From Vicious Walkers to TASEP.

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Abstract

We consider a model which interpolates between the totally asymmetric simple exclusion process and the vicious walkers model. We calculate the survival probability for this model and obtain the scaling function which describes the transition from one model to another. Via fluctuation dissipation relations the results are applied to study the current fluctuations in the totally asymmetric simple exclusion process.

1 Introduction.

Exact solutions of 1D many particle stochastic models have given much insight into the physics of non-equilibrium systems in one dimension. They serve as a testing ground for the macroscopic theories, being able to verify their predictions. The examples are the description of different kinds of non-equilibrium phase transitions, calculation of the large deviation functions for the density profile and cumulated current of particles, verification of the fluctuation dissipation relations and testing of the range of their validity.

The range of models is very broad. In context of present article we mention two of them. The first one is the vicious walkers (VW) model that has been introduced in the physical literature by M. Fisher [**?**] to describe the wetting and melting phenomena. This is a random process defined as many *non-interacting* random walks on a 1D lattice, whose time-space trajectories are forbidden to cross each other. The term non-interacting means that the probability of a particular realization of the process, which meets the latter constraint, is given by the product of the probabilities of the random walks performed by each individual walker. As there is a finite probability for non-interacting random walkers to come to the same site of the lattice, neglecting these processes violates the probability conservation. In this sense we say that the dynamics of the model leads to the probability dissipation.

Another model, the totally asymmetric simple exclusion process (TASEP), was widely discussed in connection to the Kardar-Parisi-Zhang universality class. In contrast to VW, this is the model of *interacting* random walks. The interaction prevents particles from jumping to occupied sites. Therefore, similarly to VW the statistical ensemble includes only those events where the space-time trajectories of particles do not cross. The difference from VW is that the interaction changes the statistical weights of particle trajectories, when they pass via neighboring sites, such that the total probability is conserved.

Plenty results have been obtained on both models. Deep connections to the random matrix theory, statistics of young diagrams, determinantal point processes has been revealed. At the same time it is not clear whether they have anything in common. Can the results obtained for each of them be incorporated in a more general framework?

In this article we propose a semi-vicious walkers (SVW) model, which interpolates between VW and TASEP. It is a model of interacting particles with, partial repulsion or attraction, where the trajectory crossings are forbidden. The term partial repulsion (attraction) means that the probability for the particle to jump to an occupied site is not equal to zero like in TASEP and can be less (greater) that the one of free particle. At the same time, the trajectory non-crossing constraint leads to luck of the probability conservation similarly to VW. The strength of interaction, which is also characterizes the probability dissipation, is a parameter of the model, which varying in its range has the TASEP and VW as two limiting cases. In the article we concentrate on the large time asymptotics of the survival probability, i.e. the probability for all particle not to meet each other in a large time limit. In other words, we calculate the normalization constant for the statistical ensemble containing only the processes with noncrossing trajectories. The main asymptotics of the survival probability for VW has been obtained by M. Fisher for equally spaced initial positions of walkers. He has shown that the survival probability for m particles decays with time t as a power law , $t^{-\frac{m(m-1)}{4}}$. Later, this calculation has been improved in [**?**] by obtaining also a constant prefactor. In the case of general initial configuration of walkers this prefactor depends on their initial positions. This case has been studied in [**?**]. Note that, due to the probability conservation for the TASEP, the quantity we are looking for is just a normalization factor equal to one. Our aim is to find out how the Fisher's WV asymptotics transforms into the constant normalization factor in the TASEP. The problem we consider can be divided into two parts. For generic value of interaction strength away from the point corresponding to the TASEP the probability dissipation is finite. It is intuitively clear that the main asymptotics is similar to the VW one. Indeed we obtain the Fisher's power law together with the constant prefactor that depends on the initial positions of particles and the interaction strength and diverges in the TASEP limit. The prefactor is obtained in the form of the determinant of $m \times m$ matrix for any finite number of particles m. The second and probably the most interesting case is the transition region, which interpolates between SW and TASEP behaviours. To probe into this region, we consider a scaling limit of the survival probability, where the large time limit and the TASEP limit of the interaction strength are taken together. In this way we obtain a scaling function of a single parameter that controls the transition from SW to TASEP. The function is obtained in the form of multiple integral. Interestingly, in the limiting cases of SW and TASEP this integral is reduced to the Mehta integrals $I_{m,k}$ with $k = 1/2$ and $k = 1$ respectively, which appear as normalization factors of the probability distribution function of eigenvalues in orthogonal and unitary Gaussian ensembles of random matrices. Therefore the scaling function we obtain interpolates between these two objects.

The results on SVW can be applied to study the statistics of the integrated current of particles in TASEP. Specifically, the survival probability we calculate for SVW turns out to be closely related to the generating function of the moments of the cumulative particle current in the TASEP. This relation can be used as a source of the fluctuation dissipation theorems, which relate the fluctuations of current in the system where the probability is conserved, with statistics of probability dissipation in the dissipative model. In this way we obtain the generating function of particle current. Moreover, in some cases the results we obtained allow us to derive explicit asymptotical formulas for probability distribution of the particle current in TASEP.

The article is organized as follows. In the Section 2 we formulate the SVW model, state the results obtained and discuss their interpretation in terms of the probability distribution of the particle current in TASEP. In the section 3 we solve the Master equation for SVW model. In the Section 4 we obtain the asymptotic formulas for the transition probabilities. In the Section 5 we prove the limiting properties of the function characterizing the SWV-TASEP transition. The Section 6 is Summary and conclusion.

2 Model and results.

2.1 The model.

Consider m particles on 1D infinite lattice. A configuration X of the system is specified by m-tuple of strictly increasing integers

$$X = \{x_1, < x_2, \cdots, < x_m\}, \tag{1}$$

where x_i is the coordinate of i-th particle. The strictly increasing order implies the exclusion condition, i.e. two particles can not occupy the same site. We say that the relation $X \subseteq Y$ holds if

$$x_1 \le y_1 \le x_2 \le \cdots \le x_m. \tag{2}$$

The SVW model is a random process, which is defined on a set of sequences of configurations $X^0, X^1, \cdots, X^t$, such that

$$X^0 \subseteq X^1 \cdots \subseteq X^t. \tag{3}$$

We refer to such a senescence as a trajectory of the system travelled up to time t. Every trajectory is realized with the probability

$$P(X^0, \ldots, X^t) = T(X^t, X^{t-1}) \cdots T(X^2, X^1) T(X^1, X^0) P_0(X_0). \tag{4}$$

$P_0(X)$ is the initial probability of the configuration X and the transition probability $T(X, Y)$, from the configuration Y to X, is defined as follows

$$T(X, Y) = \vartheta(x_m - y_m) \prod_{k=1}^{m-1} \theta\left(x_i - y_i, x_{i+1} - y_i\right), \tag{5}$$

where

$$\begin{aligned} \vartheta(k) &= (1-p)\,\delta_{k,0} + p\delta_{k,1}, & (6)\\ \theta\left(k, l\right) &= \left(1 - p\left(1 - \kappa\delta_{l,1}\right)\right)\delta_{k,0} + p\delta_{k,1}, & (7) \end{aligned}$$

and

$$\begin{aligned} 0 &< p < 1, & (8)\\ 1 - 1/p &\le \kappa \le 1. & (9) \end{aligned}$$

This is to say that at each time step a particle can jump forward with probability p or stay with probability $(1-p)$ provided that the next site is empty. If the next site is occupied the probability for a particle to stay is $(1-p(1-\kappa))$. The remaining probability $p(1-\kappa)$, would correspond to the process when the particle jumps to the occupied site next to it. However these processes are forbidden in our consideration. This is where the probability dissipation comes from. The form of the transition probabilities corresponds to the backward sequential update, i.e. the particles are updated starting from the m-th particles one by one in backward direction. In particular limiting cases the model is reduced to

1. $\kappa = 0$ - VW, a particle jumps forward with probability p or stays with probability $(1-p)$, irrespectively of whether the next site is occupied or not. Then those realization of the process where two particles come to the same site must be removed from the statistical ensemble.
2. $\kappa = 1$ - TASEP, a particle jumps forward with probability p or stays with probability $(1-p)$ if the next site is empty. When the next site occupied it stays with probability 1.

The TASEP with the backward sequential update was studied in [**?**] and [**?**], where it was referred to as a fragmentation model. In the case $\kappa = (1-1/p)$ the probability for particle to stay when the next site occupied is zero. Therefore, the trajectories of particles passing via the neighboring sites have zero weight, i.e. are removed from the ensemble as well as those which meet at the same site. Therefore this situation resembles the vicious walks of dimers. The range of κ, (9), comes from the requirement for $(1-p(1-\kappa))$ to be a probability. Positive values of κ correspond to the repulsive interaction, while the negative ones to the attractive one. The domain $\kappa > 1$ is also of interest connection to the current fluctuation in TASEP, though is does not have a probabilistic sense in context of SVW.

2.2 The results on SVW.

Below we calculate the quantity

$$\mathcal{P}_t\left(X^0\right) = \sum_{X^0 \subseteq X^1 \cdots \subseteq X^t} P(X^0, \dots, X^t),$$

where the sum is over all the trajectories of the system starting at the configuration X^0, i.e. $P_0\left(X\right) = \delta_{X,X^0}$. This quantity is the partition function of the statistical ensemble of the trajectories with the statistical weights defined above. In the other hand, if we add the lacking processes allowing the particles to jump to an occupied site, the value of $\mathcal{P}_t\left(X^0\right)$ will have the meaning of probability for all the particles not to meet up to time t. In Fisher's original formulation of such a process, two particles annihilate when getting into the same site. Then, $\mathcal{P}_t\left(X^0\right)$ is the probability for m particles to survive up to time t. Therefore, we refer to this quantity as a survival probability.

2.2.1 Generic values of $\kappa < 1$.

For generic values of $\kappa < 1$, the survival probability $\mathcal{P}_t\left(X^0\right)$ takes the following form as $t \to \infty$.

$$\mathcal{P}_t\left(X^0\right) = A\left(\kappa, X^0\right)\left[tp(1-p)\right]^{-\frac{m(m-1)}{4}}\left[1 + O\left(\left(\log t\right)^3 t^{-1/2}\right)\right], \quad (10)$$

where the prefactor is given by

$$A(X^0;\kappa) = \frac{2^m}{(\kappa-1)^{\frac{m(m-1)}{2}}\pi^{m/2}} \prod_{l=1}^{m} \frac{\Gamma(l/2+1)}{l!} \det[g_{i,j}(x_m^0 - x_i^0;\kappa)]_{1\le i,j\le m} \tag{11}$$

where the function $g_{i,j}(x;\kappa)$ is defined as

$$g_{i,j}(x;\kappa) = \oint_{C_0} \frac{d\xi}{2\pi \mathrm{i}} \frac{(1-\kappa-\kappa\xi)^{i-1}(1+\xi)^x}{\xi^j}. \tag{12}$$

The superscript *swv* refers to the SWV model. In the limit $\kappa \to 0$ one restores the result for WV up to the rescaling of space and time.

$$A(X^0;0) = \prod_{1\le i<j\le m} \left(x_j^0 - x_i^0\right) \begin{cases} \pi^{-m/4} 2^{-m^2/4+m/2} \prod\limits_{l=1}^{m/2} \frac{1}{(2l-1)!}; & \text{even } m \\ \frac{\pi^{1/4-m/4}}{2^{(m-1)^2/2}} \prod\limits_{l=1}^{(m-1)/2} \frac{1}{(2l)!} & \text{odd } m \end{cases} \tag{13}$$

In the limit $\kappa \to 1$, corresponding to the TASEP, the asymptotics must change, as the probability conservation must be restored. The signature of this fact is the divergence of the constant term that takes place in this limit, that has an order $(1-\kappa)^{-m(m-1)/2}$. Comparing an exponent of this term with the one of the time decay $t^{-m(m-1)/4}$, we can guess that the transition takes place on the scale $(1-\kappa) \sim 1/\sqrt{t}$. This hypothesis is justified by the next result.

2.2.2 Transition regime.

Consider a limit

$$t \to \infty, \kappa \to 1, (1-\kappa)\sqrt{t} = const. \tag{14}$$

We introduce the parameter

$$\alpha = \lim_{t\to\infty} \left[(1-\kappa)\sqrt{2tp(1-p)}\right]. \tag{15}$$

In this limit

$$\mathcal{P}_t = f_m(\alpha)\left[1 + O\left(\frac{(\log t)^3}{\sqrt{t}}\right)\right], \tag{16}$$

where the function $f_m(\alpha)$ has a form of the multiple integral

$$\begin{aligned} f_m(\alpha) &= \frac{(-2)^{\frac{m(m-1)}{2}}}{\pi^{\frac{m}{2}} 2! \cdots (m-2)!} \int_{-\infty}^{\infty} du_1 \int_{u_1}^{\infty} du_2 \cdots \int_{u_{m-1}}^{\infty} du_m \int_0^{\infty} d\nu_2 \cdots \int_0^{\infty} d\nu_m \\ &\times\ e^{-u_1^2} \prod_{i=2}^{m} \nu_i^{i-2} e^{-(u_i+\nu_i)^2 - \alpha\nu_i} \Delta(u_1, \nu_2 + u_2, \ldots, \nu_m + u_m). \end{aligned} \tag{17}$$

The argument of $\mathcal{P}_t$ in (16) is omitted as the dependence on the initial configuration is lost in the limit under consideration. The limiting behaviours of $f_m(\alpha)$ must match the TASEP and SW asymptotics. Indeed, we obtain

$$\lim_{\alpha\to\infty}\alpha^{\frac{m(m-1)}{2}}f_m(\alpha) = \frac{2^{\frac{m(m+1)}{4}}}{m!\pi^{m/2}}\prod_{l=1}^{m}\Gamma(l/2+1), \tag{18}$$

$$\lim_{\alpha\to 1}f_m(\alpha) = 1 \tag{19}$$

$$\lim_{\alpha\to-\infty}e^{-\alpha^2\frac{m(m-1)}{4}}f_m(\alpha) = \frac{m^{m-1}}{(m-1)!} \tag{20}$$

The third line does not have a probabilistic meaning in SVW model. It, however, is still meaningful for description of current fluctuation in TASEP. Remarkably, a proof these three limits can be done by reducing $f_m(\alpha)$ to different cases of Mehta integrals, $I_{m,1/2}$, $I_{m.,1}$ and $I_{m-1,1}$ respectively, which play an important role in the theory of random matrices.

2.3 Current fluctuations in TASEP.

Consider the process with the transition weights $\widetilde{T}(X,Y)$ defined like in (5) where the functions $\vartheta(k)$ and $\theta(k,l)$ are replaced by

$$\widetilde{\vartheta}(k) = (1-\widetilde{p})\,\delta_{k,0} + e^{\gamma}\widetilde{p}\delta_{k,1}, \tag{21}$$

$$\widetilde{\theta}(k,l) = (1-\widetilde{p}(1-\delta_{l,1}))\,\delta_{k,0} + e^{\gamma}\widetilde{p}\delta_{k,1}. \tag{22}$$

Here $0<\widetilde{p}<1$ and γ is a complex valued parameter. It is not difficult to see that these transition weights correspond to the TASEP with the probability of particle jump $\widetilde{p}$ supplied with an additional factor of e^{γ} for each particle jump, i.e.

$$e^{\gamma Y_t}P_{TASEP}(X^0,\dots,X^t) = \widetilde{T}(X^t,X^{t-1})\cdots\widetilde{T}(X^1,X^0)P_0(X_0), \tag{23}$$

where $P_{TASEP}(X^0,\dots,X^t)$ is the probability for the sequence of particle configurations $X^0,\dots,X^t$, to occur in the TASEP for t successive steps and Y_t is the total number of jumps made by particles when going between these configurations. Thus, one can calculate the generating function of the moment of cumulative particle current

$$\left\langle e^{\gamma Y_t}\right\rangle_{TASEP} = \sum_{X^0\subseteq X^1\cdots\subseteq X^t} e^{\gamma Y_t}P_{TASEP}(X^0,\dots,X^t). \tag{24}$$

In the other hand we can see that, if we define

$$\kappa = e^{-\gamma}, \tag{25}$$

$$p = \frac{\widetilde{p}}{(1-\widetilde{p})\,e^{-\gamma}+\widetilde{p}}, \tag{26}$$

then the relation exists between the transition weights $\widetilde{T}(X,X')$ defined in (21,22) and the ones of SWV, (6,7),

$$(1-\widetilde{p})^{-m}\,\widetilde{T}(X,X') = (1-p)^{-m}\,T(X,X'). \tag{27}$$

As a result we have

$$\left\langle e^{\gamma Y_t}\right\rangle_{TASEP} = (1+\widetilde{p}\,(e^{\gamma}-1))^{mt}\,\mathcal{P}_t\left(X^0\right). \tag{28}$$

where $\mathcal{P}_t\left(X^0\right)$ is the survival probability calculated for SVW model, and the parameters κ and p of SWV are related the parameters $\widetilde{p},\gamma$ of TASEP by (25,26). This function encodes all information about the distribution of cumulative particle current. Let us apply the results of previous subsection to obtain the form of this distribution.

2.3.1 Generic values of $\gamma > 0$.

For generic values of $\gamma > 0$, away from zero we have

$$\left\langle e^{\gamma Y_t} \right\rangle_{TASEP} = \frac{\left(1 + \widetilde{p}\left(e^{\gamma} - 1\right)\right)^{mt + \frac{m(m-1)}{2}}}{\left[te^{\gamma}\widetilde{p}(1-\widetilde{p})\right]^{\frac{m(m-1)}{4}}} A\left(e^{-\gamma}, X^0\right)$$

This distribution $P_t\left(Y_t = Y\right)$ for the particle current can be extracted from the generating function $\left\langle e^{\gamma Y_t} \right\rangle$ by the inverse Z-transform.

$$P_t\left(Y_t = Y\right) = \int_{-\pi}^{\pi} \frac{d\varphi}{2\pi} \left\langle e^{\mathrm{i}\phi Y_t} \right\rangle e^{-\mathrm{i}\phi Y}.$$

For large time, $t \to \infty$ this integral can be calculated in the saddle. The integral can be calculated in the saddle point approximation, which yields

$$P_t\left(Y_t = vmt\right) \simeq \frac{\left[\left(\frac{1-p}{1-v}\right)^{1-v}\left(\frac{p}{v}\right)^{v}\right]^{mt}}{\sqrt{2\pi v(1-v)mt}} \frac{A\left(\frac{v(1-p)}{p(1-v)}, X^0\right)}{\left[tv\left(1-v\right)\right]^{\frac{m(m-1)}{4}}} \tag{29}$$

The first multiple of this expression is exactly the current probability

distribution of m independent Bernoullian random walks, while the second one is due to the exclusion interaction.

2.3.2 Scaling limit.

The transition regime corresponds to following the scaling limit

$$t \to \infty, \gamma \to 0, \gamma\sqrt{t} = const. \tag{30}$$

To give an interpretation of the results obtained for the transition regime, consider the random variable

$$y = \frac{Y_t - m\widetilde{p}t}{\sqrt{2tp\left(1-p\right)}}. \tag{31}$$

The result obtained for the transition region

$$\lim_{t\to\infty} \left\langle e^{\alpha y} \right\rangle_{TASEP} = e^{m\alpha^2/4} f_m\left(\alpha\right) \tag{32}$$

where α is an arbitrary complex valued parameter obtained from γ by taking a limit $\alpha = \lim_{t\to\infty} \gamma\sqrt{2tp\left(1-p\right)}$. Note that the random variable y is the rescaled deviation of the integrated current Y_t from $m\widetilde{p}t$, i.e. from the average value of Y_t for m non-interacting particles jumping with probability $\widetilde{p}$. By the Central Limit Theorem the limiting probability distribution of y for such particles is Gaussian, $\lim_{t\to\infty} P_m(y = \xi) = \exp\left(-\xi^2/m\right)/\sqrt{2\pi m}$. Correspondingly the generating function of its moments is $\lim_{t\to\infty} \left\langle e^{\alpha y} \right\rangle = \exp\left(m\alpha^2/4\right)$. Therefore the scaling function $f_m(\alpha)$ characterizes the deviation of the distribution of y from the Gaussian form. Specifically its derivatives give us the cumulants of y like the average shift of the center of Gaussian distribution

$$\lim_{t\to\infty} \left\langle y \right\rangle = f'\left(0\right),$$

its dispersion

$$\lim_{t\to\infty}\left(\left\langle y^2\right\rangle - \langle y\rangle^2\right) = f''(0) - \left(f'(0)\right)^2 + m/2,$$

the other ones, which characterize the deviation of the distribution form from the Gaussian. An explicit calculation of the derivatives of $f_m(\alpha)$, requires an analysis of the integrals like (17) and is beyond the goals of present article. We are able to study the tails of the distribution $P_m(y)$ asymptotically using the results on the asymptotical behaviour of $f_m(\alpha)$, (18,20) obtained above. The limiting probability distribution $P_m(y)$ of the random variable y can be obtained from (32) as an inverse Laplace transform

$$\lim_{t\to\infty} P_m(y=\xi) = \int_{\beta-i\infty}^{\beta+i\infty} e^{m\alpha^2/4-\alpha\xi} f_m(\alpha)\,\frac{d\alpha}{2\pi},$$

where the parameter β is chosen such that the contour of integration is in the domain of absolute convergence of $f_m(\alpha)$. One could hope to use the asymptotics of the function $f_m(\alpha)$ to calculate this integral in the saddle point approximation as the number of particles m becomes large. However the above asymptotical formulas for $\alpha \to \pm\infty$ are obtained under the suggestion $m = o(|\alpha|)$. Therefore one needs to consider a double scaling limit for $f_m(\alpha)$. $m \to \infty$ and $\alpha \to \pm\infty$. We do not do this analysis here.

3 Master equation.

From now on we consider only SVW model dependent on the parameters p and κ. Consider the probability $P_t\left(X, X^0\right)$ of transition from the configuration X^0 to X for arbitrary time t.

$$P_t\left(X, X^0\right) = \sum_{X^0 \subseteq X^1 \cdots \subseteq X^t \equiv X} P(X^0, \ldots, X^t).$$

It obeys the Master equation.

$$P_t\left(X, X^0\right) = \sum_{X'} T\left(X, X'\right) P_{t-1}(X', X^0) \tag{33}$$

the transition weights $T(X, X')$ are defined above. The problem of finding the eigenvectors and eigenvalues of the matrix $T(X, X')$ can be solved by the Bethe ansatz technique. As this technique is rather standard and has been reviewed in many monographs, we do not pay much attention to its application here. For details of similar derivation the reader can consult for example with the review [**?**] As a result we obtain the solution of the left and right eigenproblems for the Markov matrix $T(X, X')$

$$\Lambda(Z)\,\Psi_Z(X) = \sum_{X'} T\left(X, X'\right) \Psi_Z\left(X'\right), \tag{34}$$

$$\Lambda(Z)\,\overline{\Psi}_Z(X) = \sum_{X'} T\left(X', X\right) \overline{\Psi}_Z\left(X'\right) \tag{35}$$

parametrized by m-tuple complex parameter $Z = \{z_1, \cdots, z_m\}$. Corresponding eigenvalue is expressed in terms of these parameters

$$\Lambda(Z) = \prod_{i=1}^{m} (1 - p + p/z_i) \tag{36}$$

and the eigenvectors are given by the following determinants

$$\Psi_Z(X) = \det\left(z_i^{x_j}(1-\kappa z_i)^{i-j}\right)_{1\le i,j\le m}, \tag{37}$$

$$\overline{\Psi}_Z(X) = \det\left(z_i^{-x_j}(1-\kappa z_i)^{j-i}\right)_{1\le i,j\le m}. \tag{38}$$

It is not difficult to check that these two eigenfunctions can be used to construct the resolution of the identity operator

$$\frac{1}{m!}\oint \Psi_Z(X)\,\overline{\Psi}_Z(X')\prod_{i=1}^{m}\frac{dz_i}{2\pi \mathrm{i} z_i} = \delta_{X,X'}, \tag{39}$$

where the integration over each $z_i, i = 1,\dots,m$, is along the contour of integration is required to leave the pole of the wave function $z = 1/\kappa$ outside. Then the solution of the initial value problem for the master equation is given by

$$P_t\left(X,X^0\right) = \frac{1}{m!}\oint \Lambda^t(Z)\,\Psi_Z(X)\,\overline{\Psi}_Z\left(X^0\right)\prod_{i=1}^{p}\frac{dz_i}{2\pi i z_i}. \tag{40}$$

Finally we end up with the following integral expression for the transition probability

$$\begin{aligned} P_t\left(X,X^0\right) &= \oint \Lambda^t(Z)\prod_{i=1}^{m}\left[\frac{z_i^{x_i-x_m^0}}{(1-\kappa z_i)^{i-1}}\right] \\ &\quad\times\det\left(z_i^{x_m^0-x_j^0}(1-\kappa z_i)^{j-1}\right)_{1\le i,j\le m}\prod_{i=1}^{p}\frac{dz_i}{2\pi \mathrm{i} z_i}. \end{aligned} \tag{41}$$

The integration can easily be performed by counting the residues. The result is the determinant of $m\times m$ matrix of the form similar to the one obtained for the discrete time TASEP with backward update ([?] , [?]). Note that in the case of vicious walkers, $\kappa = 0$, the eigenfunctions are of free fermion type

$$\Psi_Z(X) = \det\left(z_i^{x_j}\right)_{1\le i,j\le m}, \tag{42}$$

$$\overline{\Psi}_Z(X) = \det\left(z_i^{-x_j}\right)_{1\le i,j\le m}. \tag{43}$$

and the integration yields famous Lindtröm-Gessel-Viennot theorem.

$$P_t\left(X,X^0\right) = \det\left[F_0\left(x_i - x_j^0, t\right)\right]_{1\le i,j\le m}, \tag{44}$$

where

$$F_0(x,t) = p^x(1-p)^{T-x}\binom{t}{x}, \tag{45}$$

These formulas serve as a starting point for the asymptotical analysis of the survival probability.

4 Asymptotical form of survival probability.

To obtain the survival probability $\mathcal{P}_t\left(X^0\right)$ for SVW we have to sum the transition probability $P_t\left(X,X^0\right)$ over the set of all final configurations X

$$\mathcal{P}_t\left(X^0\right) = \sum_{\{X\}}^{\infty} P_t\left(X,X^0\right). \tag{46}$$

We solve this problem in the limit $t \to \infty$. For pedagogical means we first outline the derivation for VW model, which mainly reproduce the Rubey's arguments. The procedure includes the asymptotical analysis of the expression for $P_t\left(X, X^0\right)$ using the saddle point approximation for the integral (40), which helps us to reduce the sum over final configurations to known integrals. The main ingredients of the derivations for VW are used then for SVW with some modifications.

4.1 Vicious walkers.

In the case of VW, $\kappa = 0$, the integral (41) is reduced to

$$\oint \Lambda^t\left(Z\right) \prod_{k=1}^{m} z_k^{x_k - x_m^0} \det(z_i^{x_m^0 - x_i^0})_{1 \le i,j \le m} \prod_{l=1}^{p} \frac{dz_l}{2\pi \mathrm{i} z_l}. \tag{47}$$

Then the determinant under the integral can be expressed

$$\det(z_i^{x_m^0 - x_i^0})_{1 \le i,j \le m} = \Delta\left(Z\right) s_\chi\left(Z\right), \tag{48}$$

in terms of the Schur function

$$s_\chi(z_1, \ldots, z_m) \equiv \det\left(z_i^{\chi_j + m - j}\right) / \Delta\left(Z\right). \tag{49}$$

parametrized by a partition $\chi = (\chi_1, \ge \chi_2, \cdots, \ge 0)$, which is defined as follows

$$\chi = (x_m - x_1 - m + 1, x_m - x_2 - m + 2, \cdots). \tag{50}$$

and the Van der Monde determinant

$$\Delta\left(Z\right) \equiv \det(z_i^{m-j})_{1 \le i,j \le m} = \prod_{1 \le i < j \le m} \left(z_i - z_j\right). \tag{51}$$

Thus (40) can be rewritten in the following form

$$P_T\left(X, X^0\right) = \oint \Delta\left(Z\right) s_\chi\left(Z\right) \prod_{i=1}^{p} e^{t f_i(z_i)} \frac{dz_i}{2\pi \mathrm{i} z_i}, \tag{52}$$

where

$$f_i(z) = \log\left(1 - p + p/z_i\right) + v_i \log z \tag{53}$$

and

$$v_i = \frac{x_i - x_m^0}{t}. \tag{54}$$

Now we are ready to estimate the integral under the assumption $t \to \infty$, while $v_i = const > 0$ for $i = 1, \ldots, m$. In addition we assume that the difference $\left(x_i^0 - x_j^0\right)$ is kept finite for any i and j. The saddle point of the function under the integral is defined by the equation

$$f_i'(z_i^*) = 0, \tag{55}$$

which yields

$$z_i^* = \frac{\left(1 - v_i\right) p}{\left(1 - p\right) v_i}. \tag{56}$$

In the vicinity of the saddle point $f_i(z)$ has an expansion

$$\begin{aligned} f_i(z_i^* + \xi) \quad &= \quad \log\left[\left(\frac{1-p}{1-v_i}\right)^{1-v_i}\left(\frac{p}{v_i}\right)^{v_i}\right] \\ &\quad + \frac{1}{2}\left(\frac{1-p}{p}\right)^2 \frac{v_i^3}{1-v_i}\xi^2 + O(\xi^3). \end{aligned} \tag{57}$$

The integration contours can be deformed to the straight line parallel to the imaginary axis, crossing the real axis at z_i^*. Note that the pre-exponential factor is the polynomial of fixed degree, such that it contributes only a constant prefactor into the highest term. As a result the integration yields

$$P_t\left(X,X^0\right)=\left(\frac{p}{1-p}\right)^{\frac{m(m-1)}{2}}\prod_{i=1}^{m}\frac{v_i^{1-m}e^{tf_i\left(\frac{(1-v_i)p}{(1-p)v_i}\right)}}{\sqrt{2\pi T v_i(1-v_i)}}\times \tag{58}$$

$$\prod_{1\le i<j\le m}(v_j-v_i)\,s_\chi\left(\frac{(1-v_1)\,p}{(1-p)\,v_1},\ldots,\frac{(1-v_m)\,p}{(1-p)\,v_m}\right)\left(1+O\left(\frac{1}{t}\right)\right).$$

The next step is to perform the summation (46) over the range of the final configurations $X\in\left\{x_1^0\le x_1<\cdots<x_m<\infty\right\}$. It can be shown that the main contribution to the sum comes from the domain

$$pt-\sqrt{t}\log t\le x_1<\cdots<x_m\le\ pt+\sqrt{t}\log t. \tag{59}$$

Indeed, $f_i\left(\frac{(1-v_i)p}{(1-p)v_i}\right)$ is a concave function of v_i in the domain $v_i\in(0,1)$ with a single maximum $v_i=p$. It follows then for $|x_i-pt|>\sqrt{t}\log t$

$$e^{tf_i\left(\frac{(1-v_i)p}{(1-p)v_i}\right)}<e^{tf_i\left(\frac{1-\sqrt{t}\log t/(1-p)}{1+\sqrt{t}\log t/p}\right)}=e^{-\frac{(\log t)^2}{2p(1-p)}}\left[1+O\left(\frac{\log t}{\sqrt{t}}\right)\right] \tag{60}$$

All the other factors in (58) are at most of polynomial order in t, while the total number of nonzero terms in the sum of interest (46) is $O\left(t^m\right)$. Therefore, the contribution from the complement of (59) being of order of $O\left(t^s e^{-\frac{(\log t)^2}{2p(1-p)}}\right)$ for some constant s is asymptotically negligible comparing to the contribution from the interior of (59). In the latter one can approximate the function $f_i\left(\frac{(1-v_i)p}{(1-p)v_i}\right)$ by the second term of it's tailor expansion at $v_i=p$, which yields

$$P_t\left(X,X^0\right)\;=\;\frac{p^{\frac{3m^2-4m}{2}}}{(2\pi)^{\frac{m}{2}}\,t^{\frac{m(1-2m)}{2}}\,(1-p)^{\frac{m^2}{2}}}s_\chi\left(1,\ldots,1\right) \tag{61}$$

$$\prod_{i=1}^{m}\exp\left(-\frac{\left(x_i-x_m^0-pt\right)^2}{2tp\,(1-p)}\right)\prod_{1\le i<j\le m}(x_j-x_i)\left[1+O\left(\frac{(\log t)^3}{\sqrt{t}}\right)\right].$$

Then, we have to evaluate sum

$$\sum_{-\sqrt{t}\log t\le x_1<\cdots<x_m\le\ \sqrt{t}\log t}\ \prod_{i=1}^{m}e^{-\frac{x_i^2}{2tp(1-p)}}\prod_{1\le i<j\le m}(x_j-x_i)$$

which can be done applying the following Lemma [?]:

Lemma 1 *Let r be a nonnegative integer and let $b:\mathbb{N}\to\mathbb{N}$ be an arbitrary function. Furthermore, let $h:\mathbb{N}\to\mathbb{R}$ be a function of at most polynomial growth. Then as r tends to infinity ,*

$$\sum_{k=b(r)}^{\infty}h(k)e^{-\frac{k^2}{r}}=\int_{b(r)}^{\infty}dx\,h(x)e^{-\frac{x^2}{r}}+O(1),$$

where $O(1)$ can be chosen independent of b.

Applying it to the sum over x_m we get

$$\sum_{x_m=x_{m-1}}^{\left[\sqrt{t}\log t\right]} e^{-\frac{x_m^2}{2tp(1-p)}} \prod_{1\le i<m} (x_m - x_i) \tag{62a}$$

$$= \left(\sum_{x_m=x_{m-1}}^{\infty} - \sum_{x_m=\left[\sqrt{t}\log t\,\right]}^{\infty} \right) e^{-\frac{x_m^2}{2tp(1-p)}} \prod_{1\le i<m} (x_m - x_i) \tag{62b}$$

$$= \left(\int\limits_{x_{m-1}}^{\infty} dx - \int\limits_{\left[\sqrt{t}\log t\,\right]}^{\infty} dx \right) e^{-\frac{x^2}{2tp(1-p)}} \prod_{1\le i<m} (x - x_i) + O(1) \tag{62c}$$

$$= \int\limits_{x_{m-1}}^{\infty} dx e^{-\frac{x^2}{2tp(1-p)}} \prod_{1\le i<m} (x - x_i) + O(1) \tag{62d}$$

Going from (62c) to (62d) we dropped one integral which is of order of $O(t^s e^{-\frac{(\log t)^2}{2p(1-p)}})$ for some constant s. Iterating this procedure for all x_i, $i = 1, \dots, m$ we come to the following m-fold integral:

$$\int\limits_{-\infty}^{\infty} dx_1 \cdots \int\limits_{x_{m-2}}^{\infty} dx_{m-1} \int\limits_{x_{m-1}}^{\infty} dx_m \prod_{i=1}^{m} e^{-\frac{x_i^2}{2tp(1-p)}} \prod_{1\le i<j\le m} |x_j - x_i|$$

The absolute values under the second product make the expression under the integral invariant with respect to the variable permutations. Symmetrizing it over all the permutations one can extend the lower limits of integration over all the variables to minus infinity

$$\frac{1}{m!} \int\limits_{-\infty}^{\infty} dx_1 \cdots \int\limits_{-\infty}^{\infty} dx_m \prod_{i=1}^{m} e^{-\frac{x_i^2}{2tp(1-p)}} \prod_{1\le i<j\le m} |x_j - x_i| \,,$$

Then, going to rescaled variables $y_i = x_i/\sqrt{tp\,(1-p)}$ we obtain

$$\mathcal{P}_t\left(X^0\right) = \frac{1}{[p(1-p)t]^{m(m-1)/4}} \frac{I_{m,1/2}}{(2\pi)^{\frac{m}{2}}\, m!} s_\chi\left(1, \dots, 1\right) \left[1 + O\left(\frac{(\log t)^3}{\sqrt{t}}\right)\right], \tag{63}$$

where

$$I_{m,k} \equiv \int\limits_{-\infty}^{\infty} dy_p \cdots \int\limits_{-\infty}^{-\infty} dy_2 \int\limits_{-\infty}^{-\infty} dy_1 \exp\left(-\frac{1}{2}\sum\nolimits_{i=1}^{m} y_i^2\right) \prod_{1\le i<j\le m} |y_j - y_i|^{2k}$$

$$= (2\pi)^{m/2} \prod_{l=1}^{m} \frac{\Gamma\left(lk+1\right)}{\Gamma(k+1)} \tag{64}$$

is the Mehta integral [?], which first appeared in context of Gaussian random matrix ensembles. Finally one can use the following formula for Schur function [?]

$$s_\chi\left(1, \dots, 1\right) = \prod_{1\le i<j\le m} \frac{\chi_i - i - \chi_j + j}{j - i}, \tag{65}$$

which results in the expression for the survival probability.

$$
\begin{aligned}
\mathcal{P}_t\left(X^0\right) &= \frac{1}{\left[p(1-p)t\right]^{m(m-1)/4}}\frac{2^m}{\pi^{m/2}}\prod_{l=1}^{m}\frac{\Gamma\left(l/2+1\right)}{l!} \\
&\times\prod_{1\le i<j\le m}\left(x_j^0-x_i^0\right)\left[1+O\left(\frac{(\log t)^3}{\sqrt{t}}\right)\right]
\end{aligned} \tag{66}
$$

After reexpressing the gamma functions in terms of the factorials we obtain the form given in (10,13).

4.2 Semi-Vicious walkers.

4.2.1 The case of generic $\kappa < 1$.

To study the asymptotic behaviour of the survival probability for the case of general κ one can use the following integral representation for the transition probability

$$
\begin{aligned}
P_t\left(X,X^0\right) &= \prod_{i=1}^{m}\oint_{C_0}\frac{dz_i}{2\pi \mathrm{i} z_i}\frac{d\xi_i}{2\pi \mathrm{i}\xi_i}\left(\frac{1-\kappa\xi_i}{1-\kappa z_i}\right)^{i-1}\xi_i^{x_m^0-x_i^0+1}z_i^{x_i-x_m^0} \\
&\times\Lambda^t\left(Z\right)\prod_{1\le i,j\le m}\frac{1}{\xi_i-z_j}\prod_{1\le i<j\le m}(z_j-z_i)(\xi_j-\xi_i)
\end{aligned} \tag{67}
$$

where the integration in each variable is along a small circle around zero, $|z_i|<|\xi_j|$ for any $i,j=1,\dots,m$. This representation can be reduced to the form (41) by the direct integration over each ξ_j $j=1,\dots,m$. This is done by accounting a contribution to the integral coming from all the poles $\xi_j = z_i$ $i=1,\dots,m$.

Using the representation (67) one can approximate the integral over z_i by its large t saddle point asymptotics.

$$
\begin{aligned}
&\prod_{i=1}^{m}\oint_{C_0}\frac{dz_i}{2\pi \mathrm{i} z_i}\left(1-\kappa z_i\right)^{-i}z_i^{x_i-x_m^0}\Lambda^t\left(Z\right) \\
&\times\prod_{1\le i,j\le m}^{p}\frac{1}{\xi_i-z_j}\prod_{1\le i<j\le m}(z_j-z_i), \\
&=\left(\frac{p}{1-p}\right)^{\frac{m(m-1)}{2}}\prod_{i=1}^{m}\frac{v_i^{1-m}e^{tf_i\left(\frac{(1-v_i)p}{(1-p)v_i}\right)}}{\sqrt{2\pi t v_i(1-v_i)}}\prod_{1\le i<j\le m}(v_j-v_i) \\
&\times\prod_{1\le i,j\le m}\left(\xi_i-\frac{(1-v_j)\,p}{(1-p)\,v_j}\right)\prod_{i=1}^{m}\left(1-\kappa\frac{(1-v_j)\,p}{(1-p)\,v_j}\right)^{-i+1}\left(1+O\left(\frac{1}{t}\right)\right)
\end{aligned} \tag{68}
$$

When $\kappa > (1-p)\,v_j/(p\,(1-v_j))$ the contour being deformed to the steepest descent one meets the pole $z=1/\kappa$. Therefore, one has to extract the contribution coming from this pole. It however turns out to be negligible when $\kappa<1$. Indeed being given by the pole at the point $z=1/\kappa$ it contains the factor $e^{tf_i(1/\kappa)}$. Because for $(1-p)\,v_i/(p\,(1-v_i))<\kappa<1$ the inequality $f_i\,(1/\kappa)<0$ holds, the pole contribution is exponentially small in t, and, hence, is $O(1/t)$. Now, one has to calculate the sum of the r.h.s. of (68) over all final configurations X, i.e. over the range of $v_1,\dots,\nu_m$. As above, we argue that the main contribution into this sum comes from the domain (59). Using the taylor expansion at $\nu_i = p$ and replacing the sum by the integral we obtain

$$\begin{aligned} \mathcal{P}_t\left(X^0\right) &= \frac{1}{[p(1-p)t]^{m(m-1)/4}} \frac{I_{m,1/2}}{(2\pi)^{\frac{m}{2}}\, m!} \\ &\times \prod_{i=1}^{m} \oint_{C_{|\xi_i|=r>1}} \frac{d\xi_i}{2\pi \mathrm{i} \xi_i} \left(\frac{1-\kappa\xi_i}{1-\kappa}\right)^{i-1} \prod_{1\le i<j\le m} (\xi_j - \xi_i) \\ &\times \prod_{1\le i,j\le m} (\xi_i - 1)^{-m} \xi_i^{x_m^0 - x_i^0 + 1} \left[1 + O\left(\frac{(\log t)^3}{\sqrt{t}}\right)\right], \end{aligned} \tag{69}$$

Note that the r.h.s. of (68) has a pole singularity beyond the range (59) at $v_j = p\kappa/(1-p+p\kappa)$, which, at the first glance, could make a summation over X problematic. It is however "at the exponential tail", namely coming with a factor of order of $e^{tf_i(1/\kappa)}$, and, as such, is of much smaller order than any terms we are accounting for. In practice of course there must not be any singularities in the expression under the sum, which means that the one in (68) must be compensated by other correction terms of the same order. This problem can be avoided by expanding the term $(1-\kappa z_i)^{-i}$ into the power series in powers of (κz_i) before the integration over z_i and then integrating the resulting series term by term. For $\kappa < 1$ the convergence of these series will be exponentially fast, i.e. the number of terms, which will finally contribute into the leading asymptotics is $O\left(|\log \kappa|^{-1}\right)$. Thus, for $\kappa < 1$ the terms of the expansion of $(1-\kappa z_i)^{-i}$, which must be taken into account, are of finite degree. Therefore, these terms do not affect the location of the saddle point, and their value can be taken as the leading term of the expansion in the effective summation range (59), yielding the above result. The only case when this is not true is when κ approaches one, which is considered in the next subsection. Writing the above product of integrals in the determinant form we obtain

$$\begin{aligned} \mathcal{P}_t\left(X^0\right) &\simeq [p(1-p)t]^{-\frac{m(m-1)}{4}} (1-\kappa)^{-\frac{m(m+1)}{2}} \frac{2^m}{\pi^{m/2}} \\ &\times \prod_{l=1}^{m} \frac{\Gamma(l/2+1)}{l!} \det[g_{i,j}(x_m^0 - x_i^0)]_{1\le i,j\le m} \end{aligned} \tag{70}$$

where the function $g_{i,j}(x)$ is defined as follows

$$f_{i,j}(x) = \oint_{C_0} \frac{d\xi}{2\pi \mathrm{i}} \frac{(1-\kappa-\kappa\xi)^{i-1}(1+\xi)^x}{\xi^j}.$$

In the limit $\kappa \to 0$ the determinant in (70) is reduced to the Schur function

$$\lim_{\kappa\to 0} \det[g_{i,j}(x_m^0 - x_i^0)] = s_\chi(1,\ldots,1), \tag{71}$$

where χ is the partition (50), which restores the result (66).

4.2.2 The limiting case $\kappa \to 1$.

Now we consider the limiting case

$$t \to \infty, \kappa \to 1, (1-\kappa)\sqrt{t} = const.$$

To this end, we apply the program outlined above. We start with the formula (41) and expand the term $(1-\kappa z_i)^{-i+1}$ into the taylor series.

$$P_t\left(X,X^0\right) = \sum_{\{n_i\}\in\mathbb{Z}^m_{\geq 0}} \prod_{i=2}^{m} \kappa^{n_i} \binom{i+n_i-1}{n_i}$$
$$\times \det\left(\oint \frac{dz_i}{2\pi \mathrm{i} z_i}\left(1-p+\frac{p}{z_i}\right)^t z_i^{x_i-x_m^0+n_i} z_i^{x_m^0-x_j^0}(1-\kappa z_i)^{j-1}\right)_{1\leq i,j\leq m} \quad (72)$$

An integral in the determinant can be evaluated in the saddle point approximation, the analysis being similar to the one we did before, (53)-(58), with the same function $f_i(z)$, (53) and v_i depending on n_i,

$$v_i = \frac{x_i+n_i-x_m^0}{t}. \quad (73)$$

What is special about the limit $\kappa\to 1$ is that the saddle point can coincide with a zero of the term $(1-\kappa z)^j$ within the effective range of the summation over v_i. Therefore instead of expanding this term into the Taylor series, we leave it in the integral as it is, while the rest can be expanded around the saddle point as usual. Then we use the following formula for Hermit polynomials

$$\int_{-\infty}^{\infty} e^{-x^2}(x-\beta)^n\, dx = \sqrt{\pi}\left(\frac{i}{2}\right)^n H_n(i\beta). \quad (74)$$

As a result we obtain

$$\oint \frac{dz}{2\pi \mathrm{i} z}\left(1-p+\frac{p}{z}\right)^t z^{x_i-x_m^0+n_i} z^{x_m^0-x_j^0}(1-\kappa z)^{j-1} \quad (75)$$
$$= e^{-tf_i(z^*)}(z^*)^{x_m^0-x_j^0-1}\int_{-\infty}^{\infty}\frac{d\xi}{2\pi}e^{-t\left|f_i''(z^*)/2\right|\xi^2}\left(1-\kappa\left(z^*+\mathrm{i}\xi\right)\right)^{j-1}$$
$$= \frac{e^{tf_i\left(\frac{(1-v_i)p}{(1-p)v_i}\right)}}{\sqrt{\pi}\,(2t)^{j/2}} H_{j-1}\left(\sqrt{\frac{tv_i(1-v_i)}{2}}\left(\frac{1}{\kappa}\frac{(1-p)\,v_i}{(1-v_i)\,p}-1\right)\right) \quad (76)$$
$$\times\ \kappa^{j-1}\left(\frac{p}{1-p}\right)^{j-x_m^0+x_j^0}\frac{v_i^{x_m^0-x_j^0-3j/2}}{(1-v_i)^{x_m^0-x_j^0-j/2}}\left(1+O\left(\frac{1}{t}\right)\right).$$

We argue then that the dominant range of the summation over X and $\{n_i\}_{i=1,\dots,m}$ is the domain

$$pt-\sqrt{t}\log t \leq x_i+n_i \leq pt+\sqrt{t}\log t, \quad (77)$$

where x_i varies within the range

$$x_1^0 \leq x_1 \leq \dots \leq x_m \leq t, \quad (78)$$

and

$$0 \leq n_i < \infty. \quad (79)$$

To this end, consider the integral (76) for some particular i and j. After expanding $(1-\kappa z_i)^j$ into the binomial sum, it becomes a finite sum of the terms like

$$(1-p)^{t-(n_i+x_i-x_j^0+k)}\, p^{n_i+x_i-x_j^0+k}\binom{t}{n_i+x_i-x_j^0+k},$$

where k is a finite integer, $0 \leq k \leq j$. Beyond the range (77) the Stirling formula estimates this to be $O(t^{-1/2}e^{-\frac{(\log t)^2}{2p(1-p)}})$. The summation over x_i, which includes at most t nonzero terms, multiplies this estimate by a factor of t. Finally the summation over n_i yields an additional factor $(1-\kappa)^{-i}$, which shows the order of the contribution from outside of the domain (77) being

$$O\left((1-\kappa)^{-i}\, t^{1/2} e^{-\frac{(\log t)^2}{2p(1-p)}}\right). \tag{80}$$

Below, the leading term of the sum of interest will be shown to decay at most as a power of t. Therefore, when κ is such that $(1-\kappa) = O(t^{-s})$ with any fixed $s > 0$, the term (80) is asymptotically negligible. One can approximate (76) using the Taylor formula, which yields

$$\det\left[H_{j-1}\left(\sqrt{\frac{tp\,(1-p)}{2}}\left(\frac{v_i - p}{p\,(1-p)} - 1 + \frac{1}{\kappa}\right)\right)\right]_{i,j=1}^{m} \tag{81}$$
$$\times \frac{\prod_{i=1}^{m} e^{-t\frac{(v_i-p)^2}{2p(1-p)}}}{\pi^{\frac{m}{2}}\left[2tp\,(p-1)\right]^{\frac{m(m+1)}{4}}}\left(1 + O\left(\frac{(\log t)^3}{\sqrt{t}}\right)\right).$$

This form can be simplified by adding to every line the lines below it with such coefficients, that all the terms of the Hermite polynomial except the highest one cancel.

$$\det\left[H_{j-1}\left(a_i\right)\right]_{i,j=1}^{m} = (-2)^{\frac{m(m-1)}{2}}\,\Delta(a_1,\ldots,a_m)$$

Thus, the survival probability has the following form

$$\mathcal{P}_t\left(X^0\right) = \sum_{\{x_i\},\{n_i\}} \frac{(-1)^{\frac{m(m-1)}{2}} \prod_{i=2}^{m} \kappa^{n_i} e^{-\frac{\left(x_i+n_i-x_m^0-pt\right)^2}{2p(1-p)t}} \binom{i+n_i-1}{n_i}}{(2\pi)^{\frac{m}{2}}\, t^{\frac{m(m+1)}{2}}\left[p\,(p-1)\right]^{\frac{m^2}{2}}} \tag{82}$$
$$\times\Delta(x_1, x_2+n_2,\ldots,x_m+n_m)\left[1 + O\left(\frac{(\log t)^3}{\sqrt{t}}\right)\right],$$

where the summation is over the domains of $\{x_i\}_{i=1}^m$ and $\{n_i\}_{i=2}^m$ defined by the inequalities (77)-(79). Due to the presence of Gaussian term $\exp\left(-t\,(v_i-p)^2/\left(2p\,(1-p)\,t\right)\right)$ the sum over n_i converges uniformly in x_i. Therefore one can interchange the order of summations over x_i and over n_i. This allows one to apply the Lemma (1) first to the former x_i and then to n_i. One also can take a limit $t \to \infty$ in the limits of the integration, which results in addition of asymptotically negligible terms. As the characteristic scale of n_i is of order of $\sqrt{t}$ an approximation of the binomial coefficient using the Stirling formula,

$$\binom{i+n}{n} = \frac{n^{i-1}}{(i-1)!}\left(1 + O\left(\frac{1}{n}\right)\right), \tag{83}$$

will be exact up to the correction term, which yields the error of order of $O\left(t^{-1/2}\right)$ in the final result. To write down the final formula for $\mathcal{P}\left(X^0\right)$ we introduce the rescaled variables

$$u_i = \frac{(x_i - pt)}{\sqrt{2tp\,(1-p)}}, \tag{84}$$

$$\nu_i = \frac{n_i}{\sqrt{2tp\,(1-p)}}. \tag{85}$$

and the transition parameter α, (15), which is constant in the limit under consideration. Then the formula (82) takes the form (16,17).

5 Asymptotical behaviour of $f_m(\alpha)$

The limiting behaviour of $f_m(\alpha)$ for $\alpha \to \infty$ expressed in terms of t must match with the results obtained for generic values of $\kappa < 1$. In the other hand the limit $\alpha \to 0$ is just the probability normalization of the ASEP, i.e. $f_m(0)$ must be equal to 1. Another limit $\alpha \to -\infty$ though has no a probabilistic meaning, can be treated as a particular limit of the generating function of the rescaled particle current in the TASEP, see Section 2. To study the asymptotical behaviour of the $f_m(0)$ we prove the following three lemmas.

Lemma 2

$$\lim_{\alpha\to\infty} \alpha^{\frac{m(m-1)}{2}} f_m(\alpha) = C^{\infty} I_{m,1/2}, \tag{86}$$

$$C^{\infty} = \frac{2^{\frac{m(m-3)}{4}}}{m!\pi^{m/2}}$$

where $I_{m,1/2}$ is the Mehta's integral defined in (64).

Proof. Let us make a variable change under the integral (17) introducing new integration variables

$$\varphi_1 = u_1, \tag{87}$$

$$\varphi_i = \nu_i + u_i, i = 2, \dots, m, \tag{88}$$

$$\mu_{i-1} = \alpha\nu_i, i = 2, \dots, m. \tag{89}$$

In the new variables the integral (17) can be written as

$$f(\alpha) = \frac{(-2)^{\frac{m(m-1)}{2}}}{\pi^{\frac{m}{2}} 2! \cdots (m-2)!} \frac{1}{\alpha^{m(m-1)/2}} \prod_{i=1}^{m-1} \int_0^{\infty} d\mu_i \mu_i^{i-1} e^{-\mu_i} g(\mu_1, \dots, \mu_{m-1}; \alpha), \tag{90}$$

where we introduce the notation

$$g(\mu_1, \dots, \mu_{m-1}; \alpha) = \int_{-\infty}^{\infty} d\varphi_1 \int_{\varphi_1 + \frac{\mu_1}{\alpha}}^{\infty} d\varphi_2 \int_{\varphi_2 + \frac{\mu_2 - \mu_1}{\alpha}}^{\infty} d\varphi_3 \tag{91}$$
$$\cdots \int_{\varphi_{m-1} + \frac{\mu_{m-1} - \mu_{m-2}}{\alpha}}^{\infty} d\varphi_m e^{-\left(\varphi_m^2 + \cdots + \varphi_1^2\right)} \Delta(\varphi_1, \dots, \varphi_m).$$

The function $g(\mu_1, \dots, \mu_{m-1}; \alpha)$ is bounded uniformly in $\alpha \in \mathbb{R}$.

$$|g(\mu_1, \dots, \mu_{m-1}; \alpha)| \leq 2^{-m(m+1)/4} I_{m,1/2},$$

which can be shown by replacing the Van der Monde determinant under the integral by its absolute value and extending the lower integration limits to minus infinity. Therefore the function under the integral in (90) is uniformly bounded and integrable. By the dominating convergence theorem one can interchange the limit $\alpha \to \infty$ and integration. Then, for the function $g(\mu_2, \dots, \mu_m; \alpha)$ we have

$$\lim_{\alpha\to\infty} g(\mu_2, \dots, \mu_m; \alpha) = I_{m,1/2} \frac{(-1)^{\frac{m(m-1)}{2}} 2^{-m(m+1)/4}}{m!}. \tag{92}$$

Remarkably the limiting value does not depend on the variables $\{\mu_1, \ldots, \mu_{m-1}\}$. Therefore the integration in (90) can be performed independently for each $i = 2, \ldots, m$ resulting in $(i-1)!$, which yields the result (86). ■

Lemma 3

$$\lim_{\alpha \to 0} f(\alpha) = C^0 I_{m,1} \tag{93}$$

where

$$C^0 = \left[(2\pi)^{\frac{m}{2}} \, 2! \cdots m! \right]^{-1}$$

Proof. In the domain of integration the absolute value of the expression under the integral (17) is bounded by its particular case corresponding to $\alpha = 0$. Therefore, by the dominating convergence theorem we can interchange the integration and the limit $\alpha \to 0$, setting $\alpha = 0$ directly under the integral. Let us make the variable change

$$\begin{aligned} \chi_1 &= u_1 \\ \chi_i &= \nu_i + u_i, i = 2, \ldots, m. \end{aligned} \tag{94}$$

Then the integral takes the form

$$\int_{-\infty}^{\infty} d\chi_1 \int_{\chi_1}^{\infty} du_2 \int_{u_2}^{\infty} du_3 \cdots \int_{u_{m-1}}^{\infty} du_m \int_{u_2}^{\infty} d\chi_2 \cdots \int_{u_m}^{\infty} d\chi_m \tag{95}$$
$$\times e^{-\chi_1^2} \prod_{i=2}^{m} (\chi_i - u_i)^{i-2} e^{-\chi_i^2} \Delta(\chi_1, \ldots, \chi_m).$$

This integrals over u_i, for $i = 1, \ldots, M$ can be evaluated step by step. First for $i = m$ we have

$$\begin{aligned} & \int_{u_{m-1}}^{\infty} du_m \int_{u_m}^{\infty} d\chi_m e^{-\chi_m^2} (\chi_m - u_m)^{m-2} \Delta(\chi_1, \ldots, \chi_m) \\ = \; & \frac{1}{m-1} \int_{u_{m-1}}^{\infty} d\chi_m e^{-\chi_m^2} (\chi_m - u_{m-1})^{m-1} \Delta(\chi_1, \ldots, \chi_m), \end{aligned}$$

which can be checked by differentiation of both sides in u_{m-1} and noting that both sides vanish in the limit $u_{m-1} \to \infty$. On the next step the integral over u_{m-1} can be calculated by parts.

$$\begin{aligned} & \int_{u_{m-2}}^{\infty} du_{m-1} \int_{u_{m-1}}^{\infty} d\chi_{m-1} \int_{u_{m-1}}^{\infty} d\chi_m e^{-\chi_{m-1}^2 - \chi_m^2} \\ & \times (\chi_{m-1} - u_{m-1})^{m-3} (\chi_m - u_{m-1})^{m-1} \Delta(\chi_1, \ldots, \chi_m) \\ = \; & \frac{1}{m-2} [\int_{u_{m-2}}^{\infty} d\chi_{m-1} \int_{u_{m-2}}^{\infty} d\chi_m e^{-\chi_{m-1}^2 - \chi_m^2} \\ & \times (\chi_{m-1} - u_{m-2})^{m-2} (\chi_m - u_{m-2})^{m-1} \Delta(\chi_2, \ldots, \chi_m) \\ & - (m-1) \int_{u_{m-2}}^{\infty} du_{m-1} \int_{u_{m-1}}^{\infty} d\chi_{m-1} \int_{u_{m-1}}^{\infty} d\chi_m e^{-\chi_{m-1}^2 - \chi_m^2} \\ & \times (\chi_{m-1} - u_{m-1})^{m-2} (\chi_m - u_{m-1})^{m-2} \Delta(\chi_1, \ldots, \chi_m)]. \end{aligned}$$

The second term cancels because of the antisymmetry of the Van der Monde determinant with respect to interchange of χ_m and χ_{m-1}. Iterating this procedure we remove $(m-1)$ integrals in the variables $u_2, \ldots, u_M$.

$$f(0) = \frac{1}{(m-1)!} \int_{-\infty}^{\infty} d\chi_1 \int_{\chi_1}^{\infty} d\chi_2 \cdots \int_{\chi_1}^{\infty} d\chi_m \qquad (96)$$
$$\times e^{-\chi_1^2} \prod_{i=2}^{m} (\chi_i - \chi_1)^{i-1} e^{-\chi_i^2} \Delta(\chi_1, \ldots, \chi_m)$$

A symmetrization of the expression under the integral in the variables $\chi_2, \ldots, \chi_m$ yields another Van der Monde determinant.

$$f(0) = \frac{(-1)^{\frac{m(m-1)}{2}}}{((m-1)!)^2} \int_{-\infty}^{\infty} d\chi_1 \int_{\chi_1}^{\infty} d\chi_2 \cdots \int_{\chi_1}^{\infty} d\chi_m e^{-\left(\chi_1^2+\cdots+\chi_m^2\right)} \left|\Delta(\chi_1, \ldots, \chi_m)\right|^2.$$

Finally we add this integral to the $(m-1)$ similar ones, obtained by interchanging χ_1 with each of $\chi_2, \ldots, \chi_m$, and divide the sum by m.

$$f(0) = \frac{2^{\frac{m(m-1)}{2}}}{\pi^{\frac{m}{2}} 2! \cdots m!} \int_{-\infty}^{\infty} d\chi_1 \cdots \int_{-\infty}^{\infty} d\chi_m e^{-\left(\chi_1^2+\cdots+\chi_m^2\right)} \left|\Delta(\chi_1, \ldots, \chi_m)\right|^2$$

This gives us the stated result. ■

Lemma 4

$$\lim_{\alpha \to -\infty} e^{-\alpha^2 m(m-1)/4} f_m(\alpha) = C^{-\infty} I_{m-1,1} \qquad (97)$$

where

$$C^{-\infty} = \frac{(\pi 2)^{\frac{1-m}{2}} m^{m-1}}{2! \cdots (m-2)! \left((m-1)!\right)^2} \qquad (98)$$

Proof. We start from integral in (17) and make the variable change

$$\begin{aligned} x_i &= \nu_i + \frac{\alpha}{2} + u_1, i = 1, \ldots, m-1 \\ s_i &= |\alpha| (u_i - u_1), i = 1, \ldots, m-1 \\ s_1 &= u_1 - \frac{\alpha}{2}(m-1), \end{aligned}$$

which yields the integral of the following form.

$$\frac{e^{\frac{\alpha^2 m(m-1)}{4}}}{|\alpha|^{m-1}} \int_{-\infty}^{\infty} ds_1 e^{-s_1^2} \int_{0}^{\infty} ds_2 e^{-s_2} \int_{s_2}^{\infty} ds_3 e^{-s_3} \cdots \int_{s_{m-1}}^{\infty} ds_m e^{-s_m}$$
$$\times \int_{s_1+\frac{\alpha m}{2}}^{\infty} dx_2 \cdots \int_{s_1+\frac{\alpha m}{2}}^{\infty} dx_m \prod_{i=2}^{m} \left(x_i - s_1 - \frac{\alpha m}{2}\right)^{i-2} e^{-\sum_{i=2}^{m}\left(x_i^2 + \frac{s_i^2+2s_i x_i}{|\alpha|}\right)}$$
$$\times \Delta\left(s_1 + \frac{\alpha m}{2}, x_2 + \frac{s_2}{|\alpha|} \ldots, x_m + \frac{s_m}{|\alpha|}\right). \qquad (99)$$

Due to the presence of the Gaussian and exponential terms, the main contribution to the integral comes from finite values of $s_1, \ldots, s_m$ and

$x_2, \ldots, x_m$. Therefore in the leading order, up to the corrections of order of $O(1/|\alpha|)$, we can neglect the terms divided by $|\alpha|$, and extend the lower limits of integration over $x_2, \ldots, x_m$ to $-\infty$. The the integrals over $s_2, \ldots, s_m$ decouple from the other ones, and we evaluate them to $1/(m-1)!$. The Van der Monde determinant becomes antisymmetric with respect to permutations of the variables $x_2, \ldots, x_m$. As the integration is over the symmetric domain, we can leave only the antisymmetric part of the rest of the expression. The product $\prod_{i=2}^{m} \left(x_i - s_1 - \frac{\alpha m}{2}\right)^{i-2}$ then results in $(-1)^{\frac{m(m-1)}{2}} \Delta(x_2 \ldots, x_m) / (m-1)!$. After collecting the leading order terms from the first argument of

$$\Delta\left(s_1 + \frac{\alpha m}{2}, x_2 \ldots, x_m\right) \simeq \left(\frac{\alpha m}{2}\right)^{m-1} \Delta(x_2 \ldots, x_m),$$

the integral over s_1 decouples as well, and yields $\sqrt{\pi}$. We finally obtain

$$\frac{e^{\frac{\alpha^2 m(m-1)}{4}} \sqrt{\pi} (-1)^{\frac{m(m-1)}{2}}}{|\alpha|^{m-1} ((m-1)!)^2} \left(\frac{\alpha m}{2}\right)^{m-1} \int_{-\infty}^{\infty} dx_2 \cdots \int_{-\infty}^{\infty} dx_m e^{-\sum_{i=2}^{m} x_i^2} |\Delta(x_2, \ldots, x_m)|^2 . \tag{100}$$

Using the definition of Mehta integral, (64), and of the function $f_m(\alpha)$ we complete the proof. ■

The lemmas and the formula fro the Mehta (64) integral prove the results (18)-(20).

6 Conclusion and perspectives.

References