}\right)^2 + s\left(\frac{t}{n}\right). \tag{23}$$

In particular, for small γt and $\langle v^2 \rangle t^2 \ll \sigma^2$, $w_n(t)^2 \approx \sigma^2 + \frac{1}{n}\langle v^2 \rangle t^2$ so that

$$R_n(t) \approx 1 - \frac{1}{2n\sigma^2}\langle v^2 \rangle t^2 \tag{24}$$

and the decay rate becomes frozen as $n \to \infty$, which is the extreme manifestation of the QZE. Moreover, for $\gamma \to 0$, we see from (21) that the second term on the right-side goes to zero. Thus, (23) still holds but now $\langle v^2 \rangle$ is independent of both γ and t. The conclusion is that in the absence of a heat bath, the QZE always holds.

(b) $\underline{\gamma t >> 1}$

Then, from (16),

$$\sigma_q^2 \approx \frac{\hbar^2}{4m^2\sigma^2}\, \frac{1}{\gamma^2}, \tag{25}$$

and (13) reduces to [11]

$$s(t) = \frac{2\hbar}{\pi m \gamma} \left\{ \log(\gamma t) + \gamma_E \right\}. \tag{26}$$

Hence

$$w^2(t) = \sigma^2 + \frac{\hbar^2}{4m^2\sigma^2}\, \frac{1}{\gamma^2} + \frac{2\hbar}{\pi m \gamma} \left[\log(\gamma t) + \gamma_E \right]. \tag{27}$$

Since for most reasonable scenarios, $\sigma^2 >> s(t) >> \sigma_q^2$ for any t, we may write

$$\begin{aligned} R(t) &\approx 1 - \frac{s(t)}{2\sigma^2} \\ &= 1 - \frac{\hbar}{\pi m \sigma^2 \gamma} \left[\log(\gamma t) + \gamma_E \right]. \end{aligned} \tag{28}$$

Assuming $\sigma_q \ll s(t)$ we also have

$$\begin{aligned} w_n^2(t) &\approx \sigma^2 + ns\left(\frac{t}{n}\right) \\ &= \sigma^2 + \frac{2n\hbar}{\pi m\gamma}\left(\log\frac{\gamma t}{n} + \gamma_E\right). \end{aligned} \tag{29}$$

Provided n is not too large we can therefore write

$$R_n(t) \approx 1 - \frac{\hbar}{\pi m\sigma^2\gamma}\left[\log\left(\frac{\gamma t}{n}\right) + \gamma_E\right]. \tag{30}$$

Thus we see that $R_n(t)$ is much less than $R(t)$, which is a manifestation of the AZE.

The conclusion is that the QZE is characterized by small γt values whereas the AZE is characterized by large γt values. As a check on the analytic results given for $I(\gamma t)$ in (14), we carried out a numerical evaluation of the integral and obtained excellent agreement.

In order to obtain the value of γt for which the transition occurs, as well as delineating more accurately the analytic results obtained above, we now turn to a numerical evaluation of the exact expression for $R(t,\gamma)$ given in (17). Thus, in Figs 1 and 2, we plot $R_n(t) = \frac{\sigma}{w_n(t)}$, corresponding to $n = 20$ measurements, and compare it to $R(t)$, for t values ranging from 0-1 and 0-40, respectively, and taking $\gamma = 10$. We note that $R_n(t)$ is initially larger than $R(t)$, corresponding to the QZE but it becomes smaller (corresponding to the AZE) for large γt values. The value of γt where the transition from QZE to AZE takes place is in the region where on the one hand $\gamma t > 1$ but on the other hand $\gamma\tau < 1$. From Fig. 2, we see that this corresponds to a γt value of 7.

It is also of interest to return to our analytic treatment to compute the transition value of γt. It can be approximated by equating $w_n^2(t)$ obtained from (19)-(21) to $w^2(t)$ obtained from (27) to get

$$\sigma^2 + \frac{\hbar\gamma}{m\pi}\frac{t^2}{n}\left(-\log\frac{\gamma t}{n} + \frac{3}{2} - \gamma_E\right) \approx \sigma^2 + \frac{2\hbar}{m\pi\gamma}(\log\gamma t + \gamma_E). \tag{31}$$

Hence

$$(\gamma t)^2\left[-\log(\gamma t) + \log n + 0.92\right] = 2n\left[\log(\gamma t) + 0.577\right]. \tag{32}$$

Thus, for $n = 20$ the solution for the transition point is given by $\gamma t \approx 7.28$, which is close to the numerically determined value. (The approximation is better for smaller values of σ_q, i.e. for smaller values of $\frac{\hbar}{m\sigma^2}$.) We also remark that, for large n, (32) leads to

$$\gamma t = \sqrt{2n}\left(1 + \frac{\delta}{\log n}\right), \tag{33}$$

where $\delta = \log 2 + 2\gamma_E - \frac{3}{2} = 0.348$. Inserting $n = 20$ gives 7.06, in good agreement with the numerical result.

In conclusion, we have presented an exact calculation of the decay rate of a free particle that is placed initially in a Gaussian state and which is coupled to a reservoir so that the complete system in initially in equilibrium at zero temperature. The results obtained demonstrate that repeated measurements made on the system lead to a QZE effect scenario for small γt values while evolving into an AZE effect scenario for large γt values, confirming similar results obtained in [5] and [6].

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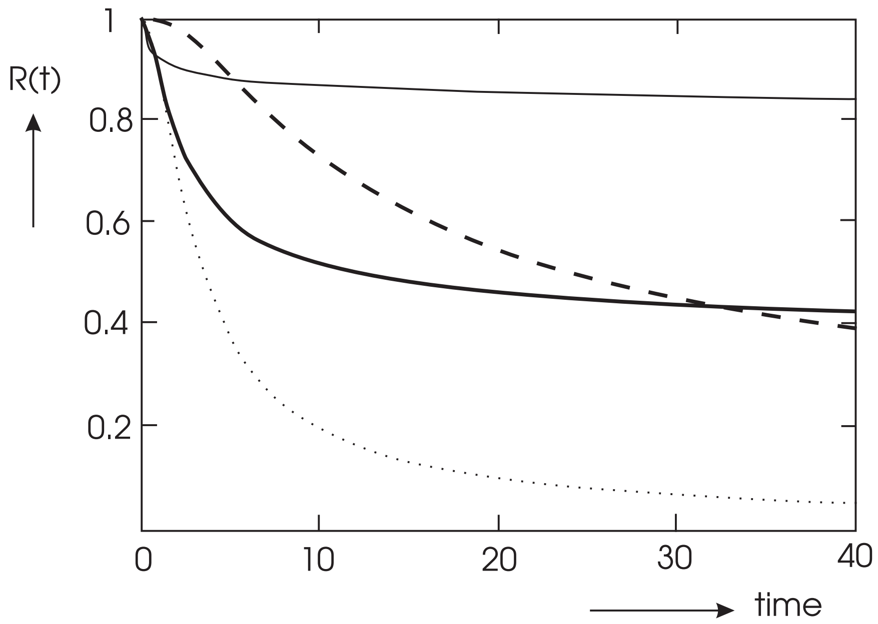

FIG. 1: *Decay rates in the case $\gamma t \gg 1$. The heavy drawn line corresponds to the decay rate with $n = 20$ intermediate measurements; the light drawn line is the decay rate without intermediate measurements. Both in the presence of dissipation. Without dissipation the corresponding rates are the heavy dashed line and the dotted line respectively. Notice that in the dissipative case, the rate with intermediate measurements is smaller than that without: this is the AZE. In the case without dissipation, the Zeno effect holds.*

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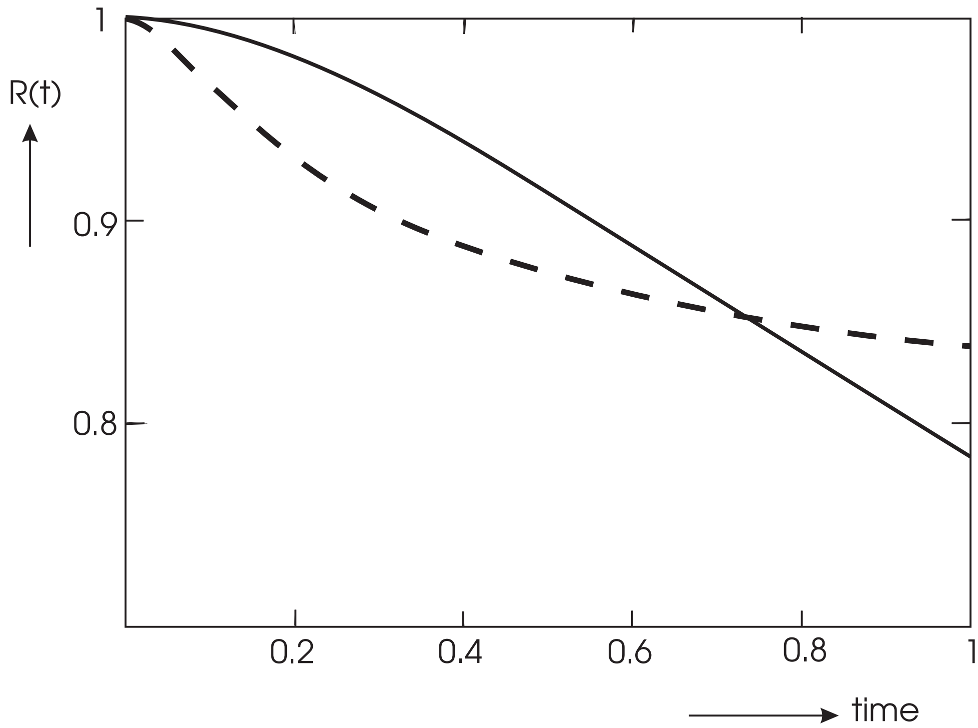

FIG. 2: *Detail of the dissipative rates of the previous figure for short times. Clearly, for* $t < 0.7$*, the Zeno effect applies.*

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