CFTs on Riemann Surfaces of Genus $g \geq 1$

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Declaration

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Summary

The purpose of this thesis is to argue that N-point functions of holomorphic fields in rational conformal field theories can be calculated by methods from algebraic geometry. We establish explicit formulae for the 2-point function of the Virasoro field on hyperelliptic Riemann surfaces of genus $g \geq 1$. N-point functions for higher N are obtained inductively, and we show that they have a nice graphical representation. We discuss the Virasoro 3-point function with application to the Virasoro $(2, 5)$ minimal model.

The formulae involve a finite number of parameters, notably the 0-point function and the Virasoro 1-point function, which depend on the moduli of the surface and can be calculated by differential equations. We propose an algebraic geometric approach that applies to any hyperelliptic Riemann surface. Our discussion includes a demonstration of our methods to the case $g = 1$.

This thesis is for my son Konrad Murchadh

Introduction

Quantum field theories are a major challenge for mathematicians. Apart from cases without interaction, the theories best understood at present are conformally invariant and do not contain massive particles. In dimension two, such *conformal field theories* (CFTs) are naturally defined on compact Riemann surfaces. This is the only case we will consider.

In order to actually compute the functions occurring in CFTs (like N-point functions $\langle \phi_1 \dots \phi_N \rangle$ of holomorphic fields, and more specifically the partition function $\langle \mathbb{1} \rangle$ for $N = 0$, where $\mathbb{1}$ is the identity field), one has to study their behaviour under changes of the conformal structure. This is done conveniently by first considering arbitrary changes of the metric. Such a change of $\langle \phi_1 \dots \phi_N \rangle$ is described by the corresponding $(N + 1)$-point function containing a copy of the Virasosoro field T. For this reason we shall investigate in Part I of this thesis the N-point functions of T (rather than of more general fields). These will then be available to our discussion of the metric dependence of N-point functions in Part II of the thesis.

The space of all possible conformal structures on the genus g surface is called the *moduli space* $\mathcal{M}_g$. Thus conformal quantum field theory is closely related to the study of functions on $\mathcal{M}_g$. For an important special class of CFTs (the rational ones) one obtains functions which are meromorphic on a compactification of $\mathcal{M}_g$ or of a finite cover.

One also needs the following generalisation: Conformal structures occur as equivalence classes of metrics, with equivalent metrics being related by Weyl transformations. The N-point functions of a CFT do depend on the Weyl transformation, but only in a way which can be described by a universal automorphy factor.

For $g = 1$ this can be made explicit as follows. The Riemann surfaces can be described as quotients $\mathbb{C}/\Lambda$, with a lattice Λ generated over $\mathbb{Z}$ by 1 and τ with $\tau \in \mathbb{H}^+$. The upper half plane $\mathbb{H}^+$ is the universal cover of $\mathcal{M}_1$, in other words its Teichmüller space. One has $\mathcal{M}_1 = SL(2,\mathbb{Z}) \setminus \mathbb{H}^+$. Meromorphic functions on finite covers of

$\mathcal{M}_1$ are called (weakly) *modular*. They can be described as functions on $\mathbb{H}^+$ which are invariant under a subgroup of $SL(2,\mathbb{Z})$ of finite index. $SL(2,\mathbb{Z})$ has therefore received the name *full modular group*.

Maps in $SL(2,\mathbb{Z})$ preserve the standard lattice $\mathbb{Z}^2$ together with its orientation and so descend to self-homeomorphisms of the torus. Inversely, every self-homeomorphism of the torus is isotopic to such a map. A modular function is a function on the space $\mathcal{L}$ of all lattices in $\mathbb{C}$ satisfying [38]

$$f(\lambda\Lambda) = f(\Lambda)\,, \quad \forall\Lambda \in \mathcal{L},\ \lambda \in \mathbb{C}^*\,. \tag{1}$$

$\mathcal{L}$ can be viewed as the space of all tori with a flat metric.

Conformal field theories on the torus provide many interesting modular functions, and modular forms. (The latter transform as $f(\lambda\Lambda) = \lambda^{-k} f(\Lambda)$ for some $k \in \mathbb{Z}$ which is specific to f, called the weight of f.)

Little work has been done so far on analogous functions for $g > 1$. The present thesis develops methods in this direction. The basic idea is that many of the relevant functions are algebraic. In order to proceed step by step, we will restrict our investigations to the locus of hyperelliptic curves, though the methods work in more general context as well.

We shall derive the ordinary differential equations that allow to compute the Virasoro N-point function on any hyperelliptic Riemann surface. For an important class of CFTs (the minimal models), the vector space of solutions is finite dimensional. It is shown that in the $(2,5)$ minimal model, our approach reproduces the known result.

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Notations and conventions

For any $k \geq 0$ and any rational function $R(x)$ of x with Laurent expansion

$$R(x) = \sum_{i \in \mathbb{Z}} a_i x^i$$

for large $|x|$, we define the polynomial

$$[R(x)]_{>k} := \sum_{i>k} a_i x^i \,. \tag{2}$$

Let $\mathbb{H}^+ := \{z \in \mathbb{C} | \, \Im(z) > 0\}$ be the complex upper half plane. $\mathbb{H}^+$ is acted upon by the *full modular group* $\Gamma_1 = SL(2,\mathbb{Z})$ with fundamental domain

$$\mathcal{F} := \left\{ z \in \mathbb{H}^+ \middle| \, |z| > 1, \, |\Re(z)| < \frac{1}{2} \right\} .$$

The operation of Γ_1 on $\mathbb{H}^+$ is not faithful whence we shall also consider the *modular group* $\overline{\Gamma}_1 := \Gamma_1/\{\pm \mathbb{I}_2\} = PSL(2,\mathbb{Z})$, (here $\mathbb{I}_2 \in GL(2,\mathbb{Z})$ is the identity matrix). We refer to S, T as the generators of Γ_1 (or of $\overline{\Gamma}_1$) given by the transformations

$$\begin{aligned} S &: z \mapsto -1/z \\ T &: z \mapsto z+1 \,. \end{aligned}$$

We shall use the convention [38]

$$G_{2k}(z) = \frac{1}{2} \sum_{n \neq 0} \frac{1}{n^{2k}} + \frac{1}{2} \sum_{m \neq 0} \sum_{n \in \mathbb{Z}} \frac{1}{(mz+n)^{2k}} \,,$$

and define E_{2k} by $G_k(z) = \zeta(k) E_k(z)$ for $\zeta(k) = \sum_{n \geq 1} \frac{1}{n^k}$, so e.g.

$$\begin{aligned} G_2(z) &= \frac{\pi^2}{6} E_2(z) \,, \\ G_4(z) &= \frac{\pi^4}{90} E_4(z) \,, \\ G_6(z) &= \frac{\pi^6}{945} E_6(z) \,. \end{aligned}$$

Let $(q)_n := \prod_{k=1}^n (1-q^k)$ be the q-Pochhammer symbol. The *Dedekind η function* is

$$\eta(z) := q^{\frac{1}{24}} (q)_\infty = q^{\frac{1}{24}} \left(1 - q + q^2 + q^5 + q^7 + \ldots\right) , \quad q = e^{2\pi i z} \,.$$

$\langle \mathbb{1} \rangle$ and $\langle T \rangle$ (or $\mathrm{A}_1, \ldots$) are parameters of central importance to this exposition. For better readibility, they appear in bold print ($\langle \mathbf{1} \rangle$ and $\langle \mathbf{T} \rangle$, or $\mathbf{A}_1, \ldots$) throughout.

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Part I

Virasoro correlation functions on hyperelliptic Riemann surfaces

Chapter 1

Introduction

Conformal Field Theories (CFTs) can be defined over arbitrary Riemann surfaces. A theory is considered to be solved once all of its N-point functions are known. We restrict our consideration to meromorphic CFTs [14] which are defined by holomorphic fields, and a rather specific class of Riemann surfaces.

The case of the Riemann sphere Σ_0 is easy, and for the torus Σ_1, one can use the standard tools of doubly periodic and modular functions ([41],[3] and more recently, e.g. [4],[29]). The case $g > 1$ is technically more demanding, however. Some progress has been made in the Vertex Operator Algebra (VOA) formalism by sewing surfaces of lower genus. There is no canonical way to do this and two different sewing procedures have been explored. Explicit formulae could be established for the genus two N-point functions for the free bosonic Heisenberg VOA and its modules ([24], [25]), and for the free fermion vertex operator superalgebra [35].

Instead, quantum field theory on a compact Riemann surface of any genus can be approached differently using methods from algebraic geometry ([30], [34], [10]) and complex analysis. N-point functions of holomorphic fields are meromorphic functions. That is, they are determined by their poles. By compactness of Σ_g, these functions are rational.

The present paper establishes explicit formulae for the 2-point functions of the Virasoro field over hyperelliptic genus-g Riemann surfaces Σ_g, where $g \geq 1$. N-point functions for $N \geq 3$ are obtained inductively from these, up to a finite number of parameters which in general cannot be determined by the methods presented in this paper. In comparison, the formulae given by the work of Mason, Tuite, and Zuevsky determine all constants, but are given in terms of infinite series.

We show that the N-point functions can be written in terms of a list of oriented

graphs. For $g = 1$ the result reduces to a formula which is very similar to eq. (3.19) in [19]. The method we used is essentially the one developed in [19] though it was found independently.

Although we deal with the Virasoro field, our method applies to more general holomorphic fields.

The material of the Chapters 2, 4, 5, and 6 is published in [22].

Chapter 2

The Virasoro OPE

In this chapter we define a global theory, a meromorphic conformal field theory on a Riemann surface, by glueing local data. For brevity, the global and the local theories will be treated on an equal footing. In order to consider the local data for themselves, it suffices to consider the Riemann surface given by the open unit disc, with its standard coordinate.

2.1 The vector bundle of holomorphic fields

For any Riemann surface S (not necessarily compact), we assume that the holomorphic fields of a meromorphic CFT on S form a vector bundle $\mathcal{F}$ over S with a *distinguished* trivialisation on every parametrized open set. More specifically, let (U, z) be a chart on S: The holomorphic map

$$U \xrightarrow{z} \mathbb{C}$$

is called a *coordinate* on U, and U will be referred to as a *coordinate patch*. We postulate that (U, z) induces a trivialization

$$\mathcal{F}|_U \xrightarrow{z^*} F \times U \,,$$

where F is the *standard fiber* of $\mathcal{F}$.

Remark 1. *An example is $\mathcal{F} = T^*S$, the cotangent bundle of S: Any chart (U, z) on S defines a nowhere vanishing section dz in T^*U and thus a trivialisation of T^*U. A different coordinate $z' = f(z)$ on U defines a different trivialisation (given by dz').*

In the present case, the fiber F is the infinite dimensional complex vector space of holomorphic fields. For $U' \subseteq U$, the trivialization corresponding to (U', z) is induced by the one for (U, z). For any coordinate patch U with coordinate z, elements of $\mathcal{F}|_U$ can be written as

$$\varphi_z(u) = (z^*)^{-1}(\varphi \times \{u\}) \,,$$

with $\varphi \in F$, $u \in U$. Abusing notations, we shall simply write $\varphi(z)$ where we actually mean $\varphi_z(u)$. (This will entail notations like $\hat{\varphi}(\hat{z})$ instead of $\varphi_{\hat{z}}(u)$ etc.). Thus an isomorphism between two coordinate patches on S induces an isomorphism between the corresponding fields. We postulate that the standard fiber F has an ascending filtration.

Remark 2. *It has been suggested to introduce the component of L_0 of the Virasoro field at this stage as F is filtrated as a result of the grading defined by the diagonalisation of L_0. As we shall see in the following section, however, L_0 is defined in terms of local coordinates while the filtration is postulated to be universal. Once everything is said and done, the two definitions are of course equivalent.*

As the base point $u \in S$ is varied, the filtration of the fibers $\mathcal{F}_u$ of $\mathcal{F}$ gives rise to a totally ordered set of finite rank subvector bundles of $\mathcal{F}$. On $\mathbb{P}^1_{\mathbb{C}}$, every such finite rank bundle admits a splitting into a direct sum of line bundles (Birkhoff-Grothendieck theorem). For $\mathbb{C} \subset \mathbb{P}^1_{\mathbb{C}}$, the degrees of the line bundles figuring in any such decomposition of finite rank subbundels of $\mathcal{F}$ define a $\mathbb{Z}$ grading on the fiber F. Thus to every (nonzero) homogeneous element $\varphi \in F$ there is associated the *(holomorphic) dimension* $h(\varphi)$ of φ. For quasi-primary (non-derivative) fields φ the degree of the corresponding line bundle is $2h(\varphi)$.

Remark 3. *All holomorphic fields can be obtained from differentiating quasi-primary fields, which implies that the action of L_0 is encoded by the line bundle structure.*

We shall assume that

$$h(\varphi) \geq 0 \,, \quad \forall\, \varphi \in F \,, \tag{2.1}$$

so that

$$F = \bigoplus_{h \in \mathbb{N}_0} F(h) \,,$$

where $F(0) \cong \mathbb{C}$ is spanned by the identity field $\mathbb{1}$, and we assume that for any

$h \in \mathbb{N}_0$, the dimension of $F(h)$ is finite. Since in a conformal field theory fields of finite dimension only are considered, it is sufficient to deal with finite sums.

It may be useful to compare our formalism to the approach by P. Goddard [14] where only the case $g = 0$ is discussed in detail. Goddard interprets F as a dense subspace of a space of states $\mathcal{H}$ using the field-state correspondence. He works on $\mathbb{C} \subset \mathbb{P}^1_{\mathbb{C}}$. In our notation this corresponds to the identity map id : $U \to \mathbb{C}$. Our field $\psi_{id}(z)$ is Goddard's $V(\psi, z)$. We will not use the field-state correspondence and reserve the word *state* for something different. Our notion of state on a Riemann surface S is a map $\langle \quad \rangle$ from products of fields $\Psi = \psi_{z_1}(p_1) \otimes \ldots \otimes \psi_{z_N}(p_N)$ to numbers $\langle \Psi \rangle \in \mathbb{C}$, in analogy to the language of operator algebra theory. We will not use the interpretation of fields as operators, however, since the necessary ordering is unnatural for $g > 1$.

2.2 Meromorphic conformal field theories

Let S_g be a connected Riemann surface of genus $g \geq 1$ (when the genus is fixed, we shall refer to S_g simply as S). We don't give a complete definition of a *meromorphic conformal field theory* here, but the most important properties are as follows [28]:

1. For $i = 1, 2$, let S_i be a Riemann surface and let $\mathcal{F}_i$ be a rank r_i vector bundle over S_i. Let $p_i^* \mathcal{F}_i$ be the pullback bundle of $\mathcal{F}_i$ by the morphism $p_i : S_1 \times S_2 \to S_i$. Let

$$\mathcal{F}_1 \boxtimes \mathcal{F}_2 := p_1^* \mathcal{F}_1 \otimes p_2^* \mathcal{F}_2$$

be the rank $r_1 r_2$ vector bundle whose fiber at $(z_1, z_2) \in S_1 \times S_2$ is $\mathcal{F}_{1,z_1} \otimes \mathcal{F}_{2,z_2}$. We are now in position to define N-point functions for bosonic fields. Let $\mathcal{F}$ be the vector bundle introduced in section 2.1. A *state* on S is a multilinear map

$$\langle \quad \rangle : \qquad \mathfrak{S}_*(\mathcal{F}) \quad \to \quad \mathbb{C},$$

where $\mathfrak{S}_*(\mathcal{F})$ denotes the set obtained by restricting the symmetric algebra $\mathfrak{S}(\mathcal{F})$ to fibers away from the partial diagonals

$$\Delta_N := \{(z_1 \ldots, z_N) \in S^N | \, z_i = z_j, \text{ for some } i \neq j\},$$

for any $N \in \mathbb{N}$. For ease of notations, when writing $\otimes$ and $\boxtimes$ we shall in the following actually mean the respective symmetrized product.

Locally, over any $U^N \subseteq S^N \setminus \Delta_N$ such that (U, z) defines a chart on S, a state is the data for any $N \in \mathbb{N}$ of an N-linear holomorphic map

$$\begin{aligned} \langle \quad \rangle : \qquad & F^{\otimes N} \times U^N \quad \to \quad \mathbb{C} \\ & (\varphi_1, z_1) \boxtimes \ldots \boxtimes (\varphi_N, z_N) \mapsto \langle \varphi_1(z_1) \otimes \ldots \otimes \varphi_N(z_N) \rangle \end{aligned}$$

satisfying the following conditions:

(a) $\langle \quad \rangle$ is compatible with the Operator Product Expansion (OPE). (The OPE is defined below in point 3, and the compatibility condition is explained in point 4.)

(b) For $\varphi_1 = \mathbb{1} \in F(0)$, the identity field, we have

$$\langle \mathbb{1}(z_1) \otimes \varphi_2(z_2) \otimes \ldots \otimes \varphi_N(z_N) \rangle = \langle \varphi_2(z_2) \otimes \ldots \otimes \varphi_N(z_N) \rangle \, .$$

Remark 4. *In standard physics' notation the symbol for the symmetric tensor product is omitted. We shall adopt this notation and write*

$$\langle \varphi_1(z_1) \ldots \varphi_N(z_N) \rangle$$

instead of $\langle \varphi_1(z_1) \otimes \ldots \otimes \varphi_N(z_N) \rangle$ *but keep in mind that each* z_i *lies in an individual copy of* U *whence the* $\varphi_i(z_i)$ *are elements in different copies of* F *and multiplication is meaningless.*

Since each φ_i is defined over U, we may view $\langle \varphi_1(z_1) \ldots \varphi_n(z_N) \rangle$ as a function of $(z_1, \ldots, z_N) \in U^N$. We call it the *$N$-point function* of the fields $\varphi_1, \ldots, \varphi_N$ over U. For example, the zero-point function[1] $\langle \mathbb{1} \rangle$ is a complex number.

Remark 5. *One can make contact to the notion of N-point function used in [14] by considering states for manifolds with boundary (see G. Segal's axioms) [32].*

2. Fields are understood by means of their N-point functions. A field ϕ is zero if all N-point functions involving ϕ vanish. That is, for any $N \in \mathbb{N}$, $N \geq 2$, and any set $\{\phi_2, \ldots \phi_N\}$ of fields,

$$\langle \phi(z_1) \, \phi_2(z_2) \, \ldots \, \phi_N(z_N) \rangle = 0 \, .$$

[1] henceforth denoted by $\langle \mathbf{1} \rangle$

3. We assume the existence of an OPE on F, in particular for any $m \in \mathbb{Z}$ of a linear degree m map

$$N_m : \quad F \otimes F \ \rightarrow \ F \, .$$

N_m has degree m if for any $\varphi_1, \varphi_2 \in F$, $N_m(\varphi_1, \varphi_2)$ has holomorphic dimension

$$m + h(\varphi_1) + h(\varphi_2) \, .$$

Note that the degree condition is void when $N_m(\varphi_1, \varphi_2)$ is the zero field.

Remark 6. *For $\varphi \in F$, the family of induced linear maps $N_m(\varphi, \ ) : F \ \rightarrow \ F$ indexed by $m \in \mathbb{Z}$ span a vertex operator algebra (VOA) (in particular a Lie algebra), with*

$$Y(\varphi, z) = \sum_{m \in \mathbb{Z}} N_m(\varphi, \ ) \, z^m$$

being the vertex operator associated with φ [11]. In particular, $L_0 = N_{-2}(T, \ )$.

4. While fields and coordinates are local objects, states should contain global information. A state is said to be *compatible with the OPE* if for any $N \in \mathbb{N}$, $N \geq 2$, and whenever $\varphi_1, \ldots, \varphi_N$ are holomorphic fields over a coordinate patch $U \subset S$, the corresponding N-point function has a Laurent series expansion in z_1 about $z_1 = z_2$ given by

$$\begin{aligned} &\langle \varphi_1(z_1) \, \varphi_2(z_2) \ldots \ \varphi_N(z_N) \rangle \\ &\qquad = \sum_{m \geq m_0} (z_1 - z_2)^m \langle N_m(\varphi_1, \varphi_2)(z_2) \, \varphi_3(z_3) \ldots \varphi_N(z_N) \rangle, \end{aligned}$$

for some $m_0 \in \mathbb{Z}$. Symbolically we write

$$\varphi_1(z_1) \, \varphi_2(z_2) \mapsto \sum_{m \geq m_0} (z_1 - z_2)^m N_m(\varphi_1, \varphi_2)(z_2) \, .$$

This arrow defines the OPE of $\varphi_1, \varphi_2 \in \mathcal{F}|_U$. We postulate that every OPE admits compatible states.

Remark 7. *Physicists write an equality here. Recall however that $\otimes$ is understood on the l.h.s.*

5. We have $N_m(\varphi, \mathbb{1}) = 0$ for $\varphi \in F$ and $m < 0$. Define the derivative of a field φ

by

$$\partial\varphi := N_1(\varphi, \mathbb{1}) .$$

Equivalently, $\partial\varphi$ is defined by prescribing

$$\langle \partial\varphi(z)\varphi_2(z_2) \ \dots \ \varphi_N(z_N)\rangle := \partial_z \langle \varphi(z)\, \varphi_2(z_2) \ \dots \ \varphi_N(z_N)\rangle ,$$

for all N-point functions involving φ.

6. In conformal field theories, one demands the existence of a *Virasoro field* $T \in F(2)$ which satisfies

$$N_{-1}(T, \varphi) = \partial\varphi , \tag{2.2}$$

whenever φ is a holomorphic section in $U \times F$.

In standard textbooks (e.g. [11]) the Virasoro algebra is required in addition to eq. (2.2) and $h(T) = 2$. The latter is equivalent to the **Virasoro OPE** [20], which is the specific arrow for the fields $\varphi_1 = \varphi_2 = T$ in point 4. The Virasoro OPE actually follows from the assumptions made in Section 2.1 and the properties 1-6 above:

Lemma 8. *In local coordinates z and w, the Virasoro OPE has the form*

$$T(z)T(w) \mapsto \frac{c/2}{(z-w)^4} .\mathbb{1} + \frac{1}{(z-w)^2} (T(z) + T(w)) + \Phi(w) + O(z-w) , \tag{2.3}$$

for some $c \in \mathbb{C}$.

The constant c is called the *central charge* of the theory. Note that

$$\Phi = N_0(T, T) - \frac{\partial^2 T}{2} .$$

Proof. (e.g. [20]) By assumption (2.1), all holomorphic fields have non-negative dimension, and $h(T) = 2$. This yields the lowest order term, since $F(0)$ is spanned by the identity field $\mathbb{1}$. Symmetry (point 1) implies the existence of a field Ω, of dimension 2, and of a constant $c \in \mathbb{C}$, such that

$$\begin{aligned} T(z)T(w) \mapsto & \frac{\frac{c}{2}.\mathbb{1}}{(z-w)^4} + \frac{\Omega(z) + \Omega(w)}{(z-w)^2} + O(1) \\ = & \frac{\frac{c}{2}.\mathbb{1}}{(z-w)^4} + \frac{2\Omega(w)}{(z-w)^2} + \frac{\partial\Omega(w)}{(z-w)} + O(1) . \end{aligned}$$

Thus $N_{-1}(T, T) = \partial\Omega$, from which (considering dimensions) we conclude $\Omega = T$. □

Example 1. *A CFT containing the identity field* $\mathbb{1}$*, the Virasoro field* T *and which is closed under* $N_m(.,.)$ *for* $m \in \mathbb{Z}$ *is said to be* ***generated by*** T*. A CFT is* ***minimal*** *if it has only finitely many non-isomorphic irreducible representations of the VOA (or the OPE). Minimal CFTs generated by* T *are called* ***minimal models****. They are parametrised by unordered pairs* (μ, ν) *of natural numbers* $\mu, \nu > 1$ *s.t.* $\gcd(\mu, \nu) = 1$*. For the* (μ, ν) *minimal model the number of such representations is (e.g., [3], [1])*

$$\frac{(\mu - 1)(\nu - 1)}{2} .$$

For $(\mu, \nu) = (2, 3)$*, one has* $F(0) = \mathbb{1}.\mathbb{C}$*, and* $F(n) = 0$ *for* $n > 0$ *(so that* $T = 0$*). The* $(2, 5)$ *minimal model has two irreducible representations, the vacuum representation* M_1 *for the lowest conformal weight (or holomorphic dimension)* $\kappa_1 - \frac{11}{60} = 0$*, and another representation* M_2 *corresponding to the conformal weight* $\kappa_2 - \frac{11}{60} = -\frac{1}{5}$ *([3], table*[2] *8.1., p. 243). (For* $g = 1$*, the* 0*-point functions are characters, and* κ_s *for* $1 \leq s \leq (\nu - 1)/2$ *is the leading power in the small q-expansion of the character* $\langle \mathbf{1} \rangle_s$ *in the* $(2, \nu)$ *minimal model. We will use* κ_s *to parametrise the characters in Chapter 8.)*

[2]Note that there is a typo in the value for the conformal weight $h_{1,2}$ of $\langle \mathbf{1} \rangle_2$ in [3].

Chapter 3

Analytic calculation of the Virasoro N-point function for some genus 1 minimal models

Virasoro N-point functions on the torus can be determined using techniques from VOA theory [41]. In this chapter we illustrate a more elementary approach using the Weierstrass $\wp$-function.

3.1 The Virasoro N-point function in the $(2,5)$ minimal model

We consider a conformal field theory (CFT) over the torus $\Sigma_1 = \mathbb{C}/\Lambda$ for $\Lambda = \mathbb{Z}.1+\mathbb{Z}.\tau$ with the property that the space $F(4)$ of the holomorphic fields of dimension $h = 4$ is one dimensional. Thus for the field Φ of the OPE (2.3), we have

$$\Phi = \left(\frac{3K}{10} - \frac{1}{2}\right) \partial^2 T, \tag{3.1}$$

for some $K \in \mathbb{C}$. The model, in which (3.1) holds true, is referred to as the $(2,5)$ **minimal model**. $K = 1$ is such a theory, but our calculations will show that anyhow. For any $N \geq 1$, the Virasoro N-point function $\langle T(z_1) \dots T(z_N) \rangle$ is an element of the field

$$\mathbb{C}(\wp(z_1|\tau), \wp'(z_1|\tau), \dots, \wp(z_N|\tau), \wp'(z_N|\tau)) \,,$$

where $\wp$ is the Weierstrass function associated to Λ,

$$\wp(z|\tau) = \frac{1}{z^2} + \sum_{m,n}{}' \left\{ \frac{1}{(z-m-n\tau)^2} - \frac{1}{(m+n\tau)^2} \right\}, \tag{3.2}$$

and $\wp' = \partial\wp/\partial z$ is its derivative. $x = \wp(z|\tau)$ and $y = \wp'(z,|\tau)$ are related by the equation $y^2 = 4(x^3 - 30G_4 x - 70G_6)$, where for $k \geq 2$, G_{2k} are the holomorphic Eisenstein series. For $N = 1$, there actually exists a covering of Σ_1 by coordinate neighbourhoods for which $\langle T(z)\rangle = \langle \mathbf{T}\rangle$ is constant. Comparison of the singularities in (3.2) and the OPE (2.3), and using that holomorphic functions on the torus are constant, yields

$$\langle T(z)T(0)\rangle = \frac{c}{12}\langle \mathbf{1}\rangle \wp''(z|\tau) + 2\langle \mathbf{T}\rangle \wp(z|\tau) + C, \tag{3.3}$$

where $C \propto \langle \mathbf{1}\rangle$ is constant in position. About $z = 0$,

$$\wp(z|\tau) = z^{-2} + 6G_4 z^2 + O(z^4)\,,$$

so

$$\langle N_0(T,T)(w)\rangle = C + cG_4\langle \mathbf{1}\rangle = \langle \Phi(w)\rangle\,.$$

But in the $(2,5)$ minimal model,

$$\langle \Phi\rangle = 0$$

by eq. (3.1), since $\langle \partial^2 T\rangle = \partial^2\langle \mathbf{T}\rangle = 0$. ($\Phi$ is referred to as the **singular vector** in M_1 of the $(2,5)$ minimal model.) We conclude that

$$C = -c\langle \mathbf{1}\rangle G_4\,. \tag{3.4}$$

The Virasoro 2-point function in the $(2,5)$ minimal model is completely determined by the 0- and 1-point function. This result has been found previously by [7].

3.2 Higher N-point functions

It is worth mentioning that the method of matching the singularities of the Virasoro OPE with suitable derivatives of the Weierstrass $\wp$-function, as demonstrated for the 2-point function in the preceding section, allows also to compute the N-point func-

tions of the Virasoro field for higher N, by recursion. For $N = 3$ we have

$$\begin{aligned}&\langle T(z)T(w)T(u)\rangle\\&= \frac{c/2}{(z-w)^4}\langle \mathbf{T}\rangle + \frac{1}{(z-w)^2}\left\{\langle T(z)T(u)\rangle + \langle T(w)T(u)\rangle\right\} + \langle \Phi(w)T(u)\rangle + O(z-w).\end{aligned}$$

On the other hand, the general form of the Virasoro 3-point function, considered as a function of z, is

$$\begin{aligned}\langle T(z)T(w)T(u)\rangle = {}& \frac{c/2}{(z-w)^4}\,\langle \mathbf{T}\rangle\\ &+ \frac{1}{(z-w)^2}\left\{\frac{c}{6}\langle \mathbf{1}\rangle \wp'' + 4\langle \mathbf{T}\rangle\wp + 2C\right\}\\ &+ \frac{1}{(z-w)}\left\{\dots\right\}\\ &+ (z-w)^0\left\{\frac{c}{24}\langle \mathbf{1}\rangle\wp^{(4)} + \langle \mathbf{T}\rangle\wp + \langle \Phi(w)T(u)\rangle\right\}\\ &+ O(z-w)\,. \end{aligned} \tag{3.5}$$

Here and henceforth we denote by $\wp^{(k)}$ for $k \geq 0$ the function $\partial^k\wp(w-u|\tau)$, where $\wp^{(0)} = \wp$. We have omitted the $(z-w)^{-1}$-term which will drop out as the symmetry between z and w is restored. On the other hand, considering the singularities and the symmetries between z, w and u yields the ansatz

$$\begin{aligned}&\langle T(z)T(w)T(u)\rangle\\ &\quad = \tilde{A}\left\{\wp''(z-w|\tau) + \wp''(z-u|\tau) + \wp''(w-u|\tau)\right\}\\ &+ \tilde{B}\left\{\wp(z-w|\tau) + \wp(z-u|\tau) + \wp(w-u|\tau)\right\}\\ &+ \tilde{C}\left\{\wp(z-w|\tau)\wp(z-u|\tau) + \wp(w-z|\tau)\wp(w-u|\tau) + \wp(u-z|\tau)\wp(u-w|\tau)\right\}\\ &+ \tilde{D}\,\wp(z-w|\tau)\wp(z-u|\tau)\wp(w-u|\tau)\\ &+ \tilde{E}, \qquad\qquad \tilde{A},\tilde{B},\tilde{C},\tilde{D},\tilde{E} \in \mathbb{C}. \end{aligned} \tag{3.6}$$

By comparison with (3.5), we conclude

$$\tilde{A} = \frac{c}{12}\langle \mathbf{T}\rangle, \quad \tilde{B} = 12C, \quad \tilde{C} = 2\langle \mathbf{T}\rangle, \quad \tilde{D} = c\langle \mathbf{1}\rangle.$$

In the $(2,5)$ minimal model, we have by eq. (3.4),

$$\tilde{B} = -12c\langle \mathbf{1}\rangle G_4\,.$$

Moreover, by eq. (3.1),

$$\langle \Phi(w)T(u)\rangle = \left(\frac{3K}{10} - \frac{1}{2}\right)\partial_w^2\langle T(w)T(u)\rangle$$

so the coefficient of $(z-w)^0$ reads

$$\frac{c}{24}\langle \mathbf{1}\rangle \wp^{(4)} + \langle \mathbf{T}\rangle \wp + \langle \Phi(w)T(u)\rangle = \frac{cK}{40}\langle \mathbf{1}\rangle \wp^{(4)} + \frac{3K}{5}\langle \mathbf{T}\rangle \wp'' \,.$$

For the term of order zero, we obtain a cubic equation in $\wp$,

$$(3c - 3Kc)\,\wp^3 + \frac{1}{5}\,(5c + 40 - 18K)\,\langle \mathbf{T}\rangle\,\wp^2 - 54c(1-K)\,G_4\,\langle \mathbf{1}\rangle\,\wp + E = 0,$$
$$\text{where} \quad E = \tilde{E} + (-9c - 60 + 36K)\;G_4\,\langle \mathbf{T}\rangle + 84cK\,G_6\,\langle \mathbf{1}\rangle.$$

The equation is satisfied iff $K = 1$, $c = -\frac{22}{5}$, and

$$\tilde{E} = -\frac{78}{5}\,G_4\,\langle \mathbf{T}\rangle + \frac{1848}{5}\,G_6\,\langle \mathbf{1}\rangle.$$

In particular, $\langle TTT\rangle$ yields 1-point functions of $N_k(T, N_\ell(T,T))$. More complicated fields are treated analogously.

3.3 The Virasoro N-point function in the $(2,7)$ minimal model

Let us now consider a CFT on $\mathbb{C}/\Lambda$ with the property that

$$\alpha\partial^4 T + \beta N_0(N_0(T,T),T) + \gamma N_0(T,\partial^2 T) + \delta N_0(\partial T,\partial T) = 0\,, \qquad (3.7)$$
$$\alpha,\beta,\gamma,\delta \in \mathbb{C} \text{ not all zero.}$$

(In the $(2,7)$ minimal model, the second singular vector occurs at level 6, cf. [3], p. 243, subsequent to table 8.1.) From the Virasoro OPE follows

$$\begin{aligned}\langle T(z)T(0)\rangle &= \frac{c}{12}\langle \mathbf{1}\rangle \wp''(z|\tau) + 2\langle \mathbf{T}\rangle \wp(z|\tau) + C \\ &= \frac{c/2}{z^4}\langle \mathbf{1}\rangle + \frac{2}{z^2}\langle \mathbf{T}\rangle + (C + cG_4\langle \mathbf{1}\rangle) + z^2(12G_4\langle \mathbf{T}\rangle + 10cG_6) + O(z^4)\,,\end{aligned}$$

so

$$\begin{aligned}\partial_z^2\langle T(z)T(0)\rangle &= \frac{c}{12}\langle \mathbf{1}\rangle \wp^{(4)}(z|\tau) + 2\langle \mathbf{T}\rangle \wp''(z|\tau) \\ &= \frac{10c}{z^6}\langle \mathbf{1}\rangle + \frac{12}{z^4}\langle \mathbf{T}\rangle + 20cG_6\langle \mathbf{1}\rangle + 24\langle \mathbf{T}\rangle G_4 + O(z^2) \\ \Rightarrow \quad \langle N_0(\partial^2 T, T)(w)\rangle &= 24\langle \mathbf{T}\rangle G_4 + 20cG_6\langle \mathbf{1}\rangle \end{aligned} \tag{3.8}$$

Since odd order derivatives of $\wp$ are odd,

$$\begin{aligned}\partial_z\partial_w\langle T(z)T(w)\rangle &= -\partial_z^2\langle T(z)T(w)\rangle \\ \Rightarrow \quad \langle N_0(\partial T, \partial T)(w)\rangle &= -\langle N_0(\partial^2 T, T)(w)\rangle\end{aligned}$$

For $\langle N_0(N_0(T,T),T)\rangle$ we have by eq. (3.7) and the fact that $\langle \mathbf{T}\rangle = \text{const.}$,

$$\langle N_0(N_0(T,T),T)\rangle = \frac{\gamma-\delta}{\beta}\langle N_0(T,\partial^2 T)\rangle\,, \quad \text{provided}\,\beta \neq 0\,. \tag{3.9}$$

The constant $\frac{\gamma-\delta}{\beta}$ can be determined independently and is thus assumed to be known. Also, the central charge of the $(2,7)$ minimal model is known to be $c = -\frac{68}{7}$. Comparison of eq. (3.5) with eq. (3.6) yields

$$\langle \Phi(w)T(u)\rangle = -2c\langle \mathbf{1}\rangle \wp^3 + (c+8)\langle \mathbf{T}\rangle \wp^2 + (24C + 60cG_4\langle \mathbf{1}\rangle)\wp + E$$

where

$$E = \tilde{E} - (9c+60)G_4\langle \mathbf{T}\rangle + 140cG_6\langle \mathbf{1}\rangle\,.$$

So $\langle \Phi(w)T(u)\rangle$ is known upon knowledge of $\langle \mathbf{1}\rangle$, $\langle \mathbf{T}\rangle$, C, and $\tilde{E}$. To determine $\tilde{E}$, it is sufficient to know $\langle N_0(\Phi,T)\rangle$. By eq. (3.9),

$$\begin{aligned}\langle N_0(\Phi,T)\rangle &= \langle N_0(N_0(T,T) - \frac{1}{2}\partial^2 T, T)\rangle \\ &= \langle N_0(N_0(T,T),T)\rangle - \frac{1}{2}\langle N_0(\partial^2 T, T)\rangle \\ &= (\frac{\gamma-\delta}{\beta} - \frac{1}{2})\,\langle N_0(\partial^2 T, T)\rangle\,.\end{aligned}$$

But $\langle N_0(\partial^2 T, T)\rangle$ is given by eq. (3.8). We conclude that in the $(2,7)$ minimal model, the Virasoro 3-point function is determined by the 0-point, and the Virasoro 1-point and 2-point function.

Chapter 4

The Virasoro 1-point function in rational coordinates, for genus $g \geq 1$

4.1 Change to rational coordinates

Let Σ_1 be a compact Riemann surface of genus $g = 1$. Such a manifold is biholomorphic to the torus $\mathbb{C}/\Lambda$ (with the induced complex structure), for the lattice Λ spanned over $\mathbb{Z}$ by 1 and some $\tau \in \mathbb{H}^+$, unique up to an $SL(2,\mathbb{Z})$ transformation. Here $\mathbb{H}^+$ denotes the upper complex half plane. We denote by z the local coordinate on Σ_1 and by $z_1, \ldots, z_N$ the corresponding variables of the N-point functions on Σ_1 [3]. Recall that N-point functions on Σ_1 are elements of $\mathbb{C}(\wp(z_1|\tau), \wp'(z_1|\tau), \ldots, \wp(z_N|\tau), \wp'(z_N|\tau))$, where $\wp$ is the Weierstrass function associated to Λ, and $\wp' = \partial\wp/\partial z$. Instead of z we shall work with the pair of complex coordinates

$$x = \wp(z|\tau), \quad y = \wp'(z|\tau)\,. \tag{4.1}$$

We compactify the variety $\{(x,y) \in \mathbb{C}^2 |\, y^2 = p(x)\}$ with

$$p(x) = 4(x^3 - 30G_4 x - 70G_6)$$

by including the point $x = \infty$ (corresponding to $z = 0 \bmod \Lambda$), and view x as a holomorphic function on $\mathbb{C}/\Lambda$ with values in $\mathbb{P}^1_{\mathbb{C}}$. Thus y defines a ramified double cover of $\mathbb{P}^1_{\mathbb{C}}$.

N-point functions can be expressed in terms of $\wp(z_1|\tau), \wp'(z_1|\tau), \ldots, \wp(z_N|\tau), \wp'(z_N|\tau)$,

or equivalently as rational functions of $x_1, y_1, \ldots, x_N, y_N$. The latter possibility generalizes much more easily to higher genus. Instead, one can try to work with the Jacobian of the curve and the corresponding ϑ functions. This would generalize to arbitrary curves, but it is unknown for which conformal field theories this is possible.

If $g > 1$, one can write Σ_g as the quotient of $\mathbb{H}^+$ by a Fuchsian group, but working with a corresponding local coordinate z becomes difficult (e.g. [8], and more recently [23]). We shall consider *hyperelliptic* Riemann surfaces Σ_g only, where $g \geq 1$. Such surfaces are defined by

$$\Sigma_g : \quad y^2 = p(x), \tag{4.2}$$

where p is a polynomial in x of degree $n = 2g + 1$ (the case $n = 2g + 2$ is equivalent and differs from the former by a rational transformation of $\mathbb{C}$ only). We assume p has no multiple zeros so that Σ_g is *regular*. Locally we will work with one complex coordinate, either x or y. None of them is a function of the other on all of the affine variety (4.2) (whence in particular we refrain from writing $y(x)^2 = p(x)$). By definition, the function x is called a *locally admissible coordinate* on an open set $U \subset \Sigma_g$ if (U, x) defines a chart, and analogously for y. Thus x is an admissible coordinate away from the ramification points (where $p = 0$), whereas y is admissible away from the locus where $p' = 0$. Let us recapitulate the behaviour of T under coordinate transformations.

Definition 1. *Given a holomorphic function f (with non-vanishing first derivative f'), we define the* ***Schwarzian derivative*** *of f by*

$$S(f) := \frac{f'''}{f'} - \frac{3[f'']^2}{2[f']^2} \, .$$

The Schwarzian derivative S has the following well-known properties:

1. $S(\lambda f) = S(f)$, $\forall\ \lambda \in \mathbb{C}$, $f \in \mathcal{D}(S)$, the domain of S (f holomorphic with $f' \neq 0$).

2. Suppose $f : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ is a linear fractional (Möbius) transformation,

$$f : \ z \mapsto f(z) = \frac{az + b}{cz + d} \, , \quad \text{where} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{C}).$$

Then $f \in \mathcal{D}(S)$, and $S(f) = 0$.

3. Let $f, g \in \mathcal{D}(S)$ be such that $f \circ g$ is defined and lies in $\mathcal{D}(S)$. Then

$$S(f \circ g) = [g']^2 \, S(f) \circ g + S(g) \, .$$

Remark 9. *Let $p, y \in \mathcal{D}(S)$ with $y^2 = p(x)$. Then by the properties 1 and 3 of the Schwarzian derivative,*

$$S(y) = S(p) + \frac{3}{8}\left[\frac{p'}{p}\right]^2 \, . \tag{4.3}$$

Direct computation yields [13]

Lemma 10. *Let T be the Virasoro field in the coordinate x. We consider a coordinate change $x \mapsto \hat{x}(x)$ such that $\hat{x} \in \mathcal{D}(S)$, and set*

$$\hat{T}(\hat{x}(x))\left[\frac{d\hat{x}}{dx}\right]^2 = T(x) - \frac{c}{12} S(\hat{x})(x).\mathbb{1} \, . \tag{4.4}$$

Then $\hat{T}$ satisfies the OPE (2.3) in $\hat{x}$. □

Corollary 11. *Let S_g be a Riemann surface of genus $g \geq 2$ with a complex projective coordinate covering (i.e. a covering by coordinate patches whose respective local coordinates differ by a Möbius transformation only). Then for any state $\langle \quad \rangle$ on S_g, and for any local coordinate x in this class, $\langle T(x) \rangle \, (dx)^2$ defines a global section of $(T^* S_g)^{\otimes 2}$.*

This section is *holomorphic* by assumption.

Proof. By property 2 of the Schwarzian derivative, and by eq. (4.4),

$$\langle T(x) \rangle \, (dx)^2 = \langle \hat{T}(\hat{x}) \rangle \, (d\hat{x})^2 \, .$$ □

The equation in the proof of Corollary 11 can be read as the cocycle condition on the 0-cochain $\langle T(x) \rangle \, (dx)^2$. For a general coordinate covering $\mathcal{U}$ of S_g, $g \geq 2$, $\langle T(x) \rangle \, (dx)^2$ will fail to define a 0-cocycle. According to eq. (4.4), however, its 1-coboundary is given by $\frac{c}{12}\langle \mathbf{1} \rangle S(\hat{x})(x)(dx)^2$ which by property 3 of the Schwarzian derivative satisfies the 1-cocycle condition

$$S(y \circ x)(z)(dz)^2 = S(y)(x(z))(dx)^2 + S(x)(z)(dz)^2 \, .$$

A 0-cochain with this property is called a *projective connection.* Thus for any projective connection $\frac{c}{12}\,C(z)$, the difference $\langle T(z)\rangle\,(dz)^2 - \frac{c}{12}\,C(z)$ defines a global holomorphic section in $(T^*S_g)^{\otimes 2}$ [16],

$$\left(\langle T(z)\rangle\,(dz)^2 - \frac{c}{12}\,C(z)\right) \in\ H^0\left(\mathcal{U}, \mathcal{O}((T^*S_g)^{\otimes 2})\right)\,. \tag{4.5}$$

Example 2. *Let S_g be a Riemann surface of arbitrary genus. Let T be defined by holomorphic fields of massless free fermions on S_g. In this case, the projective connection $\frac{c}{12}C$ is known as the Bergman projective connection ([18],[9],[31]).*

By the Riemann-Roch Theorem (e.g. [8]), the affine linear space of projective connections has dimension

$$\dim_{\mathbb{C}} H^0((T^*S_g)^{\otimes 2}) = 3(g-1)\,, \quad (g \geq 2)\,. \tag{4.6}$$

Remark 12. *Eq. (4.6) is wrong for $g = 0$ and $g = 1$. The difference from the correct result is the dimension of the automorphism group of the Riemann surface, ($dim_{\mathbb{C}} SL(2,\mathbb{C}) = 3$ and $dim_{\mathbb{C}}(\mathbb{C},+) = 1$, respectively). For $g \geq 2$, this dimension is zero.*

Example 3. *Let Σ_1 be a $g = 1$ Riemann surface. Then $T^*\Sigma_1$ is trivial. When one uses local coordinates given by the affine structure on Σ_1 [16], then $\langle T(z)\rangle$ is constant.*

4.2 Calculation of the Virasoro 1-point function

Associate to the hyperelliptic surface Σ its field of meromorphic functions $K = \mathbb{C}(x,y)$. Then K is a field extension of $\mathbb{C}$ of trancendence degree one, since $y^2 = p(x)$ where p is a polynomial in x with coefficients in $\mathbb{C}$. The two sheets (corresponding to the two signs of y) are exchanged by a Galois transformation.

In what follows, we set

$$p(x) = \sum_{k=0}^{n} a_k x^{n-k},$$

where $n = 2g+1$, or $n = 2g+2$. For convenience of application, we shall treat both cases separately throughout this section, though they are of course equivalent.

Theorem 1. ***(On the Virasoro 1-point function)***

For $g \geq 1$, let Σ_g be the genus g hyperelliptic Riemann surface

$$\Sigma : \quad y^2 = p(x),$$

where p is a polynomial with $\deg p = n$.

1. *As $x \to \infty$,*

$$\begin{aligned} \langle T(x) \rangle &\sim x^{-4}\,, && \text{for even } n\,, \\ \langle T(x) \rangle &= \frac{c}{32}\, x^{-2} \langle \mathbf{1} \rangle + O(x^{-3})\,, && \text{for odd } n\,. \end{aligned}$$

2. *We have*

$$p \langle T(x) \rangle = \frac{c}{32} \frac{[p']^2}{p} \langle \mathbf{1} \rangle + \frac{1}{4} \Theta(x,y)\,, \tag{4.7}$$

where $\Theta(x,y)$ is a ***polynomial*** *in x and y. More specifically, we have the Galois splitting*

$$\Theta(x,y) = \Theta^{[1]}(x) + y \Theta^{[y]}(x). \tag{4.8}$$

Here $\Theta^{[1]}$ is a polynomial in x of degree $n-2$ with the following property:

   (a) *If n is even,* $\left[\Theta^{[1]} + \frac{c}{8} \frac{[p']^2}{p} \langle \mathbf{1} \rangle\right]_{>n-4} = 0.$

   (b) *If n is odd,* $\left[\Theta^{[1]} + \frac{c}{8} \left(n^2 - 1\right) a_0 x^{n-2} \langle \mathbf{1} \rangle\right]_{>n-3} = 0.$

$\Theta^{[y]}$ is a polynomial in x of degree $\frac{n}{2} - 4$ if n is even, resp. $\frac{n-1}{2} - 3$ if n is odd, provided $g \geq 3$.

Proof. 1. For $x \to \infty$, we perform the coordinate change $x \mapsto \tilde{x}(x) := \frac{1}{x}$. By property 2 of the Schwarzian derivative, $S(\tilde{x}) = 0$ identically, and

$$T(x) = \tilde{T}(\tilde{x}) \left[\frac{d\tilde{x}}{dx}\right]^2,$$

where $\left[\frac{d\tilde{x}}{dx}\right]^2 = x^{-4}$. If n is even, then $\tilde{x}$ is an admissible coordinate, so $\langle \tilde{T}(\tilde{x}) \rangle$ is holomorphic in $\tilde{x}$. If n is odd, then we may take $\tilde{y} := \sqrt{\tilde{x}}$ as coordinate. $\frac{d\tilde{y}}{dx} = -\frac{1}{2} x^{-1.5}$, and according to eq. (4.4) and eq. (4.3),

$$T(x) = \frac{c}{32}\, x^{-2} + \frac{1}{4}\, \breve{T}(\tilde{y})\, x^{-3}, \tag{4.9}$$

where $\langle \check{T}(\tilde{y})\rangle$ is holomorphic in $\tilde{y}$.

2. $\langle T(x)\rangle$ is a meromorphic function of x and y over $\mathbb{C}$, whence rational in either coordinate. Thus upon multiplication by some suitable polynomial Q if necessary, we are dealing with an element in $\mathbb{C}[x, y]$, the ring of polynomials in x and y. Since $y^2 = p(x)$, $\mathbb{C}[x, y]$ is a module over $\mathbb{C}[x]$ spanned by 1 and y, so we may assume $Q \in \mathbb{C}[x]$. We conclude that the quotient field of $\mathbb{C}[x, y]$ is a vector space over the field of rational functions in x alone, spanned by 1 and y. In particular, we have a Galois-splitting

$$\langle T(x)\rangle = \langle T(x)\rangle^{[1]} + y\,\langle T(x)\rangle^{[y]}.$$

$\langle T(x)\rangle$ is $O(1)$ in x iff this holds for its Galois-even and its Galois-odd part individually, as there can't be cancellations between these. We obtain a Galois splitting for $\langle \hat{T}(y)\rangle$ by applying a rational transformation to $\langle T(x)\rangle$. From (4.4) and (4.3) follows

$$\begin{aligned} p\langle T(x)\rangle^{[1]} =& \frac{c}{32}\langle \mathbf{1}\rangle \frac{[p'(x)]^2}{p(x)} + \frac{1}{4}\,\Theta^{[1]}(x)\,, \\ p\langle T(x)\rangle^{[y]} =& \frac{1}{4}\,\Theta^{[y]}(x), \end{aligned}$$

where $\Theta^{[1]}$ and $\Theta^{[y]}$ are rational functions of x. We have

$$\begin{aligned} &\frac{1}{4}\Theta^{[1]} \\ &\quad = p\langle T(x)\rangle^{[1]} - \frac{c}{32}\langle \mathbf{1}\rangle \frac{[p']^2}{p} = \frac{1}{4}[p']^2\langle \hat{T}(y)\rangle^{[1]} + \frac{c}{12}\langle \mathbf{1}\rangle pS(p). \end{aligned}$$

The l.h.s. is $O(1)$ in x for finite x and away from $p = 0$ (so wherever x is an admissible coordinate) while the r.h.s. is holomorphic in $y(x)$ for finite x and away from $p' = 0$ (so wherever y is an admissible coordinate). The r.h.s. does not actually depend on y but is a function of x alone. Since the loci $p = 0$ and $p' = 0$ do nowhere coincide, we conclude that $\Theta^{[1]}$ is an *entire* function on $\mathbb{C}$. It remains to check that $\Theta^{[1]}$ has a pole of the correct order at $x = \infty$. We have

$$\frac{[p']^2}{p} = n^2 a_0 x^{n-2} + n(n-2)a_1 x^{n-3} + O(x^{n-4}). \tag{4.10}$$

(a) If n is even, then $p\langle T(x)\rangle^{[1]} = O(x^{n-4})$ as $x \to \infty$, by part 1. By eqs (4.7)

and (4.10), $\Theta^{[1]}(x)$ has degree $n-2$ in x. Moreover,

$$\begin{aligned}&\Theta^{[1]}(x)\\&= -\frac{c}{8}\left(n^2a_0x^{n-2} + n(n-2)a_1x^{n-3}\right)\langle\mathbf{1}\rangle + O(x^{n-4}).\end{aligned}$$

(b) If n is odd, then $p\langle T(x)\rangle^{[1]} = \frac{c}{32}a_0x^{n-2}\langle\mathbf{1}\rangle + O(x^{n-3})$ as $x \to \infty$, by eq. (4.9). Thus $\Theta^{[1]}$ has degree $n-2$ in x. Moreover, by eq. (4.7) and eq. (4.10),

$$\Theta^{[1]}(x) = -\frac{c}{8}\left(n^2-1\right)a_0x^{n-2}\langle\mathbf{1}\rangle + O(x^{n-3}).$$

Likewise, we have

$$\frac{1}{4}y\Theta^{[y]}(x) = yp\langle T(x)\rangle^{[y]} = \frac{1}{4}[p']^2y\langle\hat{T}(y)\rangle^{[y]};$$

the l.h.s. is $O(1)$ in x wherever x is an admissible coordinate while the r.h.s. is holomorphic in y wherever y is an admissible coordinate. Since y is a holomorphic function in x and in y away from $p = 0$ and away from $p' = 0$, respectively, this is also true for

$$\frac{1}{4}p\Theta^{[y]}(x) = p^2\langle T(x)\rangle^{[y]} = \frac{1}{4}p[p']^2\langle\hat{T}(y)\rangle^{[y]}.$$

Now the r.h.s. does no more depend on y but is a function of x alone, so the above argument applies to show that $p\Theta^{[y]} =: P$ is an entire function and thus a polynomial in x. We have $p|P$:

$$\frac{P}{y} = y\Theta^{[y]}(x) = y[p']^2\langle\hat{T}(y)\rangle^{[y]}$$

is holomorphic in y about $p = 0$. Since P is a polynomial in x, and p has no multiple zeros, we must actually have $y^2 = p$ divides P. This proves that $\Theta^{[y]}$ is a polynomial in x. The statement about the degree follows from part 1.

□

Remark 13. *The main purpose of Theorem 1 is to introduce the polynomial* Θ*. Part of the results actually follow from classical formulae for the projective connection.*

For instance, for n even and $g \geq 3$, we have [8]

$$p\langle T(x)\rangle\,(dx)^2 = \frac{c}{12}p\,C(x) + \langle \mathbf{1}\rangle \sum_{i=0}^{2g-2} \alpha_i x^i (dx)^2 + y\langle \mathbf{1}\rangle \sum_{j=0}^{g-3} \beta_j x^j (dx)^2$$

for constants α_i, β_j, in the notations of (4.5). Here the projective connection $\frac{c}{12}C$ on Σ is given by

$$p\,C(x) = \frac{3}{8}\left[\frac{[p']^2}{p}\right]_{\leq n-4} \langle \mathbf{1}\rangle (dx)^2\,,$$

and

$$\left[\Theta^{[1]}(x)\right]_{\leq n-4} = 4\langle \mathbf{1}\rangle \sum_{i=0}^{2g-2} \alpha_i x^i\,, \quad \Theta^{[y]}(x) = 4\langle \mathbf{1}\rangle \sum_{j=0}^{g-3} \beta_j x^j\,.$$

Eq. (4.9) (for odd n) follows from the formula for $C(x)$ on p. 20 in [9].

Chapter 5

The Virasoro 2-point function

5.1 Calculation of the 2-point function, for genus $g \geq 1$

We first need to introduce some notation. For the polynomial $\Theta = \Theta^{[1]} + y\Theta^{[y]}$ defined by eqs (4.7) and (4.8) of Theorem 1, we set

$$\Theta^{[1]} = \mathrm{A}_0 x^{n-2} + \sum_{k=1}^{n-2} \mathbf{A}_k x^{n-2-k}, \qquad \mathbf{A}_k \propto \langle \mathbf{1} \rangle \,.$$

Recall that A_0 is known constant multiple of $\langle \mathbf{1} \rangle$. In contrast, for $1 \leq k \leq n-2$, the proportionality factor in $\mathbf{A}_k \propto \langle \mathbf{1} \rangle$, though constant in position, is a yet unknown function of the moduli of the surface. Thus $\langle \mathbf{1} \rangle$ and the $\mathbf{A}_k$ define $n-1$ parameters of the theory which we shall focus on in Part II. (The parameters appear in bold print for better readiblity of the formulae.) For $g \leq 2$, $\Theta^{[y]}$ is absent so there are no additional parameters in this case.

It will be convenient to replace $\Theta^{[1]}(x) =: -\frac{c}{8}\Pi(x)$ for which we introduce even polynomials $\Pi^{[1]}$ and $\Pi^{[x]}$ such that

$$\Pi(x) =: \Pi^{[1]}(x) + x\Pi^{[x]}(x). \tag{5.1}$$

Likewise, there are even polynomials $p^{[1]}$ and $p^{[x]}$ such that

$$p(x) = p^{[1]}(x) + xp^{[x]}(x). \tag{5.2}$$

Lemma 14. *For any even polynomial q of x, we have*

$$\begin{aligned} &q(x_1)+q(x_2) + O((x_1-x_2)^4) \\ &\quad = 2q(\sqrt{x_1x_2}) + (x_1-x_2)^2\frac{1}{4}\left(\frac{q'(\sqrt{x_1x_2})}{\sqrt{x_1x_2}} + q''(\sqrt{x_1x_2})\right), \end{aligned}$$

and

$$\begin{aligned} &x_1q(x_1)+x_2q(x_2) + O((x_1-x_2)^4) \\ &= (x_1+x_2)\left\{q(\sqrt{x_1x_2}) + (x_1-x_2)^2\frac{1}{8}\left(\frac{3q'(\sqrt{x_1x_2})}{\sqrt{x_1x_2}} + q''(\sqrt{x_1x_2})\right)\right\}. \end{aligned}$$

Note that the polynomials q and q'' in $\sqrt{x_1x_2}$ are are actually polynomials in x_1x_2.

Proof. Direct computation. The calculation can be shortened by using

$$\begin{aligned} x_1 &= (1+\epsilon)\,x, \\ x_2 &= (1-\epsilon)\,x, \end{aligned}$$

where $|\epsilon| \ll 1$. □

Abusing notations, for $j = 1, 2$, we shall write $p_j = p(x_j)$ and $\Theta_j = \Theta(x_j, y_j)$ etc. For $k \geq 0$, we denote by $[R(x_1, x_2)]_{>k}$ the polynomial in $x = x_1$ defined by eq. (2), with x_2 held fixed, and let $[R(x_1, x_2)]^{>k}$ be the polynomial for the opposite choice $x = x_2$ (x_1 fixed).

Theorem 2. ***(The Virasoro 2-point function)***

For $g \geq 1$, let Σ_g be the hyperelliptic Riemann surface

$$\Sigma: \quad y^2 = p(x),$$

where p is a polynomial, $\deg p = n$ *odd. Let*

$$\langle T(x_1)T(x_2)\rangle_c := \langle \mathbf{1}\rangle^{-1}\langle T(x_1)T(x_2)\rangle - \langle \mathbf{1}\rangle^{-2}\langle T(x_1)\rangle\langle T(x_2)\rangle$$

be the connected 2-point function of the Virasoro field. We have

1.

$$\langle T(x_1)T(x_2)\rangle_c \; p_1p_2 = O(x_1^{n-3}). \tag{5.3}$$

2. *For $|x_1|, |x_2|$ small,*

$$\langle T(x_1)T(x_2)\rangle_c \, p_1 p_2 = \langle \mathbf{1}\rangle^{-1} R(x_1, x_2) + O(1)\,|_{x_1=x_2} \,,$$

where $R(x_1, x_2)$ is a rational function of x_1, x_2 and y_1, y_2, and $O(1)|_{x_1=x_2}$ denotes terms that are regular on the diagonal $x_1 = x_2$ and polynomial in x_1, x_2 and y_1, y_2.

3. *The rational function is given by*

$$\begin{aligned}
R(x_1, x_2) = {} & \frac{c}{4}\langle \mathbf{1}\rangle \frac{p_1 p_2}{(x_1 - x_2)^4} \\
& + \frac{c}{4} y_1 y_2 \langle \mathbf{1}\rangle \left(\frac{p^{[1]}(\sqrt{x_1 x_2})}{(x_1 - x_2)^4} + \frac{1}{2}(x_1 + x_2) \frac{p^{[x]}(\sqrt{x_1 x_2})}{(x_1 - x_2)^4} \right) \\
& + \frac{c}{32}\langle \mathbf{1}\rangle \frac{p_1' p_2'}{(x_1 - x_2)^2} \\
& + \frac{c}{32} y_1 y_2 \langle \mathbf{1}\rangle \left(\frac{\frac{1}{\sqrt{x_1 x_2}} (p^{[1]})'(\sqrt{x_1 x_2})}{(x_1 - x_2)^2} + \frac{3}{2}(x_1 + x_2) \frac{\frac{1}{\sqrt{x_1 x_2}} (p^{[x]})'(\sqrt{x_1 x_2})}{(x_1 - x_2)^2} \right) \\
& + \frac{1}{8} \frac{p_1 \Theta_2 + p_2 \Theta_1}{(x_1 - x_2)^2} \\
& + \frac{1}{8} \left(y_1 \Theta_2^{[y]} + y_2 \Theta_1^{[y]} \right) \left(\frac{p^{[1]}(\sqrt{x_1 x_2})}{(x_1 - x_2)^2} + \frac{1}{2}(x_1 + x_2) \frac{p^{[x]}(\sqrt{x_1 x_2})}{(x_1 - x_2)^2} \right) \\
& + \frac{c}{32} y_1 y_2 \langle \mathbf{1}\rangle \left(\frac{(p^{[1]})''(\sqrt{x_1 x_2})}{(x_1 - x_2)^2} + \frac{1}{2}(x_1 + x_2) \frac{(p^{[x]})''(\sqrt{x_1 x_2})}{(x_1 - x_2)^2} \right) \\
& - \frac{c}{32} y_1 y_2 \left(\frac{\Pi^{[1]}(\sqrt{x_1 x_2})}{(x_1 - x_2)^2} + \frac{1}{2}(x_1 + x_2) \frac{\Pi^{[x]}(\sqrt{x_1 x_2})}{(x_1 - x_2)^2} \right).
\end{aligned}$$

Here $p^{[1]}$ and $p^{[x]}$ and $\Pi^{[1]}$ and $\Pi^{[x]}$ are the even polynomials introduced in (5.2) and in (5.1), respectively.

4. For $R(x_1, x_2)$ thus defined, the connected Virasoro 2-point function reads

$$\begin{aligned}\langle \mathbf{1}\rangle\langle T(x_1)T(x_2)\rangle_c \; p_1p_2 \\ &= R(x_1,x_2) + P(x_1,x_2,y_1,y_2) \\ &- \frac{1}{8}a_0\left(x_1^{n-2}\Theta_2 + x_2^{n-2}\Theta_1\right) - \frac{c}{64}\langle\mathbf{1}\rangle(n^2-1)a_0^2x_1^{n-2}x_2^{n-2} \\ &- \frac{1}{8}y_1a_1x_1^{\frac{n}{2}-\frac{5}{2}}x_2^{\frac{n}{2}-\frac{1}{2}}\Theta_2^{[y]} - \frac{1}{8}y_2a_1x_1^{\frac{n}{2}-\frac{1}{2}}x_2^{\frac{n}{2}-\frac{5}{2}}\Theta_1^{[y]} \\ &- \frac{1}{16}y_1a_0x_1^{\frac{n}{2}-\frac{3}{2}}x_2^{\frac{n}{2}-\frac{1}{2}}\Theta_2^{[y]} - \frac{1}{16}y_2a_0x_1^{\frac{n}{2}-\frac{1}{2}}x_2^{\frac{n}{2}-\frac{3}{2}}\Theta_1^{[y]} \\ &- \frac{3}{16}y_1a_0x_1^{\frac{n}{2}-\frac{5}{2}}x_2^{\frac{n}{2}+\frac{1}{2}}\Theta_2^{[y]} - \frac{3}{16}y_2a_0x_1^{\frac{n}{2}+\frac{1}{2}}x_2^{\frac{n}{2}-\frac{5}{2}}\Theta_1^{[y]} \\ &- \frac{1}{16}y_1a_2x_1^{\frac{n}{2}-\frac{5}{2}}x_2^{\frac{n}{2}-\frac{3}{2}}\Theta_2^{[y]} - \frac{1}{16}y_2a_2x_1^{\frac{n}{2}-\frac{3}{2}}x_2^{\frac{n}{2}-\frac{5}{2}}\Theta_1^{[y]}\,,\end{aligned}$$

where

$$\begin{aligned}&P(x_1,x_2,y_1,y_2) \\ &\quad = P^{[1]}(x_1,x_2) + y_1P^{[y_1]}(x_1,x_2) + y_2P^{[y_2]}(x_1,x_2) + y_1y_2P^{[y_1y_2]}(x_1,x_2)\,.\end{aligned} \tag{5.4}$$

Here $P^{[1]}$, $P^{[y_1y_2]}$ and for $i = 1, 2$, $P^{[y_i]}$ are polynomials in x_1 and x_2 with

$$\begin{aligned}\deg_i P^{[1]} &= n-3 = \deg_i P^{[y_j]}\,, \quad \textit{for } j \neq i\,,\\ \deg_i P^{[y_i]} &= \frac{n-1}{2} - 3 = \deg_i P^{[y_1y_2]}\,.\end{aligned}$$

($\deg_i$ denotes the degree in x_i). Moreover, $P^{[1]}$, $P^{[y_1y_2]}$ and $y_1P^{[y_1]} + y_2P^{[y_2]}$ are symmetric under flipping $1 \leftrightarrow 2$. These four polynomials are specific to the state.

Proof. Direct computation (cf. Appendix). □

In the following, let $[T(x_1)T(x_2)]_{\text{reg.}} + \langle T(x_1)\rangle_c\langle T(x_2)\rangle_c.\mathbb{1}$ with $\langle T(x)\rangle_c = \langle\mathbf{1}\rangle^{-1}\langle T(x)\rangle$ be the regular part of the OPE on the hyperelliptic Riemann surface Σ,

$$\begin{aligned}T(x_1)T(x_2)\; p_1p_2 \mapsto \frac{c}{32}f(x_1,x_2)^2.\mathbb{1} + \frac{1}{4}f(x_1,x_2)\,(\vartheta_1+\vartheta_2) \\ + [T(x_1)T(x_2)]_{\text{reg.}}\; p_1p_2 + \langle T(x_1)\rangle_c\langle T(x_2)\rangle_c p_1p_2.\mathbb{1}\,.\end{aligned} \tag{5.5}$$

Here $f(x_1, x_2) := \left(\frac{y_1-y_2}{x_1-x_2}\right)^2$, and

$$\vartheta(x) := T(x)\, p - \frac{c}{32}\frac{[p']^2}{p}.\mathbb{1} \tag{5.6}$$

satisfies $\langle\vartheta(x)\rangle = \frac{1}{4}\,\Theta(x, y)$. $\vartheta(x)$ is holomorphic about $p = 0$ since by eqs (4.3) and (4.4),

$$\hat{T}(y) = \frac{4}{[p']^2}\left(\vartheta - \frac{c}{12} p\, S(p).\mathbb{1}\right),$$

where $S(p)$ is regular at $p = 0$.

5.2 Application to the $(2, 5)$ minimal model, in the case $n = 5$

In Section 2 we introduced the so-called normal ordered product

$$N_0(\varphi_1, \varphi_2)(x_2) = \lim_{x_1 \to x_2} [\varphi_1(x_1)\varphi_2(x_2)]_{\text{regular}}$$

of two fields φ_1, φ_2, where $[\varphi_1(x_1), \varphi_2(x_2)]_{\text{regular}}$ is the regular part of the OPE of φ_1, φ_2. For $\varphi_1 = \varphi_2 = T$ and the OPE (2.3), $\langle N_0(T, T)(x)\rangle$ can be determined from Theorem 2.4. To illustrate our formalism, we provide a short proof of the following well-known result ([1], Sect. 3):

Lemma 15. *The condition $N_0(T, T) \propto \partial^2 T$ implies $c = -\frac{22}{5}$ and*

$$N_0(T, T)(x) = \frac{3}{10}\,\partial^2 T(x). \tag{5.7}$$

Proof. The statement is local, so we may assume w.l.o.g. $g = 1$. In this case,

$$\Theta^{[1]}(x) = -4cx\langle\mathbf{1}\rangle + \mathbf{A}_1, \qquad \Theta^{[y]} = 0,$$

by Theorem 1.(2b). Using Corollary 11 and the transformation rule (4.4) for $x = \wp(z|\tau)$, we find

$$\langle T(x)\rangle = \frac{c}{32}\frac{[p']^2}{p^2}\langle\mathbf{1}\rangle - c\langle\mathbf{1}\rangle\frac{x}{p} + \frac{\langle\mathbf{T}\rangle}{p},$$

where by (4.7), $\langle \mathbf{T} \rangle = \frac{\mathbf{A}_1}{4}$. Direct computation shows that

$$\langle N_0(T,T)(x) \rangle = \alpha \, \partial^2 \langle T(x) \rangle$$

iff $\alpha = \frac{3}{10}$ and $c = -\frac{22}{5}$. Since by assumption the two underlying fields are proportional, the claim follows. □

The aim of this section is to determine at least some of the constants in the Virasoro 2-point function in the $(2,5)$ minimal model for $g = 2$. We will restrict our considerations to the case when n is *odd*. (Better knowledge about $\Theta^{[1]}$ when n is even doesn't actually provide more information, it just leads to longer equations.) In the first case to consider, namely $n = 5$, all Galois-odd terms are absent. Restricting to the Galois-even terms, condition (5.7) reads as follows:

Lemma 16. *In the* $(2,5)$ *minimal model for* $g \geq 1$*, we have*

$$\begin{aligned}
&\frac{7c}{640}\langle \mathbf{1} \rangle \frac{[p'']^2}{p^2} - \frac{7c}{960}\langle \mathbf{1} \rangle \frac{p' p'''}{p^2} + \frac{c}{1536}\frac{p^{(4)}}{p} \\
&+ \frac{1}{20}\frac{p''}{p^2}\Theta^{[1]} + \frac{3}{80}\frac{p'}{p^2}(\Theta^{[1]})' - \frac{3}{160}\frac{(\Theta^{[1]})''}{p} \\
&- \frac{1}{16}\langle \mathbf{1} \rangle^{-1}\left(\frac{(\Theta^{[1]})^2}{p^2} + \frac{(\Theta^{[y]})^2}{p}\right) \\
&+ \frac{1}{4}a_0\frac{x^{n-2}}{p^2}\Theta^{[1]} - \frac{1}{8}A_0 a_0 \frac{x^{2n-4}}{p^2} \\
&- \frac{c}{8\cdot 32}\,\frac{1}{xp}\left((\Pi^{[1]})' + x(\Pi^{[x]})'\right) \\
&- \frac{c}{256}\frac{1}{xp}\langle \mathbf{1} \rangle \left(-p^{(3)} - \frac{1}{2}\left(\frac{(p^{[1]})''}{x} - (p^{[x]})''\right) + \frac{1}{2x}\left(\frac{(p^{[1]})'}{x} + 5(p^{[x]})'\right)\right) \\
&= \quad \frac{P^{[1]}(x,x)}{p^2} + \frac{P^{[y_1y_2]}(x,x)}{p}.
\end{aligned}$$

Note that the equation makes good sense since the l.h.s. is regular at $x = 0$. For instance, $(\Pi^{[1]})'$ is an odd polynomial of x, so its quotient by x is regular.

Proof. Direct computation. □

Example 4. *When* $n = 5$*,*

$$\deg P^{[1]}(x,x) = 4, \quad P^{[y_1y_2]}(x,x) = 0\,.$$

Thus we have 5 *complex degrees of freedom. One of them is the constant*[1] $\langle \mathbf{1} \rangle$*, and according e. (4.6), at most* 3 *of them are given by* $\langle T(x) \rangle$ *(or by* $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3$*). Set*

$$\begin{aligned} P^{[1]}(x_1, x_2) = B_{2,2}\, x_1^2 x_2^2 \\ + B_{2,1}(x_1^2 x_2 + x_1 x_2^2) \\ + B_2(x_1^2 + x_2^2) + B_{1,1}\, x_1 x_2 \\ + B_1(x_1 + x_2) \\ + B_0\,, \end{aligned}$$

$B_0, B_1, B_{i,j} \propto \langle \mathbf{1} \rangle$ *are constant in position, for* $i, j = 1, 2$*. The additional constraint (5.7) provides knowledge of*

$$P^{[1]}(x, x) = B_{2,2}\, x^4 + 2B_{2,1}\, x^3 + (2B_2 + B_{1,1})\, x^2 + 2B_1\, x + B_0$$

only, so we are left with one unknown. As we shall argue in Section 6.2, it should be possible to fix the remaining constant using (5.7), once the Virasoro 3*-point function is taken into account.*

[1] Recall, however, that $\langle \mathbf{1} \rangle$ and $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3$ are functions of the moduli on the surface Σ_2.

Chapter 6

The Virasoro N-point function

N point functions for $N > 3$ can be established from the correlation functions for $N = 2, 3$, using symmetry arguments. Thus finding a routine to compute the N-point functions for all $N \geq 2$ goes with encoding their constituents graphically. For the Virasoro field the graphical description has been formulated by [19] for the genera $g = 0$ and $g = 1$, for which Zhu's recursion formulae were available [41]. Essentially the same inductive arguments prove our recursion formula for the Virasoro N point function for hyperelliptic Riemann surfaces of arbitrary genus.

6.1 Graph representation of the Virasoro N-point function for $g \geq 1$

For $g \geq 1$, let Σ_g be the genus g hyperelliptic Riemann surface

$$\Sigma : \quad y^2 = p(x),$$

where p is a polynomial, $\deg p = n$, with $n = 2g + 1$, or $n = 2g + 2$. Let $\mathcal{F}$ be the bundle of holomorphic fields introduced in Section 2. For $N \in \mathbb{N}$ and $j = 1, \ldots, N$, abusing notations, we shall write $p_j = p(x_j)$ and $\vartheta_j = \vartheta(x_j)$, where ϑ is the field defined by eq. (5.6).

Theorem 3. *Let $S(x_1, \ldots, x_N)$, $N \in \mathbb{N}$, be the set of oriented graphs with vertices $x_1, \ldots, x_N$, (not necessarily connected), subject to the following condition:*

> *$\forall\, i = 1, \ldots, N$, x_i has at most one ingoing and at most one outgoing line, and if (x_i, x_j) is an oriented line connecting x_i and x_j then $i \neq j$.*

Given a state $\langle \quad \rangle$ *on* Σ*, there is a multilinear map*

$$\langle \quad \rangle_r : \quad \mathfrak{S}_*(\mathcal{F}) \to \mathbb{C} ,$$

normalised such that $\langle \mathbb{1} \rangle_r = \langle \mathbf{1} \rangle$*, with the following properties:*

1. *For all* $k \in \mathbb{N}$*,* $k \geq 2$*, and any* $\varphi_2, \ldots, \varphi_k \in \{\mathbb{1}, T\}$*, we have*

$$\langle \mathbb{1}\, \varphi_2(z_2) \ldots \varphi_k(z_k) \rangle_r = \langle \varphi_2(z_2) \ldots \varphi_k(z_k) \rangle_r .$$

2. *For all* $k \in \mathbb{N}$*,* $\langle \vartheta_1 \ldots \vartheta_k \rangle_r$ *is a polynomial in* $x_1, \ldots, x_k$ *and* $y_1, \ldots, y_k$*.*

3. *We have*

$$\langle T(x_1) \ldots T(x_N) \rangle \, p_1 \ldots p_N = \sum_{\Gamma \in S(x_1, \ldots, x_N)} F(\Gamma) , \tag{6.1}$$

where for $\Gamma \in S(x_1, \ldots, x_N)$*,*

$$F(\Gamma) := \left(\frac{c}{2}\right)^{\sharp loops} \prod_{(x_i, x_j) \in \Gamma} \left(\frac{1}{4} f(x_i, x_j)\right) \left\langle \bigotimes_{k \in A_N \cap E_N{}^c} \vartheta(x_k) \bigotimes_{\ell \in (A_N \cup E_N)^c} T(x_\ell) p_\ell \right\rangle_r .$$

Here for any oriented edge $(x_i, x_j) \in \Gamma$*,*

$$f(x_i, x_j) := \left(\frac{y_i + y_j}{x_i - x_j}\right)^2 .$$

$A_N, E_N \subset \{1, \ldots, N\}$ *are the subsets*

$$A_N := \{ i \,|\, \exists\ j \text{ such that } (x_i, x_j) \in \Gamma \} ,$$
$$E_N := \{ j \,|\, \exists\ i \text{ such that } (x_i, x_j) \in \Gamma \} .$$

$\cup$ *and* $\cap$ *are the set theoretic union and intersection, respectively, and* $(\ldots)^c$ *denotes the complement in* $\{1, \ldots, N\}$*.*

Note that $\langle \quad \rangle_r$ is not a state (it is not compatible with the OPE).

Proof. We use induction on N. By multilinearity of $\langle \quad \rangle_r$ and eq. (5.6), $F(\Gamma)$ for $\Gamma \in S(x_1, \ldots, x_N)$ is determined by $\langle T(x_1) \ldots T(x_k) \rangle_r$, for $k \leq N$.

Suppose $\langle T(x_1) \ldots T(x_k) \rangle_r$, for $k \leq N$ has the required properties for $k < N$. We define

$$\langle T(x_1) \ldots T(x_N) \rangle_r$$

by (6.1) and show first that $\langle T(x_1)\dots T(x_N)\rangle_r$ is regular as two positions coincide. In other words, let $\Gamma_0(x_1,\dots,x_N) \in S(x_1,\dots,x_N)$ be the graph whose vertices are all isolated. Then $\sum_{\Gamma\neq\Gamma_0} F(\Gamma)$ reproduces the correct singular part of the Virasoro N-point function as prescribed by the OPE (5.5) on Σ.

For $N = 1$, $\Gamma_0(x)$ is the only graph, and

$$\langle T(x)\rangle\, p = F(\Gamma_0(x)) = \langle T(x)\rangle_r p\,. \tag{6.2}$$

For $N = 2$, the admissible graphs form a closed loop, a single line segment (with two possible orientations), and two isolated points. Thus by eq. (6.1),

$$\langle T(x_1)T(x_2)\rangle_r p_1 p_2 = \langle T(x_1)T(x_2)\rangle\, p_1 p_2 - \frac{c}{32} f_{12}^2 \langle \mathbb{1}\rangle_r - \frac{1}{4} f_{12}\langle \vartheta_1 + \vartheta_2\rangle_r\,,$$

where $f_{12} := f(x_1, x_2)$. According to the OPE (5.5), $\langle T(x_1)T(x_2)\rangle_r p_1 p_2$ is regular on the diagonal $x_1 = x_2$.

In order to prove regularity of $\langle T(x_1)\dots T(x_N)\rangle_r p_1 \dots p_N$ on all partial diagonals for $N > 2$, it suffices to show that the coefficients of the singularities are correct. Suppose the graph representation for the k-point function of the Virasoro field is correct for $2 \le k \le N-1$. For $1 \le i \le N$, set $S^{[i]} := S(x_i,\dots,x_N)$ and $\Gamma_0^{[i]} := \Gamma_0(x_i,\dots,x_N)$. For $1 \le i, j \le N$, $i \neq j$, define

$$\begin{aligned} S_{(ij)} &:= \{\Gamma \in S^{[1]} |\ (x_i, x_j), (x_j, x_i) \in \Gamma\}\,, \\ S_{(i,j)} &:= \{\Gamma \in S^{[1]} |\ (x_i, x_j) \in \Gamma, (x_j, x_i) \notin \Gamma\}\,, \\ S_{(i)(j)} &:= \{\Gamma \in S^{[1]} |\ (x_i, x_j), (x_j, x_i) \notin \Gamma\}\,. \end{aligned}$$

$S^{[1]}$ decomposes as

$$S^{[1]} = S_{(12)} \cup S_{(1,2)} \cup S_{(2,1)} \cup S_{(1),(2)}.$$

Since $S_{(12)} \cong S^{[3]}$, the equality

$$\sum_{\Gamma\in S_{(12)}} F(\Gamma) = \frac{c}{32}\, f_{12}^2\, \langle T(x_3)\dots T(x_N)\rangle\, p_3 \dots p_N$$

holds by the induction hypothesis. By the symmetrization argument following eq.

(A.2), it remains to show that

$$\sum_{\Gamma\in S^{[1]}\setminus S_{(12)}} F(\Gamma) = \frac{f_{12}}{2}\,\langle\vartheta_2 T(x_3)\dots T(x_N)\rangle\, p_3\dots p_N + O((x_1-x_2)^{-1})\,,$$

which under the induction hypothesis on $S^{[2]}$ and $S^{[3]}$, we reformulate as

$$\begin{aligned}&\sum_{\Gamma\in S^{[2]}} F(\varphi^{-1}(\Gamma)) + F(\bar\varphi^{-1}(\Gamma))\\ &\quad= \frac{f_{12}}{2}\left(\sum_{\Gamma\in S^{[2]}} F(\Gamma) - \frac{c}{32}\frac{[p_2']^2}{p_2}\sum_{\Gamma'\in S^{[3]}} F(\Gamma')\right) + O((x_1-x_2)^{-1})\,.\end{aligned}$$

Here φ, $\bar\varphi$ are the invertible maps

$$\begin{aligned}\varphi:&\quad S_{(1,2)}\ \to\ S^{[2]},\\ \bar\varphi:&\quad S_{(2,1)}\ \to\ S^{[2]},\end{aligned}$$

given by contracting the link (x_1,x_2) resp. (x_2,x_1) into the point x_2, and leaving the graph unchanged otherwise. Let $S_{(2)}\subset S^{[2]}$ be the subset of graphs containing x_2 as an isolated point, and let $\chi: S_{(2)}\ \to\ S^{[3]}$ be the isomorphism given by omitting the vertex x_2 from the graph. Now for $\Gamma\in S_{(2)}$, the graph representation yields

$$\begin{aligned}&F(\varphi^{-1}(\Gamma)) + F(\bar\varphi^{-1}(\Gamma))\\ &\quad=\ \frac{f_{12}}{2}\left(F(\Gamma) - \frac{c}{32}\frac{[p_2']^2}{p_2}\,F(\chi(\Gamma))\right) + O((x_1-x_2)^{-1})\,,\end{aligned}$$

while for $\Gamma\in S^{[2]}\setminus S_{(2)}$,

$$F(\varphi^{-1}(\Gamma)) + F(\bar\varphi^{-1}(\Gamma)) =\ \frac{f_{12}}{2}\,F(\Gamma) + O((x_1-x_2)^{-1})\,.$$

It remains to show part 2. It is sufficient to show that $\langle\vartheta_1\dots\vartheta_k\rangle_r$ for $1\le k\le N$ is regular at $p_1=0$. We use induction. From eqs (5.6) and (6.2) follows $\langle\vartheta\rangle_r = \langle\vartheta\rangle = \frac{1}{4}\Theta$, which is a polynomial in x and y. Now suppose $\langle\vartheta_1\dots\vartheta_k\rangle_r$ is regular at $p_1=0$,

for $k \leq N-1$. For $I \in \mathcal{P}(\{2,\dots,N\})$, the powerset of $\{2,\dots,N\}$, we have by eq. (5.6),

$$\begin{aligned}\langle \vartheta_1 \dots \vartheta_N \rangle_r &= \langle T(x_1)\dots T(x_N)\rangle_r p_1 \dots p_N - \frac{c}{32}\frac{[p_1']^2}{p_1} \sum_{I\in\mathcal{P}(\{2,\dots,N\})} \prod_{i\in I}\left(\frac{c}{32}\frac{[p_i']^2}{p_i}\right)\langle \bigotimes_{j\in I^c} \vartheta_j\rangle_r \\ &\qquad + \text{terms regular at } p_1 = 0 \\ &= \langle T(x_1)\dots T(x_N)\rangle_r p_1 \dots p_N - \frac{c}{32}\frac{[p_1']^2}{p_1}\langle T(x_2)\dots T(x_N)\rangle_r p_2 \dots p_N \\ &\qquad + \text{terms regular at } p_1 = 0\,.\end{aligned}$$

Here we have, using the graph representation,

$$\begin{aligned}\langle T(x_1)\dots T(x_N)\rangle_r p_1 \dots p_N &= \langle T(x_1)\dots T(x_N)\rangle p_1 \dots p_N - \sum_{\Gamma\in S^{[1]}\setminus\Gamma_0^{[1]}} F(\Gamma) \\ &= \frac{c}{32}\frac{[p_1']^2}{p_1}\left(\langle T(x_2)\dots T(x_N)\rangle p_2 \dots p_N - \sum_{\Gamma\in S^{[2]}\setminus\Gamma_0^{[2]}} F(\Gamma)\right) \\ &\quad + \text{terms regular at } p_1 = 0 \\ &= \frac{c}{32}\frac{[p_1']^2}{p_1}\langle T(x_2)\dots T(x_N)\rangle_r p_2 \dots p_N \\ &\quad + \text{terms regular at } p_1 = 0\,.\end{aligned}$$

We explain the second identity. Consider the augmentation map $a : S^{[2]} \setminus \Gamma_0^{[2]} \to S^{[1]} \setminus \Gamma_0^{[1]}$ defined by adjoining the isolated vertex x_1 to the graph. We have

$$F(a(\Gamma)) = \frac{c}{32}\frac{[p_1']^2}{p_1}F(\Gamma) + \{\text{terms regular at } p_1 = 0\}\,.$$

Indeed, all terms in $F(a(\Gamma))$ that involve ϑ_1 are k-point functions with $k < N$, since end points of edges are not labelled, and so are regular at $p_1 = 0$ by assumption. We conclude that $\langle \vartheta_1 \dots \vartheta_N\rangle_r$ is regular at $p_1 = 0$.

□

Since the proof is by recursion, it should generalise easily to more general Riemann surfaces.

We illustrate the theorem for the case $N = 3$. Recall that the connected 1-point,

2-point and 3-point functions are given by

$$\begin{aligned}\langle\varphi(x)\rangle_c =& \langle\mathbf{1}\rangle^{-1}\langle\varphi(x)\rangle\,,\\ \langle\varphi_1(x_1)\varphi_2(x_2)\rangle_c =& \langle\mathbf{1}\rangle^{-1}\langle\varphi_1(x_1)\varphi_2(x_2)\rangle - \langle\mathbf{1}\rangle^{-2}\langle\varphi_1(x_1)\rangle\langle\varphi_2(x_2)\rangle\,,\end{aligned}$$

and

$$\begin{aligned}\langle\varphi_1(x_1)\varphi_2(x_2)\varphi_3(x_3)\rangle_c = & \langle\mathbf{1}\rangle^{-1}\langle\varphi_1(x_1)\varphi_2(x_2)\varphi_3(x_3)\rangle\\ & - \langle\mathbf{1}\rangle^{-2}\left\{\langle\varphi_1(x_1)\varphi_2(x_2)\rangle\langle\varphi_3(x_3)\rangle + \text{cyclic}\right\}\\ & - \langle\mathbf{1}\rangle^{-3}\langle\varphi_1(x_1)\rangle\langle\varphi_2(x_2)\rangle\langle\varphi_3(x_3)\rangle\,.\end{aligned}$$

Example 5. *When* $\deg p = n$ *is odd,*

$$\langle T(x_1)T(x_2)T(x_3)\rangle_c\; p_1p_2p_3 = O(x_1^{n-3})\,.$$

In the region where x_1, x_2, x_3 *are finite, the connected Virasoro* 3*-point function is given by*

$$\langle T(x_1)T(x_1)T(x_3)\rangle_c p_1p_2p_3 = R^{(0)}(x_1,x_2,x_3) + O(1)|_{x_1,x_2,x_3}\,,$$

where $R^{(0)}$ is the rational function

$$\begin{aligned}
R^{(0)}(x_1,x_2,x_3) =& \frac{c}{64}\left(\frac{y_1+y_2}{x_1-x_2}\right)^2\left(\frac{y_1+y_3}{x_1-x_3}\right)^2\left(\frac{y_2+y_3}{x_2-x_3}\right)^2 \\
&+\frac{1}{64}\left(\frac{y_1+y_2}{x_1-x_2}\right)^2\left(\frac{y_1+y_3}{x_1-x_3}\right)^2\langle\mathbf{1}\rangle^{-1}(\Theta_2+\Theta_3) \\
&+\frac{1}{64}\left(\frac{y_1+y_2}{x_1-x_2}\right)^2\left(\frac{y_2+y_3}{x_2-x_3}\right)^2\langle\mathbf{1}\rangle^{-1}(\Theta_1+\Theta_3) \\
&+\frac{1}{64}\left(\frac{y_1+y_3}{x_1-x_3}\right)^2\left(\frac{y_2+y_3}{x_2-x_3}\right)^2\langle\mathbf{1}\rangle^{-1}(\Theta_1+\Theta_2) \\
&+\frac{1}{4}\left(\frac{y_1+y_2}{x_1-x_2}\right)^2\Big(\langle\mathbf{1}\rangle^{-1}\langle[T(x_1)T(x_3)]_{reg.}\rangle p_1p_3 \\
&\qquad\qquad +\langle\mathbf{1}\rangle^{-1}\langle[T(x_2)T(x_3)]_{reg.}\rangle p_2p_3\Big) \\
&+\frac{1}{4}\left(\frac{y_1+y_3}{x_1-x_3}\right)^2\Big(\langle\mathbf{1}\rangle^{-1}\langle[T(x_1)T(x_2)]_{reg.}\rangle p_1p_2 \\
&\qquad\qquad +\langle\mathbf{1}\rangle^{-1}\langle[T(x_2)T(x_3)]_{reg.}\rangle\, p_2p_3\Big) \\
&+\frac{1}{4}\left(\frac{y_2+y_3}{x_2-x_3}\right)^2\Big(\langle\mathbf{1}\rangle^{-1}\langle[T(x_1)T(x_2)]_{reg.}\rangle\, p_1p_2 \\
&\qquad\qquad +\langle\mathbf{1}\rangle^{-1}\langle[T(x_1)T(x_3)]_{reg.}\rangle\, p_1p_3\Big).
\end{aligned}$$

Here for $i,j\in\{1,2,3\}$, $\left[T(x_i)T(x_j)\right]_{reg.}$ is defined by (5.5). By part 2 in the Proof of Theorem 2, $\langle\left[T(x_i)T(x_j)\right]_{reg.}\rangle\, p_ip_j$ is a polynomial in x_i,x_j and y_i,y_j.

Moreover, the $O(1)|_{x_1,x_2,x_3}$ term is a polynomial in x_1,x_2,x_3 and y_1,y_2,y_3. Indeed, $\langle T(x_1)T(x_2)T(x_3)\rangle_c$ is regular at $p_1=0$ because

$$\begin{aligned}
\langle T(x_1)T(x_2)T(x_3)\rangle_c =& \langle\mathbf{1}\rangle^{-1}\langle T(x_1)T(x_2)T(x_3)\rangle - \langle\mathbf{1}\rangle^{-2}\langle T(x_1)\rangle\langle T(x_2)T(x_3)\rangle \\
&-\langle\mathbf{1}\rangle^{-1}\left\{\langle T(x_1)T(x_2)\rangle_c\langle T(x_3)\rangle + \langle T(x_3)T(x_1)\rangle_c\langle T(x_2)\rangle\right\}.
\end{aligned}$$

6.2 Application to the $(2,5)$ minimal model, for $n=5$

We consider the $(2,5)$ minimal model on a genus $g=2$ hyperelliptic Riemann surface

$$\Sigma:\quad y^2=p(x).$$

There are exactly $2^g=4$ parameters, given by $\langle\mathbf{1}\rangle$ and $\langle T(x)\rangle$ (or $\mathbf{A}_1,\mathbf{A}_2,\mathbf{A}_3$). As we shall argue now, we expect that all other constants in the Virasoro 2-point and 3-point

function are determined.

W.l.o.g. $n = 5$. In this case the 2-point function in the $(2, 5)$ minimal model has been determined previously, up to one constant, cf. Example 4. In the 3-point function, there is only one polynomial $P^{[1]}(x_1, x_2, x_3)$, of degree $n - 3$ in each of x_1, x_2, x_3, free to choose. Set

$$\begin{aligned}
P^{[1]}(x_1, x_2, x_3) =& B_{2,2,2}\, x_1^2 x_2^2 x_3^2 \\
&+ B_{2,2,1}(x_1^2 x_2^2 x_3 + x_1^2 x_2 x_3^2 + x_1 x_2^2 x_3^2) \\
&+ B_{2,1,1}(x_1^2 x_2 x_3 + x_1 x_2^2 x_3 + x_1 x_2 x_3^2) \\
&\quad + B_{2,2,0}(x_1^2 x_2^2 + x_1^2 x_3^2 + x_2^2 x_3^2) \\
&+ B_{2,1,0}(x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2) \\
&\quad + B_{1,1,1}\, x_1 x_2 x_3 \\
&+ B_{2,0,0}(x_1^2 + x_2^2 + x_3^2) + B_{1,1,0}(x_1 x_2 + x_1 x_3 + x_2 x_3) \\
&+ B_{1,0,0}(x_1 + x_2 + x_3) \\
&+ B_{0,0,0}\,,
\end{aligned}$$

where $B_{i,j,k} \propto \langle \mathbf{1} \rangle$ for $i, j, k \in \{1, 2\}$ and $k \leq j \leq i$ are constant in position. The constraint eq. (5.7) provides the knowledge of

$$\begin{aligned}
P^{[1]}(x_2, x_2, x_3) =& B_{2,2,2}\, x_2^4 x_3^2 \\
&+ B_{2,2,1}(x_2^4 x_3 + 2x_2^3 x_3^2) \\
&+ 2B_{2,1,1} x_2^3 x_3 + (B_{2,1,1} + 2B_{2,2,0}) x_2^2 x_3^2 + B_{2,2,0} x_2^4 \\
&+ 2B_{2,1,0}(x_2^3 + x_2 x_3^2) + (B_{1,1,1} + 2B_{2,1,0})\, x_2^2 x_3 \\
&+ 2B_{1,1,0} x_2 x_3 + (B_{1,1,0} + 2B_{2,0,0}) x_2^2 + B_{2,0,0} x_3^2 \\
&+ B_{1,0,0}(2x_2 + x_3) \\
&+ B_{0,0,0},
\end{aligned}$$

(obtained in the limit as $x_1 \to x_2$), and thus of all 10 coefficients. So given $\langle \mathbf{1} \rangle$ and $\langle T(x) \rangle$, the Virasoro 3-point function is uniquely determined.

Since $\langle [T(x_1)T(x_2)]_{\text{reg.}} \rangle\, p_1 p_2$ obtained from (5.5) is just the $O(1)|_{x_1 = x_2}$ part in the connected 2-point function given by eq. (A.5), the remaining unknown constant in the 2-point function is determined using Example 5.

Thus in Part I of the thesis we have formulated a set of necessary conditions on any CFT on a hyperelliptic Riemann surface. We shall not investigate the question about existence, which requires different methods.

Appendix A

A.1 Proof of Theorem 2 (Section 5.1)

1. We have

$$\begin{aligned}&\langle T(x_1)T(x_2)\rangle\, p_1p_2\\ &\qquad = \left[\langle T(x_1)T(x_2)\rangle\, p_1p_2\right]_{n-2} + \left[\langle T(x_1)T(x_2)\rangle\, p_1p_2\right]_{\leq n-3},\end{aligned}$$

where according to (4.9),

$$\begin{aligned}\left[\langle T(x_1)T(x_2)\rangle\, p_1p_2\right]_{n-2} &= \frac{c}{32}a_0x^{n-2}\langle T(x_2)\rangle\, p_2\\ &= \langle \mathbf{1}\rangle^{-1}\left[\langle T(x_1)\rangle\langle T(x_2)\rangle\, p_1p_2\right]_{n-2},\end{aligned}$$

so

$$\begin{aligned}&\langle T(x_1)T(x_2)\rangle\, p_1p_2 - \langle \mathbf{1}\rangle^{-1}\langle T(x_1)\rangle\langle T(x_2)\rangle p_1p_2\\ &\qquad = \left[\langle T(x_1)T(x_2)\rangle\, p_1p_2\right]_{\leq n-3} - \langle \mathbf{1}\rangle^{-1}\left[\langle T(x_1)\rangle\langle T(x_2)\rangle p_1p_2\right]_{\leq n-3}.\end{aligned}$$

This shows (5.3).

2. The proof is constructive. We build up a candidate and correct it subsequently so as to

   - match the singularities prescribed by the OPE,
   - behave at infinity according to (5.3),
   - be holomorphic in the appropriate coordinates away from the diagonal. Σ is covered by the coordinate patches $\{p \neq 0\}$, $\{p' \neq 0\}$ and $\{|x^{-1}| < \epsilon\}$.

   General outline: The 2-point function is meromorphic on Σ whence rational.

So once the singularities are fixed it is clear that we are left with the addition of polynomials as the only degree of freedom. The key ingredient is the use of the rational function

$$f(x_1, x_2) := \left(\frac{y_1 + y_2}{x_1 - x_2}\right)^2 , \tag{A.1}$$

which has a double pole at $x_1 = x_2$ as $y_1 = y_2 \neq 0$, and is regular for (x_1, y_1) close to $(x_2, -y_2)$.

For finite and fixed but generic x_2, and for the function f defined by eq. (A.1), we have

$$\frac{c}{32}\frac{1}{p_1 p_2} f(x_1, x_2)^2 = \frac{c/2}{(x_1 - x_2)^4} + \frac{c}{16}\frac{[p_2']^2}{p_2^2 (x_1 - x_2)^2} + O((x_1 - x_2)^{-1}) .$$

Moreover,

$$\frac{1}{4}\frac{1}{p_1 p_2} f(x_1, x_2) = \frac{1}{p_2 (x_1 - x_2)^2} + O((x_1 - x_2)^{-1}) . \tag{A.2}$$

Thus

$$\frac{[p_2']^2}{p_2^2 (x_1 - x_2)^2} = \frac{1}{8 p_1 p_2}\left(\frac{[p_1']^2}{p_1} + \frac{[p_2']^2}{p_2}\right) f(x_1, x_2) + O((x_1 - x_2)^{-1}) .$$

We conclude that

$$\begin{aligned}\frac{c/2}{(x_1 - x_2)^4}\langle \mathbf{1} \rangle = \frac{c}{32}\frac{1}{p_1 p_2}\langle \mathbf{1} \rangle \Big\{ & f(x_1, x_2)^2 \\ & - \frac{1}{4} f(x_1, x_2)\left(\frac{[p_1']^2}{p_1} + \frac{[p_2']^2}{p_2}\right)\Big\} + O(1) ,\end{aligned} \tag{A.3}$$

where $O(1)$ includes all terms regular at $x_1 = x_2$. Now by eq. (A.2),

$$\frac{\langle T(x_1) \rangle + \langle T(x_2) \rangle}{(x_1 - x_2)^2} = \frac{1}{4} f(x_1, x_2)\left(\frac{\langle T(x_1) \rangle}{p_2} + \frac{\langle T(x_2) \rangle}{p_1}\right) + O(1) . \tag{A.4}$$

From eqs (A.3) and (A.4) we obtain

$$\begin{aligned}&\frac{c/2}{(x_1 - x_2)^4}\langle \mathbf{1} \rangle + \frac{\langle T(x_1) \rangle + \langle T(x_2) \rangle}{(x_1 - x_2)^2} \\ &\qquad = \frac{1}{p_1 p_2}\left(\frac{c}{32}\langle \mathbf{1} \rangle f(x_1, x_2)^2 + \frac{1}{16} f(x_1, x_2)(\Theta_1 + \Theta_2)\right) + O(1) ,\end{aligned}$$

by eq. (4.7) Thus in the region where x_1 and x_2 are *finite*, we have

$$\langle \mathbf{1} \rangle \langle T(x_1)T(x_2) \rangle_c \; p_1 p_2 = R^{(0)}(x_1, x_2) + O\,(1)\,|_{x_1 = x_2}, \tag{A.5}$$

where

$$R^{(0)}(x_1, x_2) := \; \frac{c}{32} f(x_1, x_2)^2 \langle \mathbf{1} \rangle + \frac{1}{16} f(x_1, x_2)(\Theta_1 + \Theta_2)\,. \tag{A.6}$$

Note that the $O(1)|_{x_1 = x_2}$ terms are restricted to polynomials in x_1, x_2 and y_1, y_2. This simplification is due to the use of the connected 2-point function (the coefficient of any positive power of $f(x_1, x_2)$ in $\langle T(x_1)T(x_2\rangle \; p_1 p_2$ is regular at $p = 0$, so all singularities in at $p = 0$ drop out when $\langle T(x_1)T(x_2\rangle_c \; p_1 p_2$ is considered.)

The degree requirement (5.3) yields the upmost specification of eq. (A.5), because some terms appearing in

$$\begin{aligned} R^{(0)}(x_1, x_2) = \; & \frac{c}{32} \langle \mathbf{1} \rangle \frac{(p_1 - p_2)^2}{(x_1 - x_2)^4} \\ & + \frac{c}{8} y_1 y_2 \langle \mathbf{1} \rangle \frac{p_1 + p_2}{(x_1 - x_2)^4} + \frac{c}{4} \langle \mathbf{1} \rangle \frac{p_1 p_2}{(x_1 - x_2)^4} \\ & + \frac{1}{16} \frac{p_1 + 2y_1 y_2 + p_2}{(x_1 - x_2)^2} \left(\Theta_1^{[1]} + \Theta_2^{[1]} \right) \\ & + \frac{1}{16} \frac{p_1 + 2y_1 y_2 + p_2}{(x_1 - x_2)^2} \left(y_1 \Theta_1^{[y]} + y_2 \Theta_2^{[y]} \right) \end{aligned}$$

are absent in eq. (A.5) and so determine some of the polynomials in the connected 2-point function (which in the following we shall refer to as correction terms). To keep formulae short, we shall go over to the rational function $R(x_1, x_2)$ introduced in part 3 of Theorem 2, since it has milder divergencies for $|x|$ large than $R^{(0)}(x_1, x_2)$ does. Thus we show now that

$$R^{(0)}(x_1, x_2) = R(x_1, x_2) + \text{polynomials}, \tag{A.7}$$

where the "polynomial" part is a sum of polynomials in x_1, x_2 and in y_1, y_2. Indeed, we have the following identities:

$$\frac{(p_1 - p_2)^2}{(x_1 - x_2)^4} = \frac{p_1' p_2'}{(x_1 - x_2)^2} + \text{polynomial}.$$

Lemma 14 yields

$$\begin{aligned}
&\frac{c}{8}y_1y_2\langle\mathbf{1}\rangle\frac{p_1+p_2}{(x_1-x_2)^4}\\
&=\frac{c}{4}y_1y_2\langle\mathbf{1}\rangle\frac{p^{[1]}(\sqrt{x_1x_2})}{(x_1-x_2)^4}+\frac{c}{8}y_1y_2(x_1+x_2)\langle\mathbf{1}\rangle\frac{p^{[x]}(\sqrt{x_1x_2})}{(x_1-x_2)^4}\\
&+\frac{c}{32}y_1y_2\langle\mathbf{1}\rangle\frac{\frac{1}{\sqrt{x_1x_2}}(p^{[1]})'(\sqrt{x_1x_2})}{(x_1-x_2)^2}\\
&\qquad\qquad+\frac{3c}{64}y_1y_2(x_1+x_2)\langle\mathbf{1}\rangle\frac{\frac{1}{\sqrt{x_1x_2}}(p^{[x]})'(\sqrt{x_1x_2})}{(x_1-x_2)^2}\\
&+\frac{c}{32}y_1y_2\langle\mathbf{1}\rangle\frac{(p^{[1]})''(\sqrt{x_1x_2})}{(x_1-x_2)^2}\\
&\qquad\qquad+\frac{c}{64}y_1y_2(x_1+x_2)\langle\mathbf{1}\rangle\frac{(p^{[x]})''(\sqrt{x_1x_2})}{(x_1-x_2)^2}\\
&+\text{polynomial.}
\end{aligned}\tag{A.8}$$

Likewise,

$$\begin{aligned}
&\frac{1}{8}y_1y_2\frac{\Theta_1^{[1]}+\Theta_2^{[1]}}{(x_1-x_2)^2}\\
&\quad=-\frac{c}{32}y_1y_2\left(\frac{\Pi^{[1]}(\sqrt{x_1x_2})}{(x_1-x_2)^2}+\frac{1}{2}(x_1+x_2)\frac{\Pi^{[x]}(\sqrt{x_1x_2})}{(x_1-x_2)^2}\right)\\
&\qquad+\text{polynomial.}
\end{aligned}\tag{A.9}$$

Let r, s be polynomials in the only one variable x. Then we have

$$\frac{r_1s_1+r_2s_2}{(x_1-x_2)^2}=\frac{r_1s_2+r_2s_1}{(x_1-x_2)^2}+\text{polynomial}\,.\tag{A.10}$$

Thus

$$\frac{1}{8}\frac{p_1\Theta_1+p_2\Theta_2}{(x_1-x_2)^2}=\frac{1}{8}\frac{p_1\Theta_2+p_2\Theta_1}{(x_1-x_2)^2}+\text{polynomial.}\tag{A.11}$$

(A.10) generalises to terms including y_i as

$$\frac{y_1r_1+y_2r_2}{(x_1-x_2)^2}=\frac{y_1r_2+y_2r_1}{(x_1-x_2)^2}+\frac{p_1-p_2}{x_1-x_2}\,\frac{r_1-r_2}{x_1-x_2}\,\frac{1}{y_1+y_2}\,.$$

Thus

$$\begin{aligned}&\frac{1}{16}\frac{p_1+2y_1y_2+p_2}{(x_1-x_2)^2}(y_1\Theta_1^{[y]}+y_2\Theta_2^{[y]})\\=&\frac{1}{16}\frac{p_1+2y_1y_2+p_2}{(x_1-x_2)^2}\,(y_1\Theta_2^{[y]}+y_2\Theta_1^{[y]})+\text{polynomial},\end{aligned}$$

and Lemma 14 yields

$$\begin{aligned}&\frac{1}{16}\frac{p_1+p_2}{(x_1-x_2)^2}\,\left(y_1\Theta_2^{[y]}+y_2\Theta_1^{[y]}\right)=\\&\frac{1}{8}\left(y_1\Theta_2^{[y]}+y_2\Theta_1^{[y]}\right)\times\\&\qquad\times\left(\frac{p^{[1]}(\sqrt{x_1x_2})}{(x_1-x_2)^2}+\frac{1}{2}(x_1+x_2)\frac{p^{[x]}(\sqrt{x_1x_2})}{(x_1-x_2)^2}\right)\\&+\text{polynomial}.\end{aligned}\tag{A.12}$$

This proves eq. (A.7). Note that this result implies that in the finite region, $R(x_1,x_2)$ has the correct singularities. It remains to correct its behaviour for large $|x|$.

3. We first subtract all terms from R which are of non-admissible order in x_1. These depend polynomially on x_2 because this is true for $\left[(x_1-x_2)^{-\ell}\right]_{>k}$ with $\ell\in\mathbb{N}$, $k\in\mathbb{Z}$, (x_1 large), and may depend on y_2. The result may still be degree violating in x_2. Thus the corrected rational function reads

$$\begin{aligned}&R-[R]_{>n-3}-\left[R-[R]_{>n-3}\right]^{>n-3}\\=&R-[R]_{>n-3}-[R]^{>n-3}+\left[[R]_{>n-3}\right]^{>n-3}.\end{aligned}$$

Since the subtractions could be done in a different order, the procedure only works due to

$$\left[[R]_{>n-3}\right]^{>n-3}=\left[[R]^{>n-3}\right]_{>n-3}.\tag{A.13}$$

The connected 2-point function is thus determined up to addition of a polynomial $P(x_1,x_2,y_1,y_2)$ of the form (5.4) which is specific to the state. The degree and symmetry requirements for $P(x_1,x_2,y_1,y_2)$ are immediate.

For clarity, we first list the terms contained in $-[R]_{>n-3}$ resp. $-y_1\,[R]_{>\frac{n}{2}-3}$:

From (A.8),

$$-\frac{3c}{64}y_1y_2\left[x_1\langle\mathbf{1}\rangle\frac{\frac{1}{\sqrt{x_1x_2}}(p^{[x]})'(\sqrt{x_1x_2})}{(x_1-x_2)^2}\right]_{>\frac{n}{2}-3}, \tag{A.14}$$

$$-\frac{c}{64}y_1y_2\left[x_1\langle\mathbf{1}\rangle\frac{(p^{[x]})''(\sqrt{x_1x_2})}{(x_1-x_2)^2}\right]_{>\frac{n}{2}-3}, \tag{A.15}$$

from (A.9),

$$\frac{c}{64}y_1y_2\left[x_1\frac{\Pi^{[x]}(\sqrt{x_1x_2})}{(x_1-x_2)^2}\right]_{>\frac{n}{2}-3}, \tag{A.16}$$

from (A.11),

$$-\frac{1}{8}\Theta_2\left[\frac{p_1}{(x_1-x_2)^2}\right]_{>n-3}, \tag{A.17}$$

and from (A.12),

$$\frac{1}{8}y_1\,\Theta_2^{[y]}\left[\frac{p^{[1]}(\sqrt{x_1x_2})}{(x_1-x_2)^2}\right]_{>\frac{n}{2}-3}, \tag{A.18}$$

$$\frac{1}{16}\,y_1\Theta_2^{[y]}\left[x_1\frac{p^{[x]}(\sqrt{x_1x_2})}{(x_1-x_2)^2}\right]_{>\frac{n}{2}-3}, \tag{A.19}$$

$$\frac{1}{16}\,y_1x_2\Theta_2^{[y]}\left[\frac{p^{[x]}(\sqrt{x_1x_2})}{(x_1-x_2)^2}\right]_{>\frac{n}{2}-3}. \tag{A.20}$$

Now we give the full explicit expression for

$$-[R]_{>n-3}-[R]^{>n-3}+\left[[R]_{>n-3}\right]^{>n-3}\ .$$

(A.14) and (A.15) yield

$$\frac{c}{64}y_1y_2(n^2-1)a_0x_1^{\frac{n}{2}-\frac{3}{2}}x_2^{\frac{n}{2}-\frac{5}{2}}=-\frac{1}{8}y_1y_2\mathrm{A}_0x_1^{\frac{n}{2}-\frac{3}{2}}x_2^{\frac{n}{2}-\frac{5}{2}},$$

which cancels against the term we obtain from (A.16). For odd n, $\mathrm{A}_0=-\frac{c}{8}(n^2-1)a_0\langle\mathbf{1}\rangle$, so (A.17) yields

$$-\frac{1}{8}a_0\left(x_1^{n-2}\Theta_2+x_2^{n-2}\Theta_1\right)-\frac{c}{64}(n^2-1)a_0^2x_1^{n-2}x_2^{n-2}.$$

(A.18) yields

$$\frac{1}{8}y_1a_1x_1^{\frac{n}{2}-\frac{5}{2}}x_2^{\frac{n}{2}-\frac{1}{2}}\Theta_2^{[y]}+\frac{1}{8}y_2a_1x_1^{\frac{n}{2}-\frac{1}{2}}x_2^{\frac{n}{2}-\frac{5}{2}}\Theta_1^{[y]}\,.$$

(A.19) yields:

$$\begin{aligned}&\frac{1}{16}y_1a_0x_1^{\frac{n}{2}-\frac{3}{2}}x_2^{\frac{n}{2}-\frac{1}{2}}\Theta_2^{[y]}+\frac{1}{16}y_2a_0x_1^{\frac{n}{2}-\frac{1}{2}}x_2^{\frac{n}{2}-\frac{3}{2}}\Theta_1^{[y]}\,,\\&\frac{1}{8}y_1a_0x_1^{\frac{n}{2}-\frac{5}{2}}x_2^{\frac{n}{2}+\frac{1}{2}}\Theta_2^{[y]}+\frac{1}{8}y_2a_0x_1^{\frac{n}{2}+\frac{1}{2}}x_2^{\frac{n}{2}-\frac{5}{2}}\Theta_1^{[y]}\,,\\&\frac{1}{16}y_1a_2x_1^{\frac{n}{2}-\frac{5}{2}}x_2^{\frac{n}{2}-\frac{3}{2}}\Theta_2^{[y]}+\frac{1}{16}y_2a_2x_1^{\frac{n}{2}-\frac{3}{2}}x_2^{\frac{n}{2}-\frac{5}{2}}\Theta_1^{[y]}\,.\end{aligned}$$

(A.20) yields:

$$\frac{1}{16}y_1a_0x_1^{\frac{n}{2}-\frac{5}{2}}x_2^{\frac{n}{2}+\frac{1}{2}}\Theta_2^{[y]}+\frac{1}{16}y_2a_0x_1^{\frac{n}{2}+\frac{1}{2}}x_2^{\frac{n}{2}-\frac{5}{2}}\Theta_1^{[y]}.$$

Since all terms are symmetric w.r.t. interchange of x_1 and x_2, eq. (A.13) has been verified. This completes the proof.

A.2 Behaviour of the Virasoro 2-point function under degeneration of the surface in the case $g = 1$

As mentioned by the author in the viva, the formula for the Virasoro 2-point function is consistent w.r.t. the degeneration of the Riemann surface Σ_g.

Suppose X, X' are two different ramification points of any hyperelliptic Riemamn surface Σ_g. A linear fractional transformation on $\mathbb{P}^1_{\mathbb{C}}$ sending $X' \to X$ results in a hyperelliptic Riemann surface Σ_{g-1}, since in the limit, $X = X'$ will no more be a ramification point. Indeed, while on Σ_g a path between X and X' will change the sheet, in the limit, winding around $X' = X$ will trace a path on one single sheet.

We checked consistency for $\Sigma_1 : y^2 = p(x)$ with $\deg p = 3$. The Virasoro 1- and

2-point functions on Σ_1 are given by

$$\begin{aligned}
\langle T(x)\rangle &= \frac{c}{32}\frac{[p']^2}{p^2}\langle \mathbf{1}\rangle + \frac{-4cx+\mathbf{A}_1}{4p}\,,\\
\langle T(x_1)T(x_2)\rangle &= \frac{c}{4}\langle \mathbf{1}\rangle\frac{1}{(x_1-x_2)^4}\\
&\quad + \frac{c}{32}\langle \mathbf{1}\rangle\frac{p_1'p_2'}{(x_1-x_2)^2p_1p_2} - \frac{c}{2}\langle \mathbf{1}\rangle\frac{p_1x_2+p_2x_1}{(x_1-x_2)^2p_1p_2}\\
&\qquad + \frac{1}{8}\mathbf{A}_1\frac{p_1+p_2}{(x_1-x_2)^2p_1p_2}\\
&\quad + \frac{c}{8}y_1y_2\langle \mathbf{1}\rangle\frac{p_1+p_2}{(x_1-x_2)^4p_1p_2}\\
&\quad - \frac{c}{2}\langle \mathbf{1}\rangle y_1y_2\frac{x_1+x_2}{(x_1-x_2)^2p_1p_2} + \frac{1}{4}\mathbf{A}_1y_1y_2\frac{1}{(x_1-x_2)^2p_1p_2}\\
&\quad + 2c\langle \mathbf{1}\rangle\frac{x_1x_2}{p_1p_2} - \frac{\mathbf{A}_1}{2}\frac{(x_1+x_2)}{p_1p_2} + \frac{P^{[1]}}{p_1p_2}\\
&\quad + \langle \mathbf{1}\rangle^{-1}\langle T(x_1)\rangle\langle T(x_2)\rangle\,,
\end{aligned}$$

respectively, where $P^{[1]} \propto \langle \mathbf{1}\rangle$ is specific to the state. Denote by $\langle T(x_1)T(x_2)\rangle_{X_1,X_2,X_3}$ the 2-point function on the torus

$$\Sigma_1 : \quad y^2 = a_0(x-X_1)(x-X_2)(x-X_3)\,,$$

($a_0 \in \mathbb{C}$). A linear fractional transformation on $\mathbb{P}^1_{\mathbb{C}}$ sends $X_3 \to \infty$. Since

$$\frac{\sqrt{x_2-X_3}}{\sqrt{x_1-X_3}} = 1 - \frac{1}{2}\frac{x_1-x_2}{x_1-X_3} - \frac{1}{8}\frac{(x_1-x_2)^2}{(x_1-X_3)^2} + \ldots \overset{X_3\to\infty}{\longrightarrow} 1\,,$$

we have

$$\begin{aligned}
\lim_{X_3\to\infty}\langle T(x_1)T(x_2)\rangle_{X_1,X_2,X_3} &= \frac{c}{4}\langle \mathbf{1}\rangle_{X_1',X_2'}\frac{1}{(x_1'-x_2')^4}\\
&\quad + \frac{c}{32}\langle \mathbf{1}\rangle_{X_1',X_2'}\frac{1}{(x_1'-x_2')^2}\left(\frac{1}{(x_1'-X_1')}+\frac{1}{(x_1'-X_2')}\right)\left(\frac{1}{(x_2'-X_1')}+\frac{1}{(x_2'-X_2')}\right)\\
&\quad + \frac{c}{8}\langle \mathbf{1}\rangle_{X_1',X_2'}\frac{1}{(x_1'-x_2')^4}\left(\frac{\sqrt{(x_2'-X_1')(x_2'-X_2')}}{\sqrt{(x_1'-X_1')(x_1'-X_2')}}+\frac{\sqrt{(x_1'-X_1')(x_1'-X_2')}}{\sqrt{(x_2'-X_1')(x_2'-X_2')}}\right)\\
&\quad + \langle \mathbf{1}\rangle_{X_1',X_2'}\frac{c^2}{(32)^2}\left(\frac{1}{(x_1'-X_1')}+\frac{1}{(x_1'-X_2')}\right)^2\left(\frac{1}{(x_2'-X_1')}+\frac{1}{(x_2'-X_2')}\right)^2\,.
\end{aligned}$$

Here $\langle \mathbf{1} \rangle_{X_1', X_2'}$ is the 0-point function on the Riemann sphere

$$\Sigma_0' : \quad y'^2 = a_0(x' - X_1')(x' - X_2') \,,$$

in terms of new local coordinates x', y'. Assuming $X_1' = 0$ for simplicity, the coordinate transformation is given by

$$x'(x) = X_2 \, \frac{X_2 - X_3}{X_2 - X_1} \frac{x - X_1}{x - X_3} \,,$$

(in particular $X_2' = X_2$). To check our result, we perform another linear fractional transformation on $\mathbb{P}^1_{\mathbb{C}}$,

$$x''(x') = \frac{x'}{x' - X_2} \,,$$

sending $X_2 \to \infty$. The 2-point function on the resulting surface $\Sigma_0 : y''^2 = a_0 x''$, with 0-point function $\langle \mathbf{1} \rangle_0$ and Virasoro 2-point function $\langle T''(x_1'') T''(x_2'') \rangle_0$, reads

$$\begin{aligned} \lim_{X_2 \to \infty} \lim_{X_3 \to \infty} \langle T(x_1) T(x_2) \rangle_{X_1, X_2, X_3} = {} & \frac{c}{4} \langle \mathbf{1} \rangle_0 \frac{1}{(x_1'' - x_2'')^4} + \frac{c}{32} \langle \mathbf{1} \rangle_0 \frac{1}{(x_1'' - x_2'')^2 x_1'' x_2''} \\ & + \frac{c}{8} \langle \mathbf{1} \rangle_0 \frac{1}{(x_1'' - x_2'')^4} \left(\frac{\sqrt{x_1''}}{\sqrt{x_2''}} + \frac{\sqrt{x_2''}}{\sqrt{x_1''}} \right) \\ & + \frac{c^2}{(32)^2} \langle \mathbf{1} \rangle_0 \frac{1}{x_1''^2} \frac{1}{x_2''^2} \,. \end{aligned}$$

The arguments of the proof of Theorem 1 and 2 show that this is the correct formula,

$$\lim_{X_2 \to \infty} \lim_{X_3 \to \infty} \langle T(x_1) T(x_2) \rangle_{X_1, X_2, X_3} = \langle T''(x_1'') T''(x_2'') \rangle_0 \,.$$

Thus we have shown that our formulae for the 1- and 2-point function on the torus Σ_1 behave correctly under degeneration of Σ_1 to the sphere Σ_0.

Part II

Dependence on moduli

Chapter 7

Introduction

Let

$$\Sigma_1 := \{z \in \mathbb{C} | \, |q| \leq z \leq 1\}/\{z \sim qz\} \, ,$$

where $q = e^{2\pi i \, \tau}$ and $\tau \in \mathbb{H}^+$. Σ_1 is a torus. A *character* on Σ_1 is given by

$$\langle \mathbb{1} \rangle_{\Sigma_1} = \sum_{\substack{\varphi_j \\ \{\varphi_j\}_j \text{ basis of } F}} q^{h(\varphi_j)} \, .$$

Here F is the fiber of the bundle of holomorphic fields $\mathcal{F}$ in a rational CFT on Σ_1, as discussed in Part I of the thesis. By the fact that Part I lists necessary conditions for a CFT on a hyperelliptic Riemann surface, $\langle \mathbb{1} \rangle_{\Sigma_1}$ is in particular a 0-point function $\langle \mathbf{1} \rangle$ in the sense of Part I. On the other hand, $\langle \mathbb{1} \rangle_{\Sigma_1}$ is known to be a modular function of τ ([27], [41]). A *modular function on* a discrete subgroup Γ of $\Gamma_1 = SL(2, \mathbb{Z})$ is a Γ-invariant meromorphic function $f : \mathbb{H}^+ \to \mathbb{C}$ with at most exponential growth towards the boundary [38]. For $N \geq 1$, the *principal conguence subgroup* is the group $\Gamma(N)$ such that the short sequence

$$1 \to \Gamma(N) \hookrightarrow \Gamma_1 \xrightarrow{\pi_N} SL(2, \mathbb{Z}/N\mathbb{Z}) \to 1$$

is exact, where π_N is map given by reduction modulo N. A function that is modular on $\Gamma(N)$ is said to be *of level N*. Let $\zeta_N = e^{\frac{2\pi i}{N}}$ be the N-th root of unity with cyclotomic field $\mathbb{Q}(\zeta_N)$. Let F_N be the field of modular functions f of level N which have a Fourier expansion

$$f(\tau) = \sum_{n \geq -n_0} a_n q^{\frac{n}{N}} \, , \quad q = e^{2\pi i \, \tau} \, , \tag{7.1}$$

with $a_n \in \mathbb{Q}(\zeta_N)$, $\forall n$. The Ramanujan continued fraction

$$r(\tau) := q^{1/5} \frac{1}{1 + \frac{q}{1+\frac{q^2}{1+\ldots}}} \tag{7.2}$$

which converges for $\tau \in \mathbb{H}^+$, is an element (and actually a generator) of F_5 [2]. r is algebraic over F_1 which is generated over $\mathbb{Q}$ by the *modular j-function*,

$$j(\tau) = 12^3 \frac{g_2^3}{g_2^3 - 27 g_3^2} \, .$$

j is associated to the elliptic curve with the affine equation

$$\Sigma_1 : \quad y^2 = 4x^3 - g_2 x - g_3 \, , \quad \text{with} \quad g_2^3 - 27 g_3^2 \neq 0 \, .$$

Here g_k for $k = 2, 3$ are (specific) modular forms of weight $2k$,[1] so that j is indeed a function of the respective modulus only (the quotient $\tau = \omega_2/\omega_1$ for the lattice $\Lambda = \mathbb{Z}.\omega_1 + \mathbb{Z}.\omega_2$), or rather its orbit under Γ_1 (since we are free to change the basis (ω_1, ω_2) for Λ). In terms of the modulus, a *modular form of weight* $2k$ *on* Γ is a holomorphic function $g : \mathbb{H}^+ \to \mathbb{C}$ with subexponential growth towards the boundary [38] such that $g(\tau)\,(d\tau)^{2k}$ is Γ-invariant [33]. A modular form on Γ_1 allows a Fourier expansion of the form (7.1) with $n_0 \geq 0$.

Another way to approach modular functions is in terms of the differential equations they satisfy. The derivative of a modular function is a modular form of weight two, and higher derivatives give rise to *quasi-modular forms*, which we shall also deal with though they are not themselves of primary interest to us.

Geometrically, the conformal structure on the surface

$$\Sigma_1 : \quad y^2 = 4(x - X_1)(x - X_2)(x - X_3) \, , \quad x \in \mathbb{P}^1_{\mathbb{C}} \, ,$$

is determined by the quadrupel (X_1, X_2, X_3, ∞) of its ramification points, and we can change this structure by varying the position of X_1, X_2, X_3 infinitesimally. In this picture, the boundary of the moduli space is approached by letting two ramification points in the quadrupel run together [13].

When changing positions we may keep track of the branch points to obtain a simply connected space [6]. Thus a third way to describe modularity of the characters is by means of a subgroup of the braid group B_3 of 3 strands. The latter is the universal

[1] As mentioned earlier, a modular form of weight $2k$ transforms as $f(\lambda\Lambda) = \lambda^{-2k} f(\Lambda)$ for any $\lambda \in \mathbb{C}^*$.

central extension of the quotient group $\overline{\Gamma}_1 = \Gamma_1/\{\pm\mathbb{I}_2\}$, so that we come full circle.

Suppose $\Sigma_1 = \mathbb{C}/\Lambda$ where $\Lambda = (\mathbb{Z}.1 + \mathbb{Z}.i\beta)$ with $\beta \in \mathbb{R}$. Thus the fundamental domain is a rectangle in the (x^0, x^1) plane with length $\Delta x^0 = 1$ and width $\Delta x^1 = \beta$. The dependence of $\langle \mathbb{1} \rangle_{\Sigma_1}$ on the modulus $i\beta$ follows from the identity

$$\langle \mathbb{1} \rangle_{\Sigma_1} = \operatorname{tr} e^{-H\beta}\,, \quad H = \int T^{00} dx^0,$$

where T^{00} is a real component of the Virasoro field.[2] As mentioned above, we may regard $\langle \mathbb{1} \rangle_{\Sigma_1}$ as the 0-point function $\langle \mathbf{1} \rangle$ w.r.t. a state $\langle\ \rangle$ on Σ_1. Note that the same argument applies to N-point functions for $N > 0$.

Stretching $\beta \mapsto (1+\epsilon)\beta$ changes the Euclidean metric $G_{\mu\nu}$ $(\mu, \nu = 0, 1)$ according to

$$(ds)^2 \ \mapsto\ (ds)^2 + 2\epsilon (dx^1)^2 + O(\epsilon^2)\,.$$

Thus $dG_{11} = 2\frac{d\beta}{\beta}$, and

$$\begin{aligned} d\langle \mathbf{1} \rangle = -\operatorname{tr}(H d\beta\, e^{-H\beta}) &= -\frac{dG_{11}}{2} \left(\int \langle T^{00} \rangle dx^0 \right) \beta \\ &= -\frac{dG_{11}}{2} \iint \langle T^{00} \rangle dx^0 dx^1\,. \end{aligned} \tag{7.3}$$

The fact that $\int \langle T^{00} \rangle dx^0$ does not depend on x^1 follows from the conservation law $\partial_\mu T^{\mu\nu} = 0$:

$$\frac{d}{dx^1} \oint \langle T^{00} \rangle\, dx^0 = \oint \partial_1 \langle T^{00} \rangle\, dx^0 = -\oint \partial_0 \langle T^{10} \rangle\, dx^0 = 0\,,$$

using Stokes' Theorem.

We argue that on $S^1 \times S^1_{\beta/(2\pi)}$ (where $S^1_{\beta/(2\pi)}$ is the circle of perimeter β), states (in the sense of Part I of this thesis) are thermal states on the VOA.

When $g > 1$, equation (7.3) generalises to

$$d\langle \mathbf{1} \rangle = -\frac{1}{2} \iint dG_{\mu\nu} \langle T^{\mu\nu} \rangle\ \sqrt{G}\, dx^0 \wedge dx^1\,. \tag{7.4}$$

[2] Any dynamical quantum field theory has an energy-momentum tensor $T_{\mu\nu}$ s.t. $T_{\mu\nu} dx^\mu dx^\nu$ defines a quadratic differential, by which we mean in particular that it transforms homogeneously under coordinate changes. For coordinates $z = x^0 + ix^1$ and $\bar{z} = x^0 - ix^1$, we have [1]

$$T_{zz} = \frac{1}{4}(T_{00} - 2iT_{10} - T_{11})\,.$$

For the relation with the Virasoro field $T(z)$ discussed in Part I, cf. Section 9.1 below.

Here $G := |\det G_{\mu\nu}|$, and $dvol_2 = \sqrt{G}\, dx^0 \wedge dx^1$ is the volume form which is invariant under base change.[3] The normalisation is in agreement with eq. (7.3) (see also [3], eq. (5.140) on p. 139).

Methods that make use of the flat metric do not carry over to surfaces of higher genus. We may choose a specific metric of prescribed constant curvature to obtain mathematically correct but cumbersome formulae. Alternatively, we consider quotients of N-point functions over $\langle \mathbf{1} \rangle$ only (as done in [7]) so that the dependence on the specific metric drops out. Yet we suggest to use a singular metric that is adapted to the specific problem.

[3]The change to complex coordinates is a more intricate, however: We have $dx^0 \wedge dx^1 = iG_{z\bar{z}}\, dz \wedge d\bar{z}$ with $G_{z\bar{z}} = \frac{1}{2}$, as can be seen by setting $z = x^0 + ix^1$.

Chapter 8

Differential equations for characters in $(2, \nu)$-minimal models

8.1 Review of the differential equation for the characters of the $(2, 5)$ minimal model

The character $\langle \mathbf{1} \rangle$ of any CFT on the torus Σ_1 solves the ODE [7]

$$\frac{d}{d\tau}\langle \mathbf{1} \rangle = \frac{1}{2\pi i} \oint \langle T(z) \rangle \, dz = \frac{1}{2\pi i} \langle \mathbf{T} \rangle \, . \tag{8.1}$$

Here the contour integral is along the real period, and $\oint dz = 1$. $\langle \mathbf{T} \rangle$, while constant in position, is a modular form of weight two in the modulus. The Virasoro field generates the variation of the conformal structure [7]. In the $(2, 5)$ minimal model, we find by eqs (3.3) and (3.4) in Part I,

$$2\pi i \frac{d}{d\tau}\langle \mathbf{T} \rangle = \oint \langle T(w)T(z) \rangle \, dz = -4\langle \mathbf{T} \rangle G_2 + \frac{22}{5} G_4 \langle \mathbf{1} \rangle \, . \tag{8.2}$$

Here G_2 is the quasimodular Eisenstein series of weight 2, which enters the equation by means of the identity

$$\int_0^1 \wp(z - w|\tau) \, dz = -2G_2(\tau).$$

In terms of the *Serre derivative*

$$\mathfrak{D}_{2\ell} := \frac{1}{2\pi i}\frac{d}{d\tau} - \frac{\ell}{6}E_2\,, \tag{8.3}$$

the first order ODEs (8.1) and (8.2) combine to give the second order ODE ([26], and recently [21])

$$\mathfrak{D}_2 \circ \mathfrak{D}_0\langle\mathbf{1}\rangle = \frac{11}{3600}\,E_4\langle\mathbf{1}\rangle\,.$$

The two solutions are the well-known Rogers-Ramanujan partition functions [3]

$$\langle\mathbf{1}\rangle_1 = q^{\frac{11}{60}}\sum_{n\geq 0}\frac{q^{n^2+n}}{(q)_n} = q^{\frac{11}{60}}\left(1+q^2+q^3+q^4+q^5+2q^6+\ldots\right)\,,$$
$$\langle\mathbf{1}\rangle_2 = q^{-\frac{1}{60}}\sum_{n\geq 0}\frac{q^{n^2}}{(q)_n} = q^{-\frac{1}{60}}\left(1+q+q^2+q^3+2q^4+2q^5+3q^6+\ldots\right)\,.$$

($q = e^{2\pi i\,\tau}$) which are named after the famous Rogers-Ramanujan identities

$$q^{-\frac{11}{60}}\langle\mathbf{1}\rangle_1 = \prod_{n=\pm 2 \bmod 5}(1-q^n)^{-1}\,,\quad q^{\frac{1}{60}}\langle\mathbf{1}\rangle_2 = \prod_{n=\pm 1 \bmod 5}(1-q^n)^{-1}\,.$$

Mnemotechnically, the distribution of indices seems somewhat unfortunate. In general, however, the characters of the $(2,\nu)$ minimal model, of which there are

$$M = \frac{\nu-1}{2} \tag{8.4}$$

(ν odd) many, are ordered by their conformal weight, which is the lowest for the respective *vacuum character* $\langle\mathbf{1}\rangle_1$, having weight zero.

The Rogers-Ramanujan identity for $q^{-\frac{11}{60}}\langle\mathbf{1}\rangle_1$ provides the generating function for the partition which to a given holomorphic dimension $h \geq 0$ returns the number of linearly independent holomorphic fields present in the $(2,5)$ minimal model. Recall that this number is subject to the constraint $\partial^2 T \propto N_0(T,T)$, eq. (3.1) in Part I.

h	0	1	2	3	4	5	6
basis of $F(h)$	1	–	T	∂T	$\partial^2 T$	$\partial^3 T$	$\partial^4 T$ $N_0(T,\partial^2 T)$
dim $F(h)$	1	0	1	1	1	1	2

Holomorphic fields of dimension h in the $(2,5)$ minimal model

There is a similar combinatorical interpretation for the second Rogers-Ramanujan

identity. It involves non-holomorphic fields, however, which we disregard in this thesis.

8.2 Review the algebraic equation for the characters of the $(2, 5)$ minimal model

Besides the analytic approach, there is an algebraic approach to the characters. This is due to the fact that $\langle \mathbf{1} \rangle_1, \langle \mathbf{1} \rangle_2$, rather than being modular on the full modular group, are modular on a subgroup of Γ_1: For the generators S, T of Γ_1 we have [2]

$$T\langle \mathbf{1} \rangle_1 = {\zeta_{60}}^{11} \langle \mathbf{1} \rangle_1 \,, \quad T\langle \mathbf{1} \rangle_2 = {\zeta_{60}}^{-1} \langle \mathbf{1} \rangle_2 \,,$$

while under the operation of S, $\langle \mathbf{1} \rangle_1, \langle \mathbf{1} \rangle_2$ transform into linear combinations of one another [2],

$$S \begin{pmatrix} \langle \mathbf{1} \rangle_1 \\ \langle \mathbf{1} \rangle_2 \end{pmatrix} = \frac{2}{\sqrt{5}} \begin{pmatrix} \sin \frac{\pi}{5} & -\sin \frac{2\pi}{5} \\ \sin \frac{2\pi}{5} & \sin \frac{\pi}{5} \end{pmatrix} \begin{pmatrix} \langle \mathbf{1} \rangle_1 \\ \langle \mathbf{1} \rangle_2 \end{pmatrix} .$$

However, $\langle \mathbf{1} \rangle_1, \langle \mathbf{1} \rangle_2$ are modular under a subgroup of Γ_1 of finite index. Its fundamental domain is therefore a finite union of copies of the fundamental domain $\mathcal{F}$ of Γ_1 in $\mathbb{C}$. More specifically, if the subgroup is Γ with index $[\Gamma_1 : \Gamma]$, and if $\gamma_1, \ldots, \gamma_{[\Gamma_1:\Gamma]} \in \Gamma_1$ are the coset representatives so that $\Gamma_1 = \Gamma\gamma_1 \cup \ldots \cup \Gamma\gamma_{[\Gamma_1:\Gamma]}$, then we have

$$\mathcal{F}_\Gamma = \gamma_1 \mathcal{F} \cup \ldots \cup \gamma_{[\Gamma_1:\Gamma]} \mathcal{F} \,. \tag{8.5}$$

[15]. Thus $\langle \mathbf{1} \rangle_1$ and $\langle \mathbf{1} \rangle_2$ define meromorphic functions on a finite covering of the moduli space $\mathcal{M}_1 = \Gamma_1 \setminus \mathbb{H}^+$ and are algebraic. We can write [2]

$$\langle \mathbf{1} \rangle_1 = \frac{\theta_{5,2}}{\eta} \,, \quad \langle \mathbf{1} \rangle_2 = \frac{\theta_{5,1}}{\eta} \,,$$

where the functions $\eta, \theta_{5,1}, \theta_{5,2}$ on the r.h.s. are specific theta functions (e.g. [3])

$$\theta(\tau) = \sum_{n \in \mathbb{Z}} f(n) \,, \quad f(n) \sim q^{n^2} \,, \quad q = e^{2\pi i \, \tau} \,.$$

The characters' common denominator is the Dedekind η function. Using the Poisson transformation formula, one finds that $\eta, \theta_{5,1}, \theta_{5,2}$ are all modular forms of weight $\frac{1}{2}$

([38], Propos. 9, p. 25). For the quotient $\langle \mathbf{1} \rangle_1 / \langle \mathbf{1} \rangle_2$ and $\tau \in \mathbb{H}^+$, we find [2],

$$\frac{\langle \mathbf{1} \rangle_1}{\langle \mathbf{1} \rangle_2} = \frac{\theta_{5,2}}{\theta_{5,1}} = q^{\frac{1}{5}} \prod_{n=1}^{\infty} (1 - q^n)^{\left(\frac{n}{5}\right)} = r(\tau) \,,$$

where $r(\tau)$ is the Ramanujan continued fraction introduced in eq. (7.2). (Here $(n/5) = 1, -1, 0$ for $n = \pm 1, \pm 2, 0 \pmod 5$, respectively, is the Legendre symbol.)

$r(\tau)$ is modular on $\Gamma(5)$ with index $[\Gamma_1 : \Gamma(5)] = 120$ [16]. The quotient $\Gamma(5) \setminus \mathbb{H}^+$ can be compactified and made into a Riemann surface, which is referred to as the modular curve

$$\Sigma(5) = \Gamma(5) \setminus \mathbb{H}^* \,.$$

Here $\mathbb{H}^* := \mathbb{H}^+ \cup \mathbb{Q} \cup \{\infty\}$ is the extended complex upper half plane. $\Sigma(5)$ has genus zero and the symmetry of an icosahedron. The rotation group of the sphere leaving an inscribed icosahedron invariant is A_5, the alternating group of order 60. By means of a stereographic projection, the notion of edge center, face center and vertex are induced on the extended complex plane [5]. They are acted upon by the icosahedral group $G_{60} \subset PSL(2, \mathbb{C})$. The face centers and finite vertices define the simple roots of two monic polynomials $F(z)$ and $V(z)$ of degree 20 and 11, respectively, which transform in such a way under G_{60} that

$$J(z) := \frac{F^3(z)}{V^5(z)}$$

is invariant. It turns out that $J(r(\tau))$ for $\tau \in \mathbb{H}^+$ is $\Gamma(1)$-invariant, and in fact that $J(r(\tau)) = j(\tau)$. Thus $r(\tau)$ satisfies

$$F^3(z) - j(\tau) V^5(z) = 0$$

(for the same value of τ), which is equivalent to $r^5(\tau)$ solving the icosahedral equation

$$(X^4 - 228X^3 + 494X^2 + 228X + 1)^3 + j(\tau) X (X^2 + 11X - 1)^5 = 0 \,.$$

This is actually the minimal polynomial of r^5 over $\mathbb{Q}(j)$, so that $\mathbb{Q}(r)$ defines a function field extension of degree 60 over $\mathbb{Q}(j)$.

This construction which goes back to F. Klein, doesn't make use of a metric. In order to determine the centroid of a face (or of the image of its projection onto the sphere) only the conformal structure on S^2 is required. Indeed, the centroid of a regular polygone is its center of rotations, thus a fixed point under an operation of

$Aut(S^2) = SL(2,\mathbb{C})$.

8.3 Higher order modular ODEs

Sorting out the algebraic equations to describe the characters of the $(2,\nu)$ minimal model becomes tedious for $\nu > 5$. In contrast, the Serre derivative is a managable tool for encoding them in a compact way [26]. Since the characters are algebraic, the corresponding differential equations can not be solved numerically only, but actually analytically. We are interested in the fact that the coefficient of the respective highest order derivative can be normalised to one and all other coefficients are holomorphic in the modulus.

To the $(2,\nu)$ minimal model, where $\nu \geq 3$ is odd, we associate [3]

- the number $M = \frac{\nu-1}{2}$ introduced in eq. (8.4), which counts the characters,
- the sequence

$$\kappa_s = \frac{(\nu-2s)^2}{8\nu} - \frac{1}{24}\,, \quad s = 1,\ldots,M\,, \tag{8.6}$$

  which parametrises the characters of the $(2,\nu)$ minimal model,
- the rank $r = \frac{\nu-3}{2}$.

The character corresponding to κ_s is

$$\langle \mathbf{1} \rangle_s = f_{A,B,s} := q^{\kappa_s} \sum_{\mathbf{n}\in(\mathbb{N}_0)^r} \frac{q^{\mathbf{n}^t A\mathbf{n}+\mathbf{B}^t\mathbf{n}}}{(q)_{\mathbf{n}}}\,,$$

where

$$A = C(T_r)^{-1} \in \mathbb{Q}^{r\times r}, \quad \mathbf{B} \in \mathbb{Q}^r\,,$$

C being a Cartan matrix. The tadpole diagram of T_r is obtained from the diagram of A_{2r} by folding according to its $\mathbb{Z}_2$ symmetry.

It turns out that $\langle \mathbf{1} \rangle_s$ satisfies an Mth order ODE [26]. Given M differentiable functions $f_1,\ldots,f_M$ there always exists an ODE having these as solutions. Consider

the Wronskian determinant

$$\det\begin{pmatrix} f & \mathfrak{D}^1 f & \dots & \mathfrak{D}^M f \\ f_1 & \mathfrak{D}^1 f_1 & \dots & \mathfrak{D}^M f_1 \\ \dots & \dots & \dots & \dots \\ f_M & \mathfrak{D}^1 f_M & \dots & \mathfrak{D}^M f_M \end{pmatrix} =: \sum_{i=0}^{M} w_i \, \mathfrak{D}^i f \, .$$

Here for $m \geq 1$,

$$\mathfrak{D}^m := \mathfrak{D}_{2(m-1)} \circ \cdots \circ \mathfrak{D}_2 \circ \mathfrak{D}_0$$

is the order m differential operator which maps a modular function into a modular form of weight $2m$. ($\mathfrak{D}_k$ is the first order Serre differential operator introduced in eq. (8.3).) For $m = 0$ we set $\mathfrak{D}^0 = 1$.

Whenever f equals one of the f_i, $1 \leq i \leq M$, the determinant is zero, so we obtain an ODE in f whose coefficients are Wronskian minors containing $f_1, \dots, f_M$ and their derivatives only. These are modular when the $f_1, \dots, f_M$ and their derivatives are or when under modular transformation, they transform into linear combinations of one another (as the characters do).

Lemma 17. *Let $3 \leq \nu \leq 13$, ν odd. The characters of the $(2, \nu)$ minimal model satisfy*

$$D^{(2,\nu)}\langle \mathbf{1} \rangle = 0 \, , \tag{8.7}$$

where $D^{(2,\nu)}$ is the differential operator

$$D^{(2,\nu)} := \mathfrak{D}^M + \sum_{m=0}^{M-2} \sum_{\Omega_{2(M-m)}} \Omega_{2(M-m)} \mathfrak{D}^m \tag{8.8}$$

$$\Omega_{2(M-m)} := \alpha_m E_{2(M-m)} \, , \qquad 2 \leq M - m \leq 5 \, ,$$

$$\Omega_{12} := \alpha_0 E_{12} + \alpha_0^{(cusp)} \Delta \, .$$

Here $\Delta = \eta^{24}$ is the modular discriminant function, E_{2k} is the holomorphic Eisenstein series of weight $2k$, and the nonzero numbers α_m and $\alpha_0^{(cusp)}$ are given by the table below:

$(2,\nu)$	$(2,3)$	$(2,5)$	$(2,7)$	$(2,9)$	$(2,11)$	$(2,13)$
M	1	2	3	4	5	6
κ_M	0	$-\frac{1}{60}$	$-\frac{1}{42}$	$-\frac{1}{36}$	$-\frac{1}{33}$	$-\frac{5}{156}$
α_M	1	1	1	1	1	1
α_{M-2}		$-\frac{11}{60^2}$	$-\frac{5\cdot 7}{42^2}$	$-\frac{2\cdot 3\cdot 13}{36^2}$	$-\frac{11\cdot 53}{2^2\cdot 33^2}$	$-\frac{7\cdot 13\cdot 67}{156^2}$
α_{M-3}			$\frac{5\cdot 17}{42^3}$	$\frac{2^3\cdot 53}{36^3}$	$\frac{3\cdot 5\cdot 11\cdot 59}{2^3\cdot 33^3}$	$\frac{2^3\cdot 13\cdot 17\cdot 193}{156^3}$
α_{M-4}				$-\frac{3\cdot 11\cdot 23}{36^4}$	$-\frac{11\cdot 6151}{2^4\cdot 33^4}$	$-\frac{5\cdot 11\cdot 13\cdot 89\cdot 127}{156^4}$
α_{M-5}					$\frac{2^4\cdot 17\cdot 29}{33^5}$	$\frac{2^3\cdot 3\cdot 5\cdot 13\cdot 31\cdot 2437}{156^5}$
α_{M-6}						$-\frac{5^4\cdot 7^2\cdot 23\cdot 31\cdot 67}{156^6}$
$\alpha_{M-6}^{(\mathrm{cusp})}$						$\frac{5^2\cdot 7\cdot 11\cdot 23^2\cdot 167}{2^5\cdot 3^2\cdot 13^4\cdot 691}$

The nonzero coefficients in the order M differential operator in the $(2,\nu)$ minimal model. κ_M is displayed to explain the standard denominators of the α_m (and mark deviations from them).

Remark 18. *The prime* 691 *displayed in the denominator of* $\alpha_{M-6}^{(cusp)}$ *suggests that Bernoulli numbers are involved in the computations. This is an artefact of the choice of basis, however. Using the identity [38]*

$$E_{12} = \frac{1}{691}(441E_4^3 + 250E_6^2)\,,$$

we can write

$$\alpha_0 E_{12} + \alpha_0^{(cusp)}\Delta = -\frac{5^2\cdot 7\cdot 23}{2^7\cdot 3^5\cdot 13^6}\left(\frac{53\cdot 1069}{2^5}E_4^3 + \frac{6047}{3}E_6^2\right).$$

Only the specific values of the coefficients in eq. (8.7) seem to be new. Rather than setting up a closed formula for α_m, we shall outline the algorithm to determine these numbers, and leave the actual computation as an easy numerical exercise.

Proof. (Sketch) We first show that the highest order coefficient α_M of the ODE can

be normalised to one. For every κ_s in the list (8.6) and for $0 \le m \le M-1$, we have

$$\mathfrak{D}^m\langle \mathbf{1}\rangle_s \propto q^{\kappa_s}(1+O(q))\,. \tag{8.9}$$

Since the κ_s are all different, we know that

$$w_M \sim \prod_s q^{\kappa_s}\,, \quad q \text{ close to zero}\,,$$

where w_M is the coefficient of $\mathfrak{D}^M$ in the Wronskian. By construction, w_M has no pole at finite τ. The number of zeros can be calculated using Cauchy's Theorem [38]: Since $\mathfrak{D}^m\langle \mathbf{1}\rangle$ has weight $2m$, we find

$$\text{weight } w_M = 2\sum_{\ell=0}^{M-1}\ell = M(M-1)\,.$$

The order of vanishing $\mathrm{ord}_P(w_M)$ of w_M at a point $P \in \Gamma\setminus\mathbb{H}^+$ depends only on the orbit ΓP [38]. Denote by $\mathrm{ord}_\infty(w_M)$ the order of vanishing of w_M at ∞ (i.e. the smallest integer $n \ge 0$ such that $a_n \ne 0$ in the Fourier expansion for w_M). By eq. (8.5) for the fundamental domain of the finite index subgroup Γ of Γ_1, all orders of vanishing for Γ differ from those for Γ_1 by the same factor. Thus ([38], Propos. 2 on p. 9) generalises to subgroups $\Gamma\subset\Gamma_1$ and to

$$\mathrm{ord}_\infty(w_M) + \sum_{P\in\Gamma\setminus\mathbb{H}^+}\frac{1}{n_P}\mathrm{ord}_P(w_M) = \frac{M(M-1)}{12}\,, \tag{8.10}$$

where n_P is the order of the stabiliser. Since

$$\mathrm{ord}_\infty(w_M) = \sum_{s=1}^{M}\kappa_s = \frac{M(M-1)}{12}\,,$$

we have $\mathrm{ord}_P(w_M) = 0$ for $P\in\Gamma\setminus\mathbb{H}^+$. Thus we can divide by w_M to yield

$$\sum \tilde{\alpha}_i\mathfrak{D}^i\langle\mathbf{1}\rangle_j = 0$$

for $j = 1,\dots,M$ and the modular forms $\tilde{\alpha}_i = \frac{w_i}{w_M}$.

By (8.9), $D^{(2,\nu)}\langle\mathbf{1}\rangle_s$ is a power series of order $\ge\kappa_s$ in q. The coefficient of q^{κ_s} is a

monic degree M polynomial in κ_s, and we have

$$[D^{(2,\nu)}]_0 q^\kappa = q^\kappa \prod_{s=1}^{M} (\kappa - \kappa_s) \,, \tag{8.11}$$

since by assumption $\langle \mathbf{1} \rangle_{\kappa_s} \in \ker D^{(2,\nu)}$ for $s = 1, \dots M$. (Here $[D^{(2,\nu)}]_0$ denotes the cut-off of the differential operator $D^{(2,\nu)}$ at power zero in q.) For $2 \leq k \leq 5$, the space of modular forms of weight $2k$ is spanned by the Eisenstein series E_{2k}, while for $k = 6$, the space is two dimensional and spanned by E_{12} and Δ. However, only the Eisenstein series have a constant term, so that actually all coefficients α_m are determined by eq. (8.11). Note that vanishing of α_{M-1} (the coefficient of $\mathfrak{D}^{M-1}$ in $D^{(2,\nu)}$) implies the equality

$$-\sum_{s=1}^{M} \kappa_s = \sum_{\ell=1}^{M} \frac{1-\ell}{6} \,. \tag{8.12}$$

Indeed, the l.h.s. of eq. (8.12) equals the coefficient of κ^{M-1} in the polynomial

$$q^{-\kappa} [D^{(2,\nu)}]_0 q^\kappa$$

in eq. (8.11), while the r.h.s. equals the coefficient of κ^{M-1} in

$$q^{-\kappa} [\mathfrak{D}^M]_0 q^\kappa \,,$$

where for $0 \leq i \leq M-1$,

$$q^{-\kappa} [\mathfrak{D}^{M-i}]_0 q^\kappa = \prod_{\ell=0}^{M-i-1} (\kappa - \frac{\ell}{6}) \,.$$

Equality (8.12) thus states that $q^{-\kappa}[\mathfrak{D}^{M-1}]_0 q^\kappa$ (with leading term κ^{M-1}) does not contribute, and so is equivalent to $\alpha_{M-1} = 0$.

$\alpha_0^{(\mathrm{cusp})}$ is determined by considering the next highest order $[D^{(2,\nu)} \langle \mathbf{1} \rangle]_{\kappa+1}$ for some character. (Since modular transformations permute the characters only and have no effect on $D^{(2,\nu)}$, it is sufficient to do the computation for the vacuum character $\langle \mathbf{1} \rangle_1 = q^{\kappa_1}(1 + O(q^2))$). □

The external examiner has pointed out that the leading coefficient can also be read directly from the equation for the singular vector (Lemma 4.3 in [36]).

8.4 Generalisation to other minimal models

For $(\mu, \nu) \in \mathbb{Z}^2$, the (μ, ν)-minimal model has

$$M = \frac{(\nu - 1)(\mu - 1)}{2}$$

different characters. The set of all characters is parametrised by [3]

$$\kappa_{r,s} = \frac{(\nu r - \mu s)^2}{4\mu\nu} - \frac{1}{24}\,, \quad 1 \leq r \leq \mu - 1\,, \quad 1 \leq s \leq \nu - 1\,.$$

Due to periodicity of the conformal weights $\kappa_{r,s} + \frac{c}{24}$ (which we shall not go into here) this listing makes us count every character twice. The characters are modular functions on some finite index subgroup Γ of Γ_1 satisfying an order M differential equation, and it remains to verify that the latter has highest order coefficient $\alpha_M = 1$. We have

$$\mathrm{ord}_\infty(w_M) = \frac{1}{2} \sum_{1 \leq r \leq \mu - 1; 1 \leq s \leq \nu - 1} \kappa_{r,s} = \frac{M(M-1)}{12}\,,$$

where the factor of $1/2$ in front of the sum has been inserted to prevent the double counting mentioned above. As before, we conclude that w_M has no zeros in $\mathbb{H}^+$ and with the

Corollary 19. *The characters of the (μ, ν) minimal model satisfy an order M differential equation*

$$D^{(\nu,\mu)}\langle \mathbf{1} \rangle = 0\,,$$

where $D^{(\nu,\mu)}$ is a differential operator of the form

$$D^{(\nu,\mu)} = \mathfrak{D}^M + \sum_{m=0}^{M-2} \sum_{\Omega_{2(M-m)}} \Omega_{2(M-m)} \mathfrak{D}^m$$

where summation is over modular forms $\Omega_{2(M-m)}$ of weight $2(M-m)$.

Chapter 9

A new variation formula

The present chapter relies on joint work with W. Nahm; Sect. 9.2 is based on his ideas.

9.1 The variation formula in the literature

Formula (7.4) describes the effect on $\langle \mathbf{1} \rangle$ of a change $dG_{\mu\nu}$ in the metric. It generalises to the variation of N-point functions $\langle \varphi_1(x_1) \dots \varphi_N(x_N) \rangle$ as follows: Suppose the metric is changed on an open subset $R \subseteq S$ of the surface S. Then

$$d\langle \varphi_1(x_1) \dots \varphi_N(x_N) \rangle = -\frac{1}{2} \iint_S (dG_{\mu\nu}) \, \langle T^{\mu\nu} \varphi_1(x_1) \dots \varphi_N(x_N) \rangle \, dvol_2 \,, \tag{9.1}$$

where $dvol_2 = \sqrt{G}\, dx^0 \wedge dx^1$ ([37], eq. (12.2.2) on p. 360; see also eq. (11) in [7])[1], provided that

$$x_i \notin R \,, \qquad \text{for } i = 1, \dots, N \,. \tag{9.2}$$

Note that in order for the formula to be well-defined, $T_{\mu\nu} dx^\mu dx^\nu$ must be quadratic differential on S, i.e. one which transforms homogeneously under coordinate changes. The antiholomorphic contribution in eq. (9.1) is omitted. It is of course of the same form as the holomorphic one, up to complex conjugation.

Due to invariance of N-point functions under diffeomorphisms, $T_{\mu\nu}$ satisfies the

[1] Note that both references introduce the Virasoro field with the opposite sign. Our sign convention follows e.g. [3], cf. eq. (5.148) on p. 140.

conservation law

$$\begin{aligned} 0 = \nabla_\mu T^\mu{}_z &= \nabla_z T^z{}_z + \nabla_{\bar z} T^{\bar z}{}_z \\ &= \partial_z T^z{}_z + G^{z\bar z}\partial_{\bar z} T_{zz} \,, \end{aligned} \tag{9.3}$$

where ∇ is the covariant derivative of the Levi-Cività connection on S w.r.t. the metric $G_{\mu\nu}$. Here we have used that $T^z{}_z$ transforms like a scalar [12], whence $\nabla_z T^z{}_z = \partial_z T^z{}_z$. Moreover, $\nabla_\mu G^{\mu\nu} = 0$, and $\nabla_{\bar z} T_{zz} = \partial_{\bar z} T_{zz}$ [12], which is true since T_{zz} takes values in a holomorphic line bundle.

A Weyl transformation $G_{\mu\nu} \mapsto \mathcal{W} G_{\mu\nu}$ changes the metric only within the respective conformal class. (In any chart (U, x) on S, such transformation is given by $G_{\mu\nu}(x) \mapsto h(x) G_{\mu\nu}(x)$ with $h(x) \neq 0$ on all of U.) The effect of a Weyl transformation on N-point functions is described by the trace of T (eq. (3) on p. 310 in [7]), which equals

$$T_\mu{}^\mu = T_z{}^z + T_{\bar z}{}^{\bar z} = 2T_z{}^z = \frac{c}{24\pi}\,\mathcal{R}.\mathbb{1} \,, \tag{9.4}$$

([3], eq. (5.144) on page 140, which is actually true for the underlying fields). Here $\mathbb{1}$ is the identity field, and $\mathcal{R}$ is the scalar curvature of the Levi-Cività connection for ∇ on S. The non-vanishing of the trace (9.4) is referred to as the *trace* or *conformal anomaly*.

Since $T_\mu{}^\mu$ is a multiple of the unit field, the restriction (9.2) is unnecessary. Thus under a Weyl transformation $G_{\mu\nu} \mapsto \mathcal{W} G_{\mu\nu}$, all N-point functions change by the same factor Z (equal to $\langle \mathbf{1} \rangle$), given by

$$d \log Z = -\frac{c}{24\pi} \iint \mathcal{R}\, d\mathcal{W}\, dvol_2 \,.$$

While T_{zz} transforms as a two-form, it is not holomorphic. We will now redefine the Virasoro field to obtain a holomorphic field, but which as a result of the conformal anomaly, does not transform homogeneously in general.

Lemma 20. *[7] Suppose S has scalar curvature $\mathcal{R} = const.$ Let*

$$\frac{1}{2\pi} T(z) := T_{zz} - \frac{c}{24\pi} t_{zz} \,, \tag{9.5}$$

(with the analogous equation for $\bar T(\bar z)$), where

$$t_{zz} := \left(\partial_z \Gamma^z{}_{zz} - \frac{1}{2} (\Gamma^z{}_{zz})^2 \right) . \mathbb{1} \,.$$

Here $\Gamma^z{}_{zz} = \partial_z \log G_{z\bar{z}}$ *is the Christoffel symbol. We have*

$$\partial_{\bar{z}} T(z) = 0 \, .$$

Proof. Direct computation shows that

$$\partial_{\bar{z}} t_{zz} = -\frac{1}{2} G_{z\bar{z}} \, \partial_z (\mathcal{R}.\mathbb{1}) \, .$$

From the conservation law eq. (9.3) follows

$$\begin{aligned} \partial_{\bar{z}} T_{zz} = & -G_{z\bar{z}} \, \partial_z T^z{}_z \\ = & -\frac{c}{48\pi} G_{z\bar{z}} \, \partial_z (\sqrt{G} \, \mathcal{R}.\mathbb{1}) = \frac{c}{24\pi} \, \partial_{\bar{z}} t_{zz} \, . \end{aligned}$$

□

Thus for constant sectional curvature, $T(z)$ is a holomorphic quadratic differential.

Remark 21. *t_{zz} defines a projective connection: Under a holomorphic coordinate change, $z \mapsto w$ such that $w \in \mathcal{D}(S)$,*

$$t_{ww} \, (dw)^2 = t_{zz} \, (dz)^2 - S(w)(z).\mathbb{1} \, (dz)^2 \, ,$$

where $S(w)$ is the Schwarzian derivative. t_{zz} is known as the Miura transform *of the affine connection given by the differentials $\Gamma^z{}_{zz} dz$.*

$T(z)$ is the holomorphic field introduced in Part I.[2]

9.2 A new variation formula

Let S be a Riemann surface. We introduce

- γ : one-dimensional smooth submanifold of S , topologically isomorphic to S^1,
- R : a tubular neighbourhood of γ in S ,
- A : a vector field which conserves the metric on S and is holomorphic on R .

[2] Our notations differ from those used in [7]. Thus the standard field $T(z)$ in [7] equals $-T_{zz}$ in our exposition, and the field $\tilde{T}(z)$ in [7] equals $-\frac{1}{2\pi} T(z)$ here.

We think of $A = \alpha(z)\frac{\partial}{\partial z} \in TR$ as an infinitesimal coordinate transformation

$$\begin{aligned} z \quad &\mapsto \quad w(z) = \left(1 + \epsilon\alpha(z)\frac{\partial}{\partial z}\right) z \\ &= z + \epsilon\alpha(z)\,, \end{aligned} \tag{9.6}$$

where $|\epsilon| \ll 1$. We suppose $\alpha = 1$.

Theorem 4. *Suppose S has scalar curvature $\mathcal{R} = 0$. Let φ be a holomorphic field on S. The effect of the transformation (9.6) with $\alpha = 1$ on $\langle\varphi(w)\rangle$ is*

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \langle\varphi(w)\rangle = -i \oint_\gamma \langle T_{zz}\, \varphi(w)\rangle\, dz\,,$$

provided that

$$w \text{ does not lie on the curve } \gamma\,. \tag{9.7}$$

In particular, as w is not enclosed by γ, $\langle\varphi(w)\rangle$ doesn't change.

Proof. By property (9.7), the position of φ is not contained in a small tubular neighbourhood R of γ. Let

$$R \setminus \gamma = R_{\text{left}} \sqcup R_{\text{right}}$$

be the decomposition in connected parts left and right of γ (we assume γ has positive orientation). Let $W \subset S$ be an open set s.t.

$$\overline{W} \cap \gamma = \emptyset\,, \quad W \cup R = S\,.$$

We let $F : R \to [0,1]$ be a smooth function s.t.

$$\begin{aligned} F &= 1 \quad \text{on} \quad R_{\text{left}} \cap W\,, \\ F &= 0 \quad \text{on} \quad R_{\text{right}} \cap W\,. \end{aligned}$$

Let ϵ be so small that $z \in W^c = S \setminus W$ implies $\exp(\epsilon F)(z) \in R$. Define a new metric manifold $(S^\epsilon, G^\epsilon_{z\bar{z}})$ by

$$\begin{aligned} S^\epsilon|_W &:= S|_W \\ G^\epsilon_{z\bar{z}}(z)\,|dz|^2 &:= G_{z\bar{z}}(\exp(\epsilon F)(z))\,|d\exp(\epsilon F)(z)|^2\,, \quad z \in W^c\,. \end{aligned}$$

We have

$$dG_{\mu\nu}T^{\mu\nu} = dG_{\bar{z}\bar{z}}T^{\bar{z}\bar{z}} + \text{antiholomorphic contributions} + \text{Weyl terms}\,,$$

where we disregard the antiholomorphic contributions $\sim T_{\bar{z}\bar{z}}$, and the Weyl terms are absent since by assumption $\mathcal{R} = 0$. Alternatively, we can describe the change in the metric by the map

$$|dz|^2 \mapsto |dz + \mu d\bar{z}|^2 = dzd\bar{z} + \mu d\bar{z}d\bar{z} + \dots\,,$$

where

$$\mu = \epsilon\partial_{\bar{z}}F + O(\epsilon^2)$$

is the Beltrami differential. Thus

$$dG_{\bar{z}\bar{z}} = 2G_{z\bar{z}}\,d\mu(z,\bar{z})\,.$$

Eq. (9.1) yields

$$\begin{aligned}\frac{d\langle\varphi\rangle}{d\epsilon}|_{\epsilon=0} &= -\frac{1}{2}\iint_S \frac{\partial G_{\mu\nu}}{\partial\epsilon}|_{\epsilon=0}\,\langle T^{\mu\nu}\,\varphi\rangle\,dvol_2\\ &= -\frac{i}{2}\iint_S 2G_{z\bar{z}}\,\frac{\partial\mu(z,\bar{z})}{\partial\epsilon}|_{\epsilon=0}\,(G^{z\bar{z}})^2\langle T_{zz}\,\varphi\rangle\,G_{z\bar{z}}\,dz\wedge d\bar{z}\\ &= i\iint_R(\partial_{\bar{z}}F)\,\langle T_{zz}\,\varphi\rangle\,d\bar{z}\wedge dz\,,\end{aligned}$$

since $(G^{z\bar{z}})^k = (G_{z\bar{z}})^{-k}$ for $k\in\mathbb{Z}$. Here

$$\langle T_{zz}\,\varphi\rangle\,dz = \iota_A(\langle T_{zz}\,\varphi\rangle\,(dz)^2)$$

is the holomorphic 1-form given by the contraction of the holomorpic vector field $A = \frac{\partial}{\partial z}$ with the quadratic differential $\langle T_{zz}\,\varphi\rangle\,(dz)^2$, which is holomorphic on R. By Stokes' Theorem,

$$\begin{aligned}\frac{d\langle\varphi\rangle}{d\epsilon}|_{\epsilon=0} &= i\iint_R \partial_{\bar{z}}\left(F\,\langle T_{zz}\,\varphi\rangle\right)\,d\bar{z}\wedge dz\\ &= i\oint_{W_R} F\,\langle T_{zz}\,\varphi\rangle\,dz + i\oint_{W_L} F\,\langle T_{zz}\,\varphi\rangle\,dz\\ &= -i\oint_{W_L} F\,\langle T_{zz}\,\varphi\rangle\,dz\,.\end{aligned}$$

Here $W_R = N_R \cap \partial W$ and $W_L = N_L \cap \partial W$ are the left and right boundary, respectively, of W in R. We conclude that

$$\frac{d\langle\varphi\rangle}{d\epsilon}|_{\epsilon=0} = -i \oint_{W_L} \langle T_{zz}\, \varphi\rangle \, dz = -i \oint_{\gamma} \langle T_{zz}\, \varphi\rangle \, dz \,,$$

by holomorphicity on $R_{\text{left}} \cup \gamma$. □

Remark 22. *The construction is independent of F. When F approaches the discontinuous function defined by*

$$\begin{cases} F = 1 & \text{on} \quad R_{left} \,, \\ F = 0 & \text{on} \quad R_{right} \,, \end{cases}$$

we obtain a description of $(S^\epsilon, G^\epsilon_{z\bar{z}})$ by cutting along γ and pasting back after a transformation by $\exp(\epsilon)$ *on the left.*

There is a way to check the result of Theorem 4: Let φ be a holomorphic field whose position lies in a sufficiently small open set $U \subset S$ with boundary $\partial U = \gamma$. We can use a translationally invariant metric in U and corresponding coordinates $z, \bar{z}$. Then

$$T_{zz} = \frac{1}{2\pi} T(z)$$

in eq. (9.5). For $A = \frac{d}{dw}$, we have

$$\langle A\varphi(w)\ldots\rangle = \frac{1}{2\pi i} \oint_{\gamma} \langle T(z)\varphi(w)\ldots\rangle \, dz \,, \tag{9.8}$$

This can be seen in two ways.

1. Eq. (9.8) follows from the residue theorem for the OPE of $T(z) \otimes \varphi(w)$. Indeed, the Laurent coefficient of the first order pole at $z = w$ is $N_{-1}(T, \varphi)(w) = \partial_w \varphi$, which is holomorphic.

2. Alternatively, by Theorem 4,

$$\frac{d}{d\epsilon}\bigg|_{\epsilon} \langle \varphi(w+\epsilon)\ldots\rangle = \frac{1}{2\pi i} \oint_{\gamma} \langle T(z)\varphi(w)\ldots\rangle \, dz \,.$$

The two approaches are compatible!

9.3 Discussion of the metric

Let Σ_g be the genus g hyperelliptic Riemann surface

$$\Sigma_g : \quad y^2 = p(x) , \quad \deg p = n = 2g + 1 .$$

Recall that x which varies over the Riemann sphere, defines a complex coordinate on Σ_g, outside the ramification points where we must change to the y coordinate. $\mathbb{P}^1_{\mathbb{C}}$ does not allow for a constant curvature metric but we shall define a metric on $\mathbb{P}^1_{\mathbb{C}}$ which is flat almost everywhere.

Suppose we consider a genus one surface with $n = 3$. By means of the isomorphism $\mathbb{P}^1_{\mathbb{C}} \cong \mathbb{C} \cup \{\infty\}$, we may identify the branch points of Σ_1 with points $X_1, X_2, X_3 \in \mathbb{C}$ and $X_4 = \{\infty\}$, respectively.

Let $\theta \gg 1$, but finite, such that in the flat metric of $\mathbb{C}$,

$$|X_i| < \theta , \quad i = 1, 2, 3 .$$

We define $|X_4| := \infty$. For $\epsilon > 0$, define a metric

$$(ds(\epsilon))^2 = 2G_{z\bar{z}}(\epsilon)\, dz \otimes d\bar{z} \tag{9.9}$$

on $\mathbb{P}^1_{\mathbb{C}}$ by

$$2G_{z\bar{z}}(\epsilon) := \begin{cases} (1 + \epsilon\theta^2)^{-2} & \text{for} \quad |z| \leq \theta , \\ (1 + \epsilon z\bar{z})^{-2} & \text{for} \quad |z| \geq \theta . \end{cases}$$

The metric on Σ_1 is obtained by lifting.

Lemma 23. *In the disc $|z| \leq \theta$, the metric is flat, while in the area $|z| \geq \theta$, it is of Fubini-Study type of Gauss curvature $\mathcal{K} = 4\epsilon$.*

Proof. For $\rho = 2G_{z'\bar{z}'}(\epsilon)$ with

$$G_{z'\bar{z}'}(\epsilon) := \frac{1}{2\epsilon}(1 + z'\bar{z}')^{-2} \quad \text{for} \quad |z'| \geq \sqrt{\epsilon}\theta ,$$

we have [12]

$$\mathcal{R} = \rho^{-1}(-4\partial_z\partial_{\bar{z}} \log\rho) = \epsilon(1 + z'\bar{z}')^2(8\partial_{z'}\partial_{\bar{z}'} \log(1 + z'\bar{z}')^2) = 8\epsilon ,$$

and $\mathcal{R} = 2\mathcal{K}$. □

Definition 2. *Let Σ be a genus $g = 1$ Riemann surface with conformal structure defined by the position of the ramification points $\{X_i\}_{i=1}^3$ with finite relative distance on $\mathbb{P}^1_{\mathbb{C}}$. Let $G_{z\bar{z}}(\epsilon)$ be the metric defined by eq. (9.9). We define $\langle \mathbf{1} \rangle_{\{X_i\}_{i=1}^3,\epsilon,\theta}$ to be the zero-point function on $(\Sigma, G_{z\bar{z}}(\epsilon))$.*

By eq. (9.4) and the fact that on any surface, $\mathcal{R} = 2\mathcal{K}$,

$$T_{z\bar{z}} = \frac{c}{24\pi} G_{z\bar{z}} \mathcal{K}.\mathbb{1} \ ,$$

where $\mathbb{1}$ is the identity field. So according to eq. (7.4) we have for the 2-sphere S^2_θ of radius θ,

$$d \log \langle \mathbf{1} \rangle_{\{X_i\}_{i=1}^3,\epsilon,\theta} = \frac{c}{48\pi} \iint_{S^2_\theta} (d \log G_{z\bar{z}}(\epsilon)) \, \mathcal{K} \, dvol_2 \ .$$

Since $G(\epsilon) = (G_{z\bar{z}}(\epsilon))^2$, for $|z| > \theta$, the two-dimensional volume form is

$$dvol_2 = G_{z\bar{z}}(\epsilon) \, dz \wedge d\bar{z} = \frac{1}{2} \frac{\pi d(r^2)}{(1 + \epsilon r^2)^2} \ .$$

Now

$$d \log \langle \mathbf{1} \rangle_{\{X_i\}_{i=1}^3,\epsilon,\theta} = dI_{|z|<\theta} + dI_{|z|>\theta} \ ,$$

where for $\varrho_0^2 := \epsilon \theta^2$, the integrals yield

$$\begin{aligned} dI_{|z|<\theta} &= -\frac{c\theta^2}{12} \, d(\epsilon) \, \frac{\varrho_0^2}{(1+\varrho_0^2)^3} \ , \\ dI_{|z|>\theta} &= -\frac{c}{12} \, (d \log \epsilon) \int_{|\varrho|^2 > \varrho_0^2} \frac{\varrho^2 \, d(\varrho^2)}{(1+\varrho^2)^3} = -\frac{c}{24} \, (d \log \epsilon) \, (1 + O(\varrho_0^4)) \ . \end{aligned}$$

So for $|\varrho_0| \ll 1$,

$$\langle \mathbf{1} \rangle_{\{X_i\}_{i=1}^3,\epsilon,\theta} = \epsilon^{-\frac{c}{24}(1+O(\varrho_0^4))} \, Z \, \exp\left(-\frac{c}{12} \, \frac{\varrho_0^4}{(1+\varrho_0^2)^3} \right) , \tag{9.10}$$

where $Z \in \mathbb{C}$ is an integration constant.

Variation of ϵ rescales the metric within the conformal class defined by the branch points. In the limit as $\epsilon \searrow 0$,

$$G_{z\bar{z}} := \lim_{\epsilon \searrow 0} G_{z\bar{z}}(\epsilon) = \frac{1}{2} \quad \text{for} \quad |z| < \infty \ , \tag{9.11}$$

(and is undefined for $|z| = \infty$). Thus $\mathbb{P}^1_{\mathbb{C}}$ becomes an everywhere flat surface except for the point at infinity, which is a singularity for the metric.

Definition 3. *Let Σ_1 be a genus $g = 1$ Riemann surface with conformal structure defined by the position of the ramification points $\{X_i\}_{i=1}^3$ with finite relative distance on $\mathbb{P}^1_{\mathbb{C}}$. Let $G_{z\bar{z}}$ be the metric on Σ defined by eq. (9.11). We define the zero-point function on $(\Sigma_1, G_{z\bar{z}})$ by*

$$\langle \mathbf{1} \rangle_{\{X_i\}_{i=1}^3} := \lim_{\rho_0 \searrow 0} \epsilon^{\frac{c}{24}(1+O(\varrho_0^4))} \langle \mathbf{1} \rangle_{\{X_i\}_{i=1}^3, \epsilon, \theta} \, .$$

Thus $\langle \mathbf{1} \rangle_{\{X_i\}_{i=1}^3} = Z$. We shall also write $\langle \mathbf{1} \rangle_{\text{sing.}}$ to emphasise distinction from the 0-point function on the flat torus $(\Sigma_1, |dz|^2)$, which we denote by $\langle \mathbf{1} \rangle_{\text{flat}}$.

Remark 24. *The reason for introducing ϵ and performing $\lim_{\epsilon \searrow 0}$ is the fact that the logarithm of the Weyl factor $\mathcal{W}$ is not defined for surfaces with a singular metric and infinite volume. We have*

$$d \log \frac{\langle \mathbf{1} \rangle_{sing.}}{\langle \mathbf{1} \rangle_{flat}} = d \log \mathcal{W} \, ,$$

so $\mathcal{W}$ is determined only up to a multiplicative constant, which is infinite for $\epsilon = 0$.

Our method is available for any surface $\Sigma_g : y^2 = p(x)$ with $\deg p = n \geq 3$. When n is odd, the point at infinity is a non-distinguished element in the set of ramification points on Σ_g. We shall distribute the curvature of Σ_g evenly over these. Using the Gauss-Bonnet theorem, the total curvature is recovered as

$$\int_{\Sigma_g} \mathcal{K} \, dvol_2 = 2\pi \, \chi(\Sigma_g) = 4\pi(1-g) = 8\pi - 2\pi(2g+2) \, .$$

We interpret 8π as the contribution to the curvature from the $g = 0$ double covering and -2π from any branch point.

The method is now available for arbitrary genus $g \geq 1$ hyperelliptic Riemann surfaces and will in the following be checked against the case $g = 1$.

9.4 The main theorem

We now get to an algebraic description of the effect on an N-point function as the position of the ramification points of the surface is changed.

Theorem 5. *Let Σ_g be the hyperelliptic Riemann surface*

$$\Sigma_g : \quad y^2 = p(x) \, , \quad n = \deg p = 2g+1 \, ,$$

with roots X_j. We equip the $\mathbb{P}^1_{\mathbb{C}}$ underlying Σ_g with the singular metric which is equal to

$$|dz|^2 \quad \text{on } \mathbb{P}^1_{\mathbb{C}} \setminus \{X_1 \ \ldots, X_n\} \, .$$

Let $\langle \, \rangle_{sing}$ be a state on Σ_g with the singular metric. We define a deformation of the conformal structure by

$$\xi_j = dX_j \quad \text{for} \quad j = 1, \ldots, n \, .$$

Let (U_j, z) be a chart on Σ_g containing X_j but no field position. We have

$$d\langle \varphi \ldots \rangle_{sing} = \sum_{j=1}^{n} \left(\frac{1}{2\pi i} \oint_{\gamma_j} \langle T(z) \varphi \ldots \rangle_{sing} \, dz \right) \xi_j \, , \tag{9.12}$$

where γ_j is a closed path around X_j in U_j.

Proof. On the chart (U, z), we have $\frac{1}{2\pi} \, T(z) = T_{zz}$ in eq. (9.5), outside the points which project onto one of the X_j for $j = 1, \ldots, n$ on $\mathbb{P}^1_{\mathbb{C}}$. Moreover, γ does not pick up any curvature for whatever path γ we choose. Since

$$d\langle \mathbf{1} \rangle_{\text{sing.}} = \sum_{i=1}^{n} \xi_i \frac{\partial}{\partial X_i} \langle \mathbf{1} \rangle_{\text{sing.}} \, ,$$

formula (9.12) follows from Theorem 4. □

Chapter 10

Application to the case $g = 1$

10.1 Algebraic approach

Let Σ_1 be the genus 1 Riemann surface

$$\Sigma_1 : \quad y^2 = p(x) , \quad \deg p = n = 3 ,$$

with ramification points $X_1, X_2.X_3$. Throughout this section, we shall assume that

$$\sum_{i=1}^{3} X_i = 0 . \tag{10.1}$$

We introduce some notation: Let $m(X_1, \xi_1, \ldots, X_n, \xi_n)$ be a monomial. We denote by

$$\overline{m(X_1, \xi_1, \ldots, X_n, \xi_n)}$$

the sum over all distinct monomials $m(X_{\sigma(1)}, \xi_{\sigma(1)}, \ldots, X_{\sigma(n)}, \xi_{\sigma(n)})$, where σ is a permutation of $\{1, \ldots, n\}$. E.g. eq. (10.1) reads $\overline{X}_1 = 0$, and

$$\overline{X_1 X_2} = \sum_{\substack{i,j=1 \\ i<j}}^{3} X_i X_j , = X_1X_2 + X_1X_3 + X_2X_3 ,$$

(for $n = 3$). For any state $\langle\ \rangle$ on Σ_1, the Virasoro 1-point function on Σ is given by Theorem 1 of Part I,

$$\langle T(x) \rangle = \frac{c}{32} \frac{[p']^2}{p^2} \langle \mathbf{1} \rangle + \frac{\Theta(x)}{4p} , \tag{10.2}$$

where $\Theta(x) = \Theta^{[1]}(x)$ in the notations of Part I (the polynomial $\Theta^{[y]}$ is absent),

$$\Theta(x) = -ca_0 x\langle \mathbf{1}\rangle + \mathbf{A}_1 \,, \tag{10.3}$$

where a_0 is the leading coefficient of p, and $\mathbf{A}_1 \propto \langle \mathbf{1}\rangle$ is constant in x. The connected Virasoro 2-point function for the state $\langle\,\rangle$ on Σ_1 is given by Theorem 2 in Part I. Here we note that

$$P(x_1, x_2, y_1, y_2) = P^{[1]}(x_1, x_2) \tag{10.4}$$

is constant in position, but depends on $\langle \mathbf{1}\rangle$ and $\mathbf{A}_1$. For the 1-forms $\xi_j = dX_j$ ($j = 1, 2, 3$) we introduce the matrices

$$\Xi_{3,0} = \begin{pmatrix} X_1 & X_2 & X_3 \\ 1 & 1 & 1 \\ \xi_1 & \xi_2 & \xi_3 \end{pmatrix}, \quad \Xi_{3,1} = \begin{pmatrix} X_1 & X_2 & X_3 \\ 1 & 1 & 1 \\ \xi_1 X_1 & \xi_2 X_2 & \xi_3 X_3 \end{pmatrix},$$

and the 3×3 Vandermonde matrix

$$V_3 := \begin{pmatrix} 1 & X_1 & X_1^2 \\ 1 & X_2 & X_2^2 \\ 1 & X_3 & X_3^2 \end{pmatrix}.$$

For later use, we note that

$$\begin{aligned} \det V_3 &= \prod_{1 \le i < j \le 3} (X_j - X_i) \\ &= (X_1 - X_2)(X_2 - X_3)(X_3 - X_1)\,, \\ \frac{\det \Xi_{3,0}}{\det V_3} &= \frac{\xi_1}{(X_1 - X_2)(X_3 - X_1)} + \text{cyclic}\,, \qquad (10.5) \\ \frac{\det \Xi_{3,1}}{\det V_3} &= \frac{\xi_1 X_1}{(X_1 - X_2)(X_3 - X_1)} + \text{cyclic}\,. \qquad (10.6) \end{aligned}$$

We let

$$\Delta^{(0)} := (\det V_3)^2\,.$$

It shall be convenient to work with the 1-form

$$\omega := -3\frac{\det \Xi_{3,1}}{\det V_3}\,. \tag{10.7}$$

A simple calculation using eq. (10.1) shows that

$$d \det V_3 = -3X_1(dX_1)(X_2 - X_3) + \text{cyclic} = -3 \det \Xi_{3,1} ,$$

so that

$$\omega = \frac{1}{2} d \log \Delta^{(0)} \tag{10.8}$$

$$= \frac{\xi_1 - \xi_2}{X_1 - X_2} + \text{cyclic} . \tag{10.9}$$

Lemma 25. *Let* $\Sigma_1 : y^2 = p(x)$*, where*

$$p = 4(x - X_1)(x - X_2)(x - X_3) ,$$

where we assume (10.1) to hold. Define a deformation of Σ_1 *by*

$$\xi_j = dX_j , \quad j = 1, 2, 3.$$

In terms of the modulus τ *and the scaling parameter* λ *(the inverse length) of the real period, we have*

$$\omega = \pi i \, E_2 \, d\tau - 6 \frac{d\lambda}{\lambda} .$$

Proof. By assumption (10.1), we can write

$$p(x) = 4(x^3 + ax + b) ,$$

where on the one hand,

$$a = \overline{X_1 X_2} , \qquad b = -X_1 X_2 X_3 .$$

and [33]

$$\Delta^{(0)} = -4a^3 - 27b^2 . \tag{10.10}$$

On the other hand, [33]

$$a = -\frac{\pi^4}{3} \lambda^4 E_4 , \qquad b = -\frac{2\pi^6}{27} \lambda^6 E_6 , \tag{10.11}$$

so

$$\Delta^{(0)} = \frac{4\pi^{12}}{27}\lambda^{12}(E_4^3 - E_6^2)\,, \tag{10.12}$$

We expand the fraction defining ω in eq. (10.7) by $\det V_3$ and show that for a, b introduced above, we have

$$\det(\Xi_{3,1} V_3) = 2a^2\, da + 9b\, db\,. \tag{10.13}$$

We now establish eq. (10.13) under the additional assumption that $\xi \propto X$. In this case both sides of eq. (10.13) are proportional to $\Delta^{(0)}$, with the same proportionality factor: On the l.h.s.,

$$\det \Xi_{3,1}|_{\xi=X} \det V_3 \propto -\det \begin{pmatrix} 1 & X_1 & X_1^2 \\ 1 & X_2 & X_2^2 \\ 1 & X_3 & X_3^2 \end{pmatrix}^2 = -\Delta^{(0)}\,.$$

On the r.h.s.,

$$\begin{aligned} da &= \overline{\xi_1 X_2} \propto 2\overline{X_1 X_2} = 2a\,, \\ db &= -\overline{\xi_1 X_2 X_3} \propto -3\overline{X_1 X_2 X_3} = 3b\,. \end{aligned}$$

From this and eq. (10.10) follows eq. (10.13). Using (10.11), (10.12), and

$$\mathfrak{D}_4 E_4 = -\frac{E_6}{3}\,, \qquad \mathfrak{D}_6 E_6 = -\frac{E_4^2}{2} \tag{10.14}$$

([38], Proposition 15, p. 49), where $\mathfrak{D}_{2\ell}$ is the Serre derivative (8.3), we find

$$2\,a^2 \frac{\partial}{\partial\tau} a + 9\,b\,\frac{\partial}{\partial\tau} b = -\frac{i\,\pi}{3} E_2 \Delta^{(0)}\,.$$

For the λ derivative, we use the description of ω by eq. (10.8). From eq. (10.12) follows

$$\frac{\partial}{\partial\lambda} \log \Delta^{(0)} = \frac{12}{\lambda}\,.$$

The last two equations prove the lemma under the assumption $\xi \propto X$. For the general case we refer to Appendix B.1. □

Under variation of the ramification points, the modulus changes according to

Lemma 26. *Under the conditions of Lemma 25, we have*

$$d\tau = -i\,\pi\lambda^2\,\frac{\det \Xi_{3,0}}{\det V_3}\,. \tag{10.15}$$

Proof. We first show that for

$$p(x) = 4(x^3 + ax + b)\,,$$

we have

$$\det(\Xi_{3,0} V_3) = 9b\,da - 6a\,db\,. \tag{10.16}$$

Indeed, if we set

$$\xi_i \propto X_i^2 - \xi_0\,, \quad \xi_0 := \frac{1}{3}\left(\sum_{i=1}^{3} X_i^2\right) = \frac{1}{3}\,\overline{X_1^2}$$

then the condition (10.1) continues to hold, and both sides of eq. (10.16) are proportional to $\Delta^{(0)}$, with the same proportionality factor: On the l.h.s.,

$$\det \Xi_{3,0}|_{\xi = X^2 - \xi_0}\,\det V_3 \propto \det\begin{pmatrix} \overline{\xi_1} & \overline{\xi_1 X_1} & \overline{\xi_1 X_1^2} \\ \overline{X_1} & \overline{X_1^2} & \overline{X_1^3} \\ 3 & \overline{X_1} & \overline{X_1^2} \end{pmatrix} = -\Delta^{(0)}\,,$$

since

$$\det\begin{pmatrix} \xi_1 & \xi_2 & \xi_3 \\ X_1 & X_2 & X_3 \\ 1 & 1 & 1 \end{pmatrix} \propto \det\begin{pmatrix} X_1^2 & X_2^2 & X_3^2 \\ X_1 & X_2 & X_3 \\ 1 & 1 & 1 \end{pmatrix} - \det\begin{pmatrix} \xi_0 & \xi_0 & \xi_0 \\ X_1 & X_2 & X_3 \\ 1 & 1 & 1 \end{pmatrix},$$

where for the present choice of ξ, the latter determinant is zero. On the r.h.s., by the fact that $\overline{X_1} = 0$,

$$\begin{aligned} \xi_0 &= \frac{1}{3}\,\overline{X_1^2} = -\frac{2}{3}\,\overline{X_1 X_2} = -\frac{2a}{3}\,, \\ \overline{X_1^3} &= -3\overline{X_1^2 X_2} - 6b\,, \\ \overline{X_1^2 X_2} &= \overline{X_1 X_2 (X_1 + X_2)} = -3b\,, \end{aligned}$$

so

$$da = -\overline{\xi_1 X_1} \propto -\overline{X_1^3} + \xi_0 \overline{X_1} = -\overline{X_1^3} = -3b\,,$$
$$db = -\overline{\xi_1 X_2 X_3} \propto -\overline{X_1^2 X_2 X_3} + \xi_0 \overline{X_1 X_2} = b\overline{X_1} + \xi_0 a = \xi_0 a = -\frac{2}{3}\,a^2\,.$$

From this and eq. (10.10) follows eq. (10.16). Now by eqs (10.11), (10.12), and (10.14),

$$9\,b\,\frac{\partial}{\partial\tau}a - 6\,a\,\frac{\partial}{\partial\tau}b = 2\pi i\,(9\,b\,\mathfrak{D}_4 a - 6\,a\,\mathfrak{D}_6 b) = \frac{i}{\pi\lambda^2}\Delta^{(0)}\,.$$

The partial derivatives are actually ordinary derivatives since from eqs (10.11) follows

$$9\,b\,\frac{\partial}{\partial\lambda}a - 6\,a\,\frac{\partial}{\partial\lambda}b = 0\,.$$

Factoring out $d\tau$ in eq. (10.16) and dividing both sides by $\Delta^{(0)}/(-i\pi\lambda^2)$ yields the claimed formula. The general case (with the assumption $\xi_i \propto X_i^2 - \frac{1}{3}\,\overline{X_1^2}$ omitted) is proved in Appendix B.2. □

Theorem 6. *Let*

$$\Sigma_1 : \quad y^2 = 4x^3 + a_2 x + a_3\,.$$

We equip the underlying $\mathbb{P}^1_{\mathbb{C}}$ *with the singular metric defined in Section 9.3. Let* $\langle\,\rangle_{sing}$ *be a state on* Σ_1 *w.r.t. this metric. Define a deformation of* Σ_1 *by*

$$\xi_j = dX_j\,, \quad j = 1, 2, 3\,.$$

Let ω *be the corresponding* 1*-form*

$$\omega = -3\frac{\det \Xi_{3,1}}{\det V_3}\,.$$

We have the following system of linear differential equations

$$(d + \frac{c}{24}\,\omega)\langle \mathbf{1}\rangle_{sing.} = -\frac{1}{8}(\mathbf{A}_1)_{sing.}\,\frac{\det \Xi_{3,0}}{\det V_3}\,, \tag{10.17}$$
$$(d + \frac{c-8}{24}\omega)\,(\mathbf{A}_1)_{sing.} = C_{sing.}\,\frac{\det \Xi_{3,0}}{\det V_3}\,,$$

where for $P^{[1]}$ from eq. (10.4),

$$C_{sing.} := -2P^{[1]} - \frac{1}{8}\langle \mathbf{1}\rangle_{sing.}^{-1}(\mathbf{A}_1)^2_{sing.} - \frac{2ca_2}{3}\langle \mathbf{1}\rangle_{sing.} .$$

In particular, in the $(2,5)$-minimal model,

$$C_{sing.} = \frac{11}{150}\langle \mathbf{1}\rangle_{sing.} a_2 .$$

In general, $C_{\mathrm{sing.}}$ is a function of $\langle \mathbf{1}\rangle_{\mathrm{sing.}}$ and $(\mathbf{A}_1)_{\mathrm{sing.}}$. Note that the occurrence of a term $\sim (\mathbf{A}_1)^2_{\mathrm{sing.}}$ in the definition of $C_{\mathrm{sing.}}$ is an artefact of our presentation since $P^{[1]}$ has been defined by means of the *connected* Virasoro 2-point function.

Remark 27. *In contrast to the ODE (8.1) for the zero-point function $\langle \mathbf{1}\rangle_{flat}$ on $(\Sigma_1, |dz|^2)$, the corresponding differential equation (10.17) for $\langle \mathbf{1}\rangle_{sing.}$ w.r.t the singular metric comes with a covariant derivative. Denote by*

$$(\mathbf{A}_1)_{flat} = 4\langle \mathbf{T}\rangle_{flat} =: \alpha_{flat}\langle \mathbf{1}\rangle_{flat} , \quad (\mathbf{A}_1)_{sing.} =: \alpha_{sing.}\langle \mathbf{1}\rangle_{sing.}$$

the parameters w.r.t. the flat and the singular metric, respectively. By eqs (8.1), (10.17) and (10.15),

$$d\log\frac{\langle \mathbf{1}\rangle_{sing.}}{\langle \mathbf{1}\rangle_{flat}} = -\frac{c}{24}\omega + \frac{1}{8\pi i\,\lambda^2}\left(\alpha_{sing.} - \alpha_{flat}\right)d\tau .$$

Using eq. (10.8), we obtain

$$\langle \mathbf{1}\rangle_{sing.} \propto (\Delta^{(0)})^{-\frac{c}{48}}\,\langle \mathbf{1}\rangle_{flat} , \tag{10.18}$$

with proportionality factor equal to $\exp\left\{\frac{1}{8\pi i\,\lambda^2}\int\left(\alpha_{sing.} - \alpha_{flat}\right)d\tau\right\}$. In particular, $\langle \mathbf{1}\rangle_{sing.}$ is not a modular function. This is due to the non-vanishing of the scalar curvature $\mathcal{R}$ in the Weyl factor $\mathcal{W}$ (cf. Remark 24).

Proof. (of the Theorem)
Notations: All state-dependent objects are understood to refer to the singular metric on Σ_1.

For $j = 1,2,3$, let γ_j be a closed path enclosing $X_j \in \mathbb{P}^1_{\mathbb{C}}$ and no other zero of p. x does not define a coordinate close to X_j, however y does. On the ramified covering, a closed path has to wind around X_j by an angle of 4π. We shall be working with the x coordinate, and mark the double circulation along γ_j in $\mathbb{P}^1_{\mathbb{C}}$ by a symbolic $2\times\gamma_j$

under the integral. Thus for $j = 1$ we have

$$\begin{aligned}\frac{1}{2\pi i}\oint_{2\times\gamma_1}\langle T(x)\rangle\ dx &= 2\lim_{x\to X_1}(x-X_1)\langle T(x)\rangle\\ &= \frac{1}{8}\left(\frac{c\langle\mathbf{1}\rangle}{X_1-X_2}+\frac{c\langle\mathbf{1}\rangle}{X_1-X_3}+\frac{\Theta(X_1)}{(X_1-X_2)(X_1-X_3)}\right)\\ &= \frac{1}{8}\frac{c(-2X_1+X_2+X_3)\langle\mathbf{1}\rangle-\mathrm{A}_0X_1-\mathbf{A}_1}{(X_1-X_2)(X_3-X_1)}\\ &= -\left(\frac{c}{4}\langle\mathbf{1}\rangle+\frac{\mathrm{A}_0}{8}\right)\frac{X_1\langle\mathbf{1}\rangle}{(X_1-X_2)(X_3-X_1)}\\ &\quad-\frac{1}{8}\frac{\mathbf{A}_1}{(X_1-X_2)(X_3-X_1)}+\frac{c}{8}\langle\mathbf{1}\rangle\frac{X_2+X_3}{(X_1-X_2)(X_3-X_1)}\,.\end{aligned}$$

So

$$\begin{aligned}d\langle\mathbf{1}\rangle=\sum_{i=1}^{3}\left(\frac{1}{2\pi i}\oint_{2\times\gamma_i}\langle T(x)\rangle\,dx\right)dX_i &= -\left(\frac{c}{4}\langle\mathbf{1}\rangle+\frac{\mathrm{A}_0}{8}\right)\frac{\det\Xi_{3,1}}{\det V_3}-\frac{1}{8}\mathbf{A}_1\frac{\det\Xi_{3,0}}{\det V_3}\\ &\quad+\frac{c}{8}\langle\mathbf{1}\rangle\left(\frac{\xi_1(X_2+X_3)}{(X_1-X_2)(X_3-X_1)}+\text{cyclic}\right),\end{aligned}$$

using eqs (10.5) and (10.6). When (10.1) is imposed and $\mathrm{A}_0 = -4c\langle\mathbf{1}\rangle$ is used, we obtain the differential equation (10.17) for $\langle\mathbf{1}\rangle$. When $\langle T(x)\rangle$ is varied by changing all ramifications points X_1, X_2, X_3 simultaneously, we must require the position x not to lie on or be enclosed by any of the corresponding three curves γ_1, γ_2 and γ_3. Then we have

$$\begin{aligned}d\langle T(x)\rangle &= \sum_{j=1}^{3}\left(\frac{1}{2\pi i}\oint_{2\times\gamma_j}\langle T(x')T(x)\rangle\,dx'\right)dX_j\\ &= \sum_{j=1}^{3}\left(\frac{\langle\mathbf{1}\rangle}{2\pi i}\oint_{2\times\gamma_j}\langle T(x')T(x)\rangle_c\,dx'\right)dX_j+\langle\mathbf{1}\rangle^{-1}\langle T(x)\rangle\,d\langle\mathbf{1}\rangle.\end{aligned}$$

Here $\langle T(x)\rangle$ is given by formula (10.2). For $\langle T(x)T(x')\rangle_c$, we use Theorem 2 in Part I. The terms $\propto yy'$ (with $y'^2 = p(x')$) do not contribute: As $X_j \in \mathbb{P}^1_{\mathbb{C}}$ is wound around twice along the closed curve γ_j, the square root y' changes sign after one tour, so the

corresponding terms cancel. Thus for $j = 1$ we have, using eq. (10.3) for $\Theta(x')$,

$$\frac{\langle \mathbf{1} \rangle}{2\pi i} \oint_{2\times\gamma_1} \langle T(x')T(x) \rangle_c \, dx' \tag{10.19}$$

$$\begin{aligned} &= 2 \lim_{x' \to X_1} (x' - X_1) \Big\{ \frac{c}{4} \frac{\langle \mathbf{1} \rangle}{(x'-x)^4} + \frac{c}{32} \frac{p'(x')p'\langle \mathbf{1} \rangle}{(x'-x)^2 p(x')p} + \frac{1}{8} \frac{p(x')\Theta + p\Theta(x')}{(x'-x)^2 p(x')p} \\ &\qquad + \frac{P^{[1]}}{p(x')p} - \frac{a_0}{8} \frac{x'\Theta + x\Theta(x')}{p(x')p} - \frac{a_0^2 c}{8} \frac{x'x\langle \mathbf{1} \rangle}{p(x')p} \Big\} \\ &= \frac{c}{16} \frac{\langle \mathbf{1} \rangle}{(X_1 - x)^2} \frac{p'}{p} + \frac{1}{4} \frac{\Theta(X_1)}{(X_1 - x)^2 p'(X_1)} \\ &\quad + \frac{2P^{[1]}}{p'(X_1)p} - \frac{a_0}{4} \frac{X_1 \mathbf{A}_1}{p'(X_1)p} - \frac{a_0}{4} \frac{x\Theta(X_1)}{p'(X_1)p} \, . \end{aligned} \tag{10.20}$$

Multiplying the first term on the r.h.s. of eq. (10.20) by ξ_1 and adding the corresponding terms as j takes the values $2, 3$ yields

$$\frac{c}{16} \langle \mathbf{1} \rangle \frac{p'}{p} \left(\frac{\xi_1}{(x - X_1)^2} + \text{cyclic} \right) = \frac{c}{32} \langle \mathbf{1} \rangle d \left(\frac{p'}{p} \right)^2 \, .$$

The cyclic symmetrisation of the remaining four terms on the r.h.s. of eq. (10.20) gives $d\left(\frac{\Theta(x)}{4p}\right) - \frac{\Theta(x)}{4p} \, d \log \langle \mathbf{1} \rangle$. We deduce the differential equation for $\mathbf{A}_1$. Firstly,

$$d\Theta(x) = 4p \, d\left(\frac{\Theta}{4p} \right) + \Theta \, \frac{dp}{p} \, .$$

By the above, using $p'(X_1) = -a_0(X_1 - X_2)(X_3 - X_1)$ with $a_0 = 4$,

$$\begin{aligned} 4p \, d\left(\frac{\Theta}{4p} \right) = &- \frac{p}{4} \Big(\frac{1}{(x - X_1)^2} \frac{\xi_1 \Theta(X_1)}{(X_1 - X_2)(X_3 - X_1)} + \text{cyclic} \Big) \\ &+ x \left(\frac{\xi_1 \Theta(X_1)}{(X_1 - X_2)(X_3 - X_1)} + \text{cyclic} \right) \\ &- 2P^{[1]} \, \frac{\det \Xi_{3,0}}{\det V_3} + \mathbf{A}_1 \frac{\det \Xi_{3,1}}{\det V_3} + \Theta(x) \, d \log \langle \mathbf{1} \rangle \, . \end{aligned} \tag{10.21}$$

Secondly, using partial fraction decomposition,

$$\frac{\Theta(x)}{p} = -\frac{1}{(x - X_1)} \, \frac{\Theta(X_1)}{4(X_1 - X_2)(X_3 - X_1)} + \text{cyclic.}$$

Solving for Θ and using that

$$\frac{dp}{p} = -\left(\frac{\xi_1}{x - X_1} + \text{cyclic} \right) \, ,$$

yields

$$\Theta(x)\,\frac{dp}{p} = \frac{p}{4}\left(\frac{\Theta(X_1)}{(x-X_1)(X_1-X_2)(X_3-X_1)} + \text{cyclic}\right)\sum_{j=1}^{3}\frac{\xi_j}{(x-X_j)}\,. \tag{10.22}$$

Note that three terms in the sum on the r.h.s. of eq. (10.22) are equal but opposite to the first term on the r.h.s. of eq. (10.21). Since $\overline{\xi_1} = 0$, we have for the remaining sum

$$\begin{aligned}&\frac{p}{4}\left(\frac{\Theta(X_1)}{(x-X_1)(X_1-X_2)(X_3-X_1)}\sum_{j\neq 1}\frac{\xi_j}{(x-X_j)} + \text{cyclic}\right)\\ &= -\left(\frac{\Theta(X_1)(\xi_2 X_3+\xi_3 X_2)}{(X_1-X_2)(X_3-X_1)} + \text{cyclic}\right) - x\left(\frac{\xi_1\Theta(X_1)}{(X_1-X_2)(X_3-X_1)} + \text{cyclic}\right),\end{aligned}$$

where the second term on the r.h.s. is equal but opposite to the one before last on the r.h.s. of eq. (10.21). For the first term we have (cf. Appendix B.3)

$$-\frac{\Theta(X_1)(\xi_2 X_3+\xi_3 X_2)}{(X_1-X_2)(X_3-X_1)} + \text{cyclic} = -\frac{2}{3}ca_2\langle\mathbf{1}\rangle\frac{\det\Xi_{3,0}}{\det V_3} - 2\mathbf{A}_1\frac{\det\Xi_{3,1}}{\det V_3}\,.$$

Using $\Theta(X_1) = -4cX_1\langle\mathbf{1}\rangle + \mathbf{A}_1$, we conclude that

$$d\mathbf{A}_1 = -\mathbf{A}_1\frac{\det\Xi_{3,1}}{\det V_3} + (-2P^{[1]} - \frac{2ca_2}{3}\langle\mathbf{1}\rangle)\frac{\det\Xi_{3,0}}{\det V_3} + \mathbf{A}_1\, d\log\langle\mathbf{1}\rangle\,.$$

Plugging in eq. (10.17) yields the claimed formula. To determine the constant in the $(2,5)$-minimal model, we write

$$p = 4x^3 + a_1x^2 + a_2x + a_3\,.$$

By Lemma 16 in Part I, using $c = -\frac{22}{5}$, we find

$$P^{[1]} = -\frac{77}{400}a_1^2\langle\mathbf{1}\rangle + \frac{2}{20}a_1\mathbf{A}_1 + \frac{143}{100}a_2\langle\mathbf{1}\rangle - \frac{1}{16}\langle\mathbf{1}\rangle^{-1}\mathbf{A}_1^2\,.$$

□

The formulation of the differential equations using determinants relies on the permutation symmetry of the equations' constituent parts. This symmetry will continue to be present as the number of ramification points increases. With the genus, however, also the degree of the polynomial Θ will grow and give rise to additional terms having no lower genus counterpart (cf. Section 10.3).

10.2 Comparison with the analytic approach, for the $(2,5)$ minimal model

We provide a rough check that the system of linear differential equations obtained from Theorem 6 for the $(2,5)$ minimal model is consistent with the system discussed in Section 8.1. By formula (10.18), we have

$$\langle \mathbf{1} \rangle = \Delta^{(0)^{-\frac{c}{48}}} f \,, \quad \mathbf{A}_1 = \Delta^{(0)^{-\frac{c}{48}}} g \,, \tag{10.23}$$

for some functions f, g of τ, with $f, g \propto \langle \mathbf{1} \rangle_z$. We have [38]

$$\Delta^{(0)} = \prod_{i<j} (X_i - X_j)^2 \sim \eta^{24} = q - 24q^2 + O(q^3) \,,$$

and so close to the boundary of the moduli space where $X_1 \approx X_2$, we have

$$(X_1 - X_2) \sim q^{\frac{1}{2}} = e^{\pi i \, \tau} \,. \tag{10.24}$$

As before, we shall work with assumption (10.1). Since in this region only the difference $X_1 - X_2$ matters, we may w.l.o.g. suppose that

$$X_2 = \text{const.}$$

($\xi_2 = 0$). In view of (10.24) on the one hand, and the series expansion of the Rogers-Ramanujan partition functions $\langle \mathbf{1} \rangle_z$ on the other, we have to show that

$$f \sim (X_1 - X_2)^{-\frac{1}{30}} \,, \quad \text{or} \quad f \sim (X_1 - X_2)^{\frac{11}{30}} \,. \tag{10.25}$$

Eq. (10.23) yields

$$d\langle \mathbf{1} \rangle = \Delta^{(0)^{-\frac{c}{48}}} df - \frac{c}{24} \omega f \Delta^{(0)^{-\frac{c}{48}}} \,,$$

using eq. (10.8), and a similar equation is obtained for $d\mathbf{A}_1$. So by Theorem 6,

$$\begin{aligned} df &= -\frac{1}{8} g \, \frac{\det \Xi_{3,0}}{\det V_3} \,, \\ (d - \frac{1}{3}\omega) g &= \frac{11}{150} a_2 f \, \frac{\det \Xi_{3,0}}{\det V_3} \,. \end{aligned} \tag{10.26}$$

Since $f \sim (X_1 - X_2)^\alpha$ for some $\alpha \in \mathbb{R}$,

$$df \sim \frac{\xi_1 \alpha}{X_1 - X_2} f \,. \tag{10.27}$$

On the r.h.s. of eq. (10.26), we have by the assumption (10.1),

$$\frac{\det \Xi_{3,0}}{\det V_3} = \frac{\xi_1}{(X_1 - X_2)(X_3 - X_1)} + \text{cyclic} \sim \frac{\xi_1}{(X_1 - X_2)(-3X_2)} \sim \frac{\omega}{(-3X_2)}$$

since $X_1 \approx X_2$, and we have omitted the regular terms. Eq. (10.26) thus yields

$$g \approx 24 X_2 \alpha f \,.$$

Now we use the differential equation for g,

$$24 X_2 \alpha (d - \frac{1}{3}\omega) f \sim \frac{11}{150} f a_2 \, \frac{\omega}{(-3X_2)}$$

which by eq. (10.27) and $a_2 \sim -12 X_2^2$ reduces to the quadratic equation

$$\alpha(\alpha - \frac{1}{3}) \sim \frac{11}{900}$$

and is solved by $\alpha = -\frac{1}{30}$ and $\frac{11}{30}$. This yields (10.25), so the check works.

10.3 Outlook: Generalisation to higher genus

For $\Sigma_g : y^2 = p(x)$ with $\deg p = n \geq 3$, we have from eq. (4.8) in Theorem 1 of Part I,

$$\Theta(x, y) = \Theta^{[1]}(x) + y\Theta^{[y]}(x) \,, \quad \deg \Theta^{[1]}(x) = n - 2 \,.$$

$\Theta^{[y]}$ does not contribute to the contour integral as $y\frac{dx}{p} = \frac{dx}{y}$ is a holomorphic differential on Σ_g. As stated in the viva, the author has established a preliminary formulation of the differential equations for $\langle \mathbf{1} \rangle_{\text{sing.}}$ and $\langle T(x) \rangle_{\text{sing}}$ for the case $n = 5$ ($g = 2$). In the following, all state-dependent objects are understood to refer to the singular metric on Σ_2. In the present case, $\Theta^{[y]}$ is absent, so

$$\langle T(x) \rangle = \frac{c}{32} \frac{[p']^2}{p^2} \langle \mathbf{1} \rangle + \frac{1}{4} \, \frac{\mathrm{A}_0 x^3 + \mathbf{A}_1 x^2 + \mathbf{A}_2 x + \mathbf{A}_3}{p} \,,$$

where A_0 is known in terms of $\langle \mathbf{1} \rangle$ by Theorem 1 of Part I, while $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3$ are parameters of the theory, in addition to $\langle \mathbf{1} \rangle$. Eq. (10.9) is adapted to $n = 5$ as

$$\omega = \sum_{s=1}^{n} \sum_{j \neq s} \frac{\xi_s}{X_s - X_j} \, .$$

The differential equation for $\langle \mathbf{1} \rangle$ now reads

$$\left(d - \frac{c}{8} \omega \right) \langle \mathbf{1} \rangle = \frac{1}{2 a_0} \left(\mathrm{A}_0 \frac{\det \Xi_{5,3}}{\det V_5} + \sum_{k=1}^{3} \mathbf{A}_k \frac{\det \Xi_{5,3-k}}{\det V_5} \right) ,$$

where V_5 is the 5×5 Vandermonde matrix and

$$\Xi_{5,k} := \begin{pmatrix} X_1^3 & X_2^3 & X_3^3 & X_4^3 & X_5^3 \\ X_1^2 & X_2^2 & X_3^2 & X_4^2 & X_5^2 \\ X_1 & X_2 & X_3^3 & X_4^3 & X_5 \\ 1 & 1 & 1 & 1 & 1 \\ \xi_1 X_1^k & \xi_2 X_2^k & \xi_3 X_3^k & \xi_4 X_4^k & \xi_5 X_5^k \end{pmatrix} , \quad k = 0, \ldots, 3 \, .$$

The derivation of the differential equation for $\langle T(x) \rangle_{\mathrm{sing}}$ has been based on the connected Virasoro 2-point function (computed in Theorem 2 of Part I) which resulted in a non-linear differential equation. An improved formulation reestablishing linearity, and the individual equations for the parameters $\mathbf{A}_i$ ($i = 1, 2, 3$), were not completed by the time of the viva.

Future work will deal with a variation formula for the Virasoro N-point function for arbitrary g and $N > 1$.

Appendix B

B.1 Completion of the Proof of Lemma 25 (Section 10.1)

It remains to show eq. (10.13) for general deformations $\xi_i = dX_i$, assuming that $\overline{X_1} = 0$, eq. (10.1). We have

$$\begin{aligned} a = \overline{X_1X_2}\,, \qquad & da = d(\overline{X_1X_2}) \\ & = \xi_1X_2 + \xi_1X_3 + \xi_2X_1 + \xi_2X_3 + \xi_3X_1 + \xi_3X_2 = \overline{\xi_1X_2} \\ b = -X_1X_2X_3\,, \qquad & db = -d(X_1X_2X_3) \\ & = -\xi_1X_2X_3 - \xi_2X_1X_3 - \xi_3X_1X_2 = -\overline{\xi_1X_2X_3}\,. \end{aligned}$$

Let $\alpha, \beta \in \mathbb{Q}$. On the one hand, since $\overline{X_1} = 0$, we have

$$(\overline{X_1X_2})^2 = \overline{X_1^2X_2^2} + 2X_1X_2X_3 \cdot \overline{X_1} = \overline{X_1^2X_2^2}\,, \tag{B.1}$$

so

$$\alpha a^2 da + \beta b\, db = \alpha\, \overline{X_1^2X_2^2} \cdot \overline{\xi_1X_2} + \beta X_1X_2X_3 \cdot \overline{\xi_1X_2X_3}\,.$$

On the other hand,

$$\begin{aligned} \det \Xi_{3,1} \det V_3 &= \det \begin{pmatrix} \xi_1X_1 & \xi_2X_2 & \xi_3X_3 \\ X_1 & X_2 & X_3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & X_1 & X_1^2 \\ 1 & X_2 & X_2^2 \\ 1 & X_3 & X_3^2 \end{pmatrix} \\ &= \det \begin{pmatrix} \overline{\xi_1X_1} & \overline{\xi_1X_1^2} & \overline{\xi_1X_1^3} \\ 0 & \overline{X_1^2} & \overline{X_1^3} \\ 3 & 0 & \overline{X_1^2} \end{pmatrix} \\ &= 3\left(\overline{X_1^3} \cdot \overline{\xi_1X_1^2} - \overline{X_1^2} \cdot \overline{\xi_1X_1^3}\right) + \left(\overline{X_1^2}\right)^2 \cdot \overline{\xi_1X_1}\,. \end{aligned}$$

Here

$$\left(\overline{X_1^2}\right)^2 = 4\left(\overline{X_1 X_2}\right)^2 = 4\,\overline{X_1^2 X_2^2} \qquad \text{(by eq. (B.1))}$$
$$\overline{\xi_1 X_1} = -\overline{\xi_1 X_2}\,, \tag{B.2}$$

and

$$\begin{aligned}\overline{\xi_1 X_1^2} &= -\overline{\xi_1 X_1 X_2}\\ &= -\xi_1 X_1(X_2+X_3) + \text{cyclic} = -\overline{X_1 X_2}\cdot\overline{\xi_1} + \overline{\xi_1 X_2 X_3} = \overline{\xi_1 X_2 X_3} \qquad \text{(B.3)}\\ \overline{X_1^3} &= X_1(X_2+X_3)^2 + \text{cyclic} = \overline{X_1 X_2^2} + 6X_1X_2X_3 = 3X_1X_2X_3\,, \qquad \text{(B.4)}\end{aligned}$$

since

$$\overline{X_1 X_2^2} = -X_1X_2(X_1+X_3) - X_1X_2(X_2+X_3) + \text{cyclic} = -6X_1X_2X_3 - \overline{X_1^2 X_2} = -3X_1X_2X_3\,.$$

Moreover,

$$\begin{aligned}\overline{\xi_1 X_1^3} &= \xi_1 X_1(X_2+X_3)^2 + \text{cyclic} = \overline{\xi_1 X_1 X_2^2} + 2X_1X_2X_3\cdot\overline{\xi_1} = \overline{\xi_1 X_1 X_2^2}\\ \overline{X_1^2} &= \overline{\xi_1 X_1^2 X_2^3} + \overline{\xi_1 X_1^2 X_2 X_3^2} + \overline{\xi_1 X_2^3} - X_1(X_2+X_3) + \text{cyclic} = -2\,\overline{X_1 X_2}\,, \qquad \text{(B.5)}\end{aligned}$$

and

$$\begin{aligned}\overline{X_1X_2}\cdot\overline{\xi_1 X_1 X_2^2} &= (X_1X_2 + X_1X_3 + X_2X_3)(\xi_1X_1X_2^2 + \xi_1X_1X_3^2 + \text{cyclic})\\ &= X_1^2X_2^2\cdot\xi_1X_2 + X_1^2X_3^2\cdot\xi_1X_3 + \text{cyclic}\\ &\quad + X_1X_2\cdot\xi_1X_1X_3^2 + X_1X_3\cdot\xi_1X_1X_2^2 + \text{cyclic}\\ &\quad + X_2X_3\cdot(\xi_1X_1X_2^2 + \xi_1X_1X_3^2) + \text{cyclic}\\ &= \overline{X_1^2X_2^2}\cdot\overline{\xi_1X_2} + X_1X_2X_3\cdot\overline{\xi_1X_1X_2} + X_1X_2X_3\cdot\overline{\xi_1X_2^2}\\ &= \overline{X_1^2X_2^2}\cdot\overline{\xi_1X_2}\,,\end{aligned}$$

by eq. (B.3) and

$$\begin{aligned}\overline{\xi_1 X_2^2} &= -\xi_1X_2(X_1+X_3) - \xi_1(X_1+X_2)X_3 + \text{cyclic}\\ &= -\overline{\xi_1X_1X_2} - 2\,\overline{\xi_1X_2X_3} = \overline{\xi_1X_2X_3}\,.\end{aligned}$$

We conclude that

$$\begin{aligned}\det \Xi_{3,1} \det V_3 &= 9X_1X_2X_3 \cdot \overline{\xi_1 X_2 X_3} + 6\overline{X_1X_2} \cdot \overline{\xi_1 X_1 X_2^2} - 4\overline{X_1^2X_2^2} \cdot \overline{\xi_1 X_2} \\ &= 9X_1X_2X_3 \cdot \overline{\xi_1 X_2 X_3} + 2\overline{X_1^2X_2^2} \cdot \overline{\xi_1 X_2}\,,\end{aligned}$$

and so $\alpha = 2, \beta = 9$, as required.

B.2 Completion of the Proof of Lemma 26 (Section 10.1)

It remains to show eq. (10.16) for general deformations $\xi_i = dX_i$, assuming that $\overline{X_1} = 0$, eq. (10.1).

We use the expressions for a, b, da, db listed at the beginning of Appendix B.1.

Let $\alpha, \beta \in \mathbb{Q}$. On the one hand,

$$\begin{aligned}\alpha\, a\, db + \beta\, b\, da &= -\alpha\overline{X_1X_2} \cdot \overline{\xi_1 X_2 X_3} - \beta\, X_1X_2X_3 \cdot \overline{\xi_1 X_2} \\ &= -(\alpha+\beta)\, X_1X_2X_3 \cdot \overline{\xi_1 X_2} - \overline{\xi_1 X_1^2 X_2^3} + \overline{\xi_1 X_1^2 X_2 X_3^2} + \overline{\xi_1 X_2^3}\alpha\, \overline{\xi_1 X_2^2 X_3^2}\,.\end{aligned}$$

On the other hand,

$$\begin{aligned}\det \Xi_{3,0} \det V_3 &= \det \begin{pmatrix} \xi_1 & \xi_2 & \xi_3 \\ X_1 & X_2 & X_3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & X_1 & X_1^2 \\ 1 & X_2 & X_2^2 \\ 1 & X_3 & X_3^2 \end{pmatrix} \\ &= \det \begin{pmatrix} 0 & \overline{\xi_1 X_1} & \overline{\xi_1 X_1^2} \\ 0 & \overline{\xi_1 X_1^2 X_2^3} + \overline{\xi_1 X_1^2 X_2 X_3^2} + \overline{\xi_1 X_2^3 X_1^2} & \overline{X_1^3} \\ 3 & 0 & \overline{X_1^2} \end{pmatrix} = 3\left(\overline{X_1^3} \cdot \overline{\xi_1 X_1} - \overline{X_1^2} \cdot \overline{\xi_1 X_1^2}\right).\end{aligned}$$

Eqs (B.4), (B.2), (B.5) and (B.3) from Appendix B.1 yield

$$\begin{aligned}\det \Xi_{3,0} \det V_3 &= 3\left(-3X_1X_2X_3 \cdot \overline{\xi_1 X_2} + 2\,\overline{X_1X_2} \cdot \overline{\xi_1 X_2 X_3}\right) \\ &= 3\left(-3X_1X_2X_3 \cdot \overline{\xi_1 X_2} + 2\,\overline{\xi_1 X_2^2 X_3^2} + 2X_1X_2X_3 \cdot \overline{\xi_1 X_2}\right) \\ &= -3X_1X_2X_3 \cdot \overline{\xi_1 X_2} + 6\,\overline{\xi_1 X_2^2 X_3^2}\,.\end{aligned}$$

We conclude that $\alpha = -6$, $\alpha + \beta = 3$, so $\beta = 9$. This completes the proof.

B.3 Completion of the proof of Theorem 6 (Section 10.1)

It remains to show that

$$-\frac{\Theta(X_1)(\xi_2 X_3+\xi_3 X_2)}{(X_1-X_2)(X_3-X_1)}+\text{cyclic} = -\frac{2}{3}ca_2\langle\mathbf{1}\rangle\frac{\det\Xi_{3,0}}{\det V_3}-2\mathbf{A}_1\frac{\det\Xi_{3,1}}{\det V_3}\,.$$

We have

$$\begin{aligned}\xi_2X_3+\xi_3X_2 &= (\xi_2+\xi_3)(X_2+X_3)-(\xi_2X_2+\xi_3X_3)\\ &= \xi_1X_1-(\xi_2X_2+\xi_3X_3)\\ &= 2\xi_1X_1-\overline{\xi_1X_1}\,.\end{aligned}$$

It follows that

$$-\frac{\Theta(X_1)(\xi_2X_3+\xi_3X_2)}{(X_1-X_2)(X_3-X_1)}+\text{cyclic} = \frac{8c\langle\mathbf{1}\rangle\,\xi_1X_1^2-2\mathbf{A}_1\xi_1X_1}{(X_1-X_2)(X_3-X_1)}+\text{cyclic}\,,$$

since $\overline{\xi_1X_1}$ is symmetric and both

$$\frac{1}{(X_1-X_2)(X_3-X_1)}+\text{cyclic} = 0\,, \tag{B.6}$$

$$\frac{X_1}{(X_1-X_2)(X_3-X_1)}+\text{cyclic} = 0\,. \tag{B.7}$$

Now

$$X_1^2 = -X_1(X_2+X_3) = -\frac{a_2}{4}+X_2X_3\,; \tag{B.8}$$

we claim that

$$\frac{\xi_1X_2X_3}{(X_1-X_2)(X_3-X_1)}+\text{cyclic} = \frac{a_2}{6}\frac{\det\Xi_{3,0}}{\det V_3}\,. \tag{B.9}$$

Indeed, since $\xi_1X_2X_3+\text{cyclic} = \overline{\xi_1X_2X_3}$ is symmetric, we have by eq. (B.6),

$$\frac{\xi_1X_2X_3}{(X_1-X_2)(X_3-X_1)}+\text{cyclic} = -\frac{\xi_2X_3X_1+\xi_3X_1X_2}{(X_1-X_2)(X_3-X_1)}+\text{cyclic}\,.$$

Since $\overline{\xi_1} = 0$, we have

$$
\begin{aligned}
-\frac{\xi_2 X_3 X_1 + \xi_3 X_1 X_2}{(X_1 - X_2)(X_3 - X_1)} + \text{cyclic} &= \left(\frac{\xi_1 (X_3 X_1 + X_1 X_2)}{(X_1 - X_2)(X_3 - X_1)} + \text{cyclic} \right) + \left(\frac{(\xi_3 X_3 + \xi_2 X_2) X_1}{(X_1 - X_2)(X_3 - X_1)} + \text{cyclic} \right) \\
&= \frac{a_2}{4} \frac{\det \Xi_{3,0}}{\det V_3} - \left(\frac{\xi_1 X_2 X_3}{(X_1 - X_2)(X_3 - X_1)} + \text{cyclic} \right) \\
&\quad - \left(\frac{\xi_1 X_1^2}{(X_1 - X_2)(X_3 - X_1)} + \text{cyclic} \right),
\end{aligned}
$$

using symmetry of $\overline{\xi_1 X_1}$ and eq. (B.7) again. From eq. (B.8) follows eq. (B.9), and the proof of Theorem 6 is complete.

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