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Summary: When light passes through media which are all at rest, the frequency is constant, and so, in the geometrical optics of fixed media, frequency is merely a passive parameter, with one value in one problem and another value in another. This is changed if the media are in relative motion, for now the frequency varies along a ray, and dispersion must be taken into account. Dispersion is usually small in practice, and one might be tempted to regard it as a troublesome complication, to be dealt with by some suitable approximation. But this would be a mistake from a theoretical standpoint, because it is dispersion that brings Hamiltonian dynamics into optics, and enables us to link together, as two aspects of a single mathematical theory, physical theories seemingly distinct, namely, geometrical mechanics (particles and associated de Broglie waves) and geometrical optics (phase waves and associated photons).

In this paper Hamilton's method in geometrical optics, suitably generalized, is used as a basis for relativistic geometrical optics in moving dispersive media. In mechanics it is natural to start with a particle, and develop secondarily the associated de Broglie waves. In optics, on the other hand, the natural datum is a refractive index, expressing phase velocity in terms of frequency and direction of propagation, and so one starts with waves and develops secondarily the associated photons.

As for the physical validity of the theory here developed, no new assumption is added to existing geometrical optics in the case of media

which move without acceleration; all the theory does is to replace by a single systematized plan the known method of reducing each medium to rest in turn by a Lorentz transformation. The results for accelerated media, on the other hand, must be regarded as tentative; the assumption is made that, in the local rest frame of the medium, phase velocity is given by the refractive index for the medium when completely at rest.

T A B L E   O F   C O N T E N T S

GEOMETRICAL OPTICS IN MOVING DISPERSIVE MEDIA

1. Introduction

Geometrical optics, as ordinarily understood and as developed by Hamilton in particular, deals with the propagation of light in fixed media. The frequency of the light plays a minor part; it is a fixed parameter in any optical problem, since frequency is unchanged by passage through fixed media.

The special theory of relativity shows us how to treat media in uniform motion. We can reduce each medium in turn to rest by a Lorentz transformation, investigate the optical problem for this medium by the classical method, and then, by the inverse Lorentz transformation, restore the original frame which shows the medium in motion. Frequency is no longer a fixed parameter, for it changes when light enters a moving medium.

There is a case not covered by the above plan, namely, when two media slide past one another with no intervening vacuum. There is then no frame in which both media are at rest. We may meet this difficulty by inserting a fictitious layer of vacuum between the media, a vacuum being essentially a "fixed medium" for any Galileian observer. This device may fail, however, for the passage of light from one medium to the other may be prevented by total reflection which keeps the light from entering the fictitious vacuum.

At best, this method of reducing each medium in turn to rest is mathematically clumsy, and we are tempted to embark on a general and comprehensive scheme for geometrical optics in moving media, the case of media in uniform motion being contained as a special case, but with accelerated motions also included. That is what the present paper purports to give. The method is essentially Hamilton's method in geometrical optics, now formulated in the four-dimensional space-time of Minkowski instead of in three-dimensional space, with frequency elevated from its unimportant role of constant parameter to be the fourth partner in the slowness-frequency tetrad, or, equivalently, fourth partner in the momentum-energy tetrad of the photon.

How far is this mathematical theory physical? To what extent can it be used to predict the results of optical experiments performed with moving media?

We are familiar with the physical limitations of the geometrical optics of fixed media. The wave lengths involved must be very small relative to the dimensions of the apparatus (macroscopic lengths), and they must at the same time be large relative to the microscopic structure of the medium; the frequency must be small compared with the absorption frequencies of the medium. These limitations apply, naturally, in the case of moving media.

But there is another limitation, present in the case of fixed media but somewhat more important in the case of accelerated media, and that is the contribution which physical optics must make to geometrical optics before the latter can make physical predictions. Consider a fixed medium. Geometrical optics is a mathematical theory involving a certain function, the refractive index n characterizing the medium, depending on position for a heterogeneous medium and on direction for anisotropic medium. It is not the business of geometrical optics to say what form that function n has; that is a datum to be supplied by physical optics. Likewise, in the case of moving media, a datum must be supplied.

In developing the geometrical optics of fixed media, Hamilton offered two alternative approaches corresponding respectively to corpuscular and wave theories. In the second method, he postulated a slowness-frequency equation of the form

$$\Omega(\sigma, \tau, \upsilon, \nu, x, y, z) = 0, \tag{1.1}$$

where (σ, τ, υ) is the slowness vector for waves, ν the frequency and (x, y, z) coordinates of position. This equation is the datum which physical optics must supply. For example, for an isotropic medium it reads

$$\sigma^2 + \tau^2 + \upsilon^2 - n^2/c^2 = 0, \tag{1.2}$$

where c is the velocity in vacuo and n is the refractive index, which may be a constant, or, more generally, some specified function of position.

The formal passage from fixed to moving media is extremely simple: we insert t as an eighth argument in (1.1), so that we have an equation of the form

$$\Omega(\sigma, \tau, \upsilon, \nu, x, y, z, t) = 0 \tag{1.3}$$

(we shall change the notation slightly later). It may even happen that t is absent; this occurs when the motion of the media is steady. But once again it is the duty of physical optics to supply the form of the function Ω, now complicated by the fact that it depends not only on the properties that the medium would have if at rest, but also on its motion.

Ignorance of the forms of the function Ω which occur in nature does not prevent us from developing a general theory, valid no matter what that form may be. Here are some physical situations to which the general method is applicable or may be applicable:

(a) A set of unaccelerated media, separated by vacuum, the optical properties of each medium (when at rest) being known. The media may be heterogeneous and anisotropic.

(b) As in (a) but without the vacuum separation, the media sliding past one another in contact.

(c) Fluids in accelerated motion, assuming them optically isotropic in the local rest frame.

We can feel quite confident about (a); nothing is involved but the geometrical optics of media at rest plus the special theory of relativity. It is unlikely that any one would bother to test theoretical conclusions in (a) by experiment. There is on the other hand an element of speculation in (b) and (c); the present theory here offers predictions which might be verified by experiment, or might not.

We would like to be able to follow light through a rapidly spinning disc of glass by the methods of geometrical optics, and we certainly could if the equation (1.3) was given to us in explicit form. This problem will not be treated in the present paper, but there is no harm in sketching an approach to it. First, we would have to solve the elastic problem and find the stress in the glass. Secondly, we would write down the equation of form (1.1) expressing the optical properties of stressed glass at rest (it is optically anisotropic). Thirdly, we would assume that the statical slowness-frequency equation held in the local rest frame. Thus we would arrive at an equation of the form (1.3) (in which t would not actually appear if the angular velocity was constant).

When viewed in full generality, the mathematical structure of the geometrical optics of moving dispersive media is essentially the same as that of geometrical mechanics as I have developed it recently (Synge, 1954). But there are two differences, the first psychological, the second a matter of sign, trivial and yet at times confusing.

The first difference is that for historical reasons the particle comes easily to our minds in mechanics, and the de Broglie waves associated with it are regarded as a secondary and rather surprising phenomenon. This means that geometrical mechanics is most naturally developed by means of Hamilton's first method of approach (refractive index, principle of Maupertuis). But in optics we think first of waves, and derive from them the photon concept, still very surprising to us. This indicates an approach through Hamilton's second method, the equation (1.3).

The second difference is that in mechanics we generally have de Broglie waves which travel faster than light in vacuo, whereas the speed of optical phase waves in transparent media is generally less than this fundamental velocity. Equivalently, the momentum-energy 4-vector is timelike for a particle, but spacelike for a photon in matter. For neither particle nor photon can the speed exceed the fundamental velocity; that would be inconsistent with relativity, since they might be used as signals.

On account of these differences it seems best to set up the geometrical optics of moving dispersive media as a theory in its own right and not as a corollary to geometrical mechanics, in spite of the fact that they are different aspects of a single general mathematical theory.

The theory of the present paper is set in the special theory of relativity. The extension to general relativity would involve some formal complications but no essential change. There is however one point that should be mentioned. It is commonly stated in general relativity that the history of a light ray is a null geodesic. Does this mean that light uses a null geodesic to traverse a transparent gravitating mass? That seems most unlikely. Rather, the null geodesic hypothesis, when applied inside a transparent medium, would seem to mean that, if we bore in the medium a hole having the form of a null geodesic, all matter being removed from the hole, then light will travel through that hole without running into the walls, provided the ray is aimed properly at the beginning. In fact, the null geodesic hypothesis, whether outside matter or inside it, refers to light propagation in vacuo [cf. Synge (1937)]. The treatment of reflection in vacuo in general relativity given by Synge and McConnell (1928) is in accordance with the theory of the present paper, but their treatment of refraction is not, since it is based on the null geodesic hypothesis for the propagation of light in a medium.

The material which follows was presented in Seminar lectures at the Dublin Institute for Advanced Studies in 1954, in lectures at the St. Andrews Mathematical Colloquium in 1955, and, in brief form, in lectures at several places in the United States and Canada in 1956.

2. Kinematics of a 3-wave. Lamination.

We shall use the flat space-time of Minkowski with coordinates x_r ($x_4 = ict$) such that the fundamental form is $dx_n dx_n$. Latin suffixes take the values 1, 2, 3, 4, Greek suffixes the values 1, 2, 3, and the summation convention is understood for both, unless it is indicated to the contrary. We note that

$$\begin{aligned} dx_n dx_n &> 0 \text{ for spacelike } dx_r \\ dx_n dx_n &= 0 \text{ for null } dx_r \qquad (2.1) \\ ds^2 = - dx_n dx_n &< 0 \text{ for timelike } dx_r ; \end{aligned}$$

the element of proper time is $d\tau = ds/c$.

The motion of a transparent medium may be described by giving the world lines of its particles. Let $\mu_r = dx_r/ds$ be the unit tangent vector to the world line at any event; then the moving medium is described kinematically by the unit vector field

$$\mu_r = \mu_r(x), \qquad \mu_n \mu_n = -1. \tag{2.2}$$

This is the 4-velocity of the medium; its 3-velocity w_ρ is related to the 4-velocity by

$$\begin{aligned} \mu_\rho &= \gamma_w w_\rho / c, \qquad \mu_4 = i\gamma_w, \\ \gamma_w &= (1 - w^2/c^2)^{-\frac{1}{2}}, \qquad w^2 = w_\rho w_\rho. \end{aligned} \tag{2.3}$$

At any event there is a local rest frame, say $S^{(o)}$, and we shall use (o) to indicate components relative to it; we have

$$\mu_\rho^{(o)} = 0, \qquad \mu_4^{(o)} = i. \tag{2.4}$$

If $F(x)$ is any function of the space-time coordinates and C a constant, the equation

$$F(x) = C \tag{2.5}$$

defines a 3-space. Its intersection with x_4 = constant is a 2-space which changes with x_4, and so we may regard (2.5) as a <u>3-wave</u>, which is the history of a moving <u>2-wave</u>. Note that the 3-wave is an absolute thing, but the 2-waves which compose it depend on the frame of reference. It is easy to show (cf. Synge, 1954, p.27) that the 3-velocity u_ρ of the 2-wave (taken normal to it) is

$$u_\rho = - ic \frac{F_{,\rho} F_{,4}}{F_{,\pi} F_{,\pi}} , \tag{2.6}$$

the commas indicating partial derivatives. For the speed u of the 2-wave we have

$$u^2 = - c^2 \frac{F^2_{,4}}{F_{,\pi} F_{,\pi}} . \tag{2.7}$$

It is immediately obvious that $u > c$ if $F_{,r} F_{,r} < 0$, i.e. if $F_{,r}$ is timelike, and that $u < c$ if $F_{,r}$ is spacelike.

Now u^2 as in (2.7) has not an invariant meaning, but $u^{(o)2}$ has, this being the speed of the 2-wave in the local rest frame of the medium. We easily verify that this invariant is

$$u^{(o)2} = c^2 \frac{(\mu_r F_{,r})^2}{F_{,s} F_{,s} + (\mu_s F_{,s})^2} , \tag{2.8}$$

for this is an invariant expression which reduces to (2.7) when we substitute from (2.4).

If we give a continuous range of values to the constant C in (2.5), we get ∞^1 3-waves, ordered in the sense of C increasing. This we may call a lamination of space-time.

To bring out a distinction between a set of phase 3-waves (see next section) and a lamination, we note that the function F (x) which defines a lamination is by no means unique. For, if G is any monotonic function, then the lamination (2.5) is equally well described by the equation

$$K(x) = D, \tag{2.9}$$

where

$$K(x) = G[F(x)], \quad D = G(C). \tag{2.10}$$

3. Phase 3-waves. Frequency.

Let C (Fig. 1) be the (timelike) world line of a source which emits light

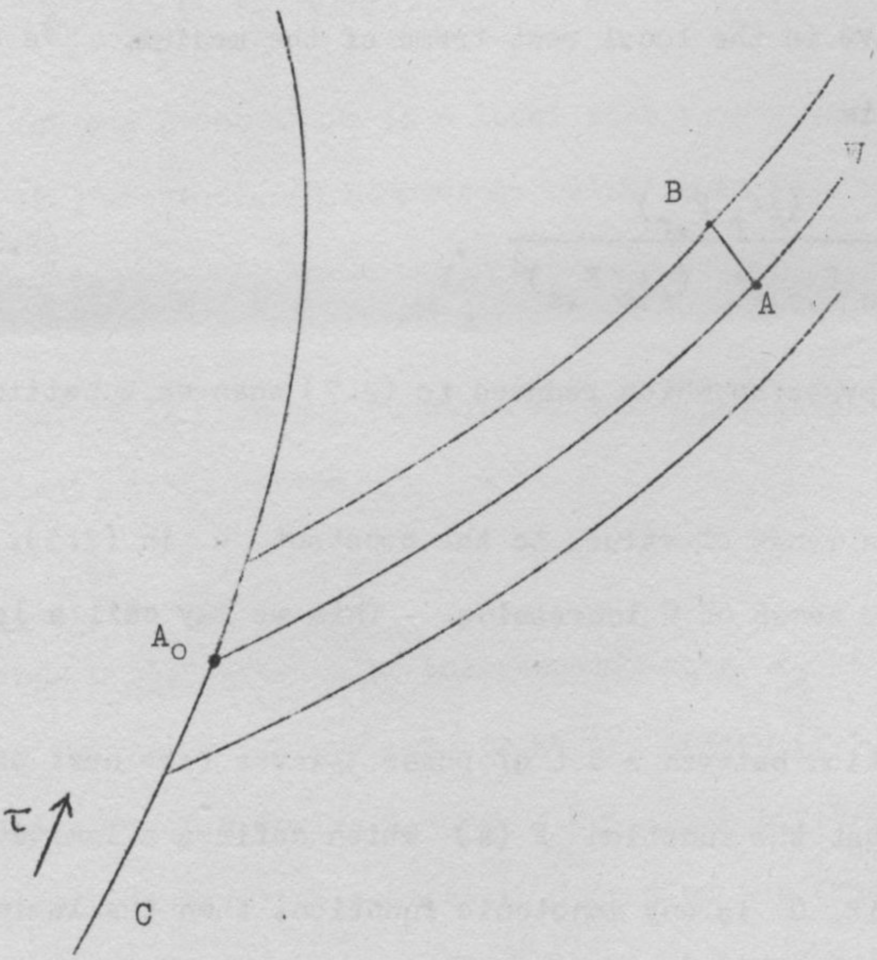

Fig. 1

Phase 3-waves emitted by a source with world line C.

of period T, measured in the proper time of C. The phase angle ϕ of emission is

$$\phi = 2\pi \tau / T , \tag{3.1}$$

where τ is proper time on C. The phase is $\exp i\phi$ or, equivalently, the number pair $(\cos \phi , \sin \phi)$.

We think of the phase angle as propagated out from C according to some law not yet specified, with the result that space-time is filled with <u>phase 3-waves</u> on each of which the phase angle ϕ is constant. This means that to any given event x_r there corresponds a value of ϕ, so that we can write $\phi = \phi(x)$; it is convenient to introduce the <u>phase function</u> $F(x)$ defined as

$$F(x) = -c\,\phi / 2\pi = -c\,\tau / T . \tag{3.2}$$

Note that if x_r is A (Fig. 1), then τ is the proper time on C of that event A_0 where the phase 3-wave W through A cuts C.

By giving all values to τ in (3.2) we get a lamination, but a set of phase 3-waves is something more than a lamination because a definite value of ϕ is attached to each 3-wave; in fact, the function $F(x)$ is physically significant, and we cannot make a transformation as in (2.10) (preserving the lamination) without losing this significance.

Note that the phase 3-waves have a natural order, viz. that of increasing τ, or equivalently increasing ϕ.

The formula (2.6) now gives the <u>phase velocity</u> u_ρ ; (2.7) gives the <u>phase speed</u> u ; (2.8) gives the phase speed $u^{(o)}$ in the local rest frame. We recall that

$$u > c \text{ if } F_{,r} F_{,r} < 0 \quad (F_{,r} \text{ timelike}),$$

$$u = c \text{ if } F_{,r} F_{,r} = 0 \quad (F_{,r} \text{ null}), \qquad (3.3)$$

$$u < c \text{ if } F_{,r} F_{,r} > 0 \quad (F_{,r} \text{ spacelike}).$$

Naturally these also hold for $u^{(o)}$. The second case corresponds to a vacuum; the third is the most usual in optical media.

If we pass from an event A (Fig. 1) to an event B on the next phase 3-wave with the same phase, we have

$$\int_A^B \phi_{,r}\, dx_r = 2\pi .$$

Here we encounter the fundamental restriction of geometrical optics (high frequency or short wave length), for we wish to replace this equation by an apparently ridiculous one:

$$\phi_{,r}\, dx_r = 2\pi , \qquad (3.3a)$$

dx_r being the displacement AB .

This last equation is accurately true for finite increments dx_r provided $\phi(x)$ is a linear function, so that $\phi_{,rs} = 0$ and $\phi_{,r}$ are constants. We have, then, a choice in making (3.3a) acceptable: either the wavelength is very small (then dx_r is very small and $\phi_{,r}$ very large), or ϕ is approximately linear in the range of the finite step dx_r. The unattained limit here present is an unavoidable source of confusion in geometrical optics, and we have to put up with it.

Accepting (3.3a), we translate it, by (3.2), into

$$F_{,r}\, dx_r = -c < 0 . \qquad (3.4)$$

Note the minus sign here. Had we defined $F(x)$ with the opposite sign in (3.2), we would have got c instead of $-c$. But to discuss all cases, we must choose a definition of $F(x)$ and stick to it; (3.2) seems to be the most convenient in later work.

Let us see what (3.4) implies, taking $F_{,r}$ to be in turn timelike, null and spacelike; in the first two cases we whall take $F_{,r}$ pointing into the future (it is easy to make the requisite changes if it points into the past).

(i) $F_{,r}$ timelike, future-pointing.

A timelike $F_{,r}$ implies $u > c$. Let W be the phase 3-wave (orthogonal to $F_{,r}$). Then (3.4) implies that dx_r and $F_{,r}$ point to the same side of W (Fig. 2).

(ii) $F_{,r}$ null, future-pointing.

Now $u = c$. The phase 3-wave W touches the null cone, $F_{,r}$ being the null vector on the line of tangency. By (3.4) dx_r points out from that side of W on which the future null cone lies (Fig. 3).

(iii) $F_{,r}$ spacelike (usual case in a medium).

A spacelike $F_{,r}$ implies $u < c$. By (3.4) dx_r and $F_{,r}$ point to the same side of W (Fig. 4).

The above results are easily established by using special frames of reference and continuity.

Fig. 5 illustrates a phenomenon which may appear strange but is by no means impossible. C is a source in vacuo, emitting phase 3-waves. Two adjacent 3-waves of the same phase are shown, the waves being curved in space-time on

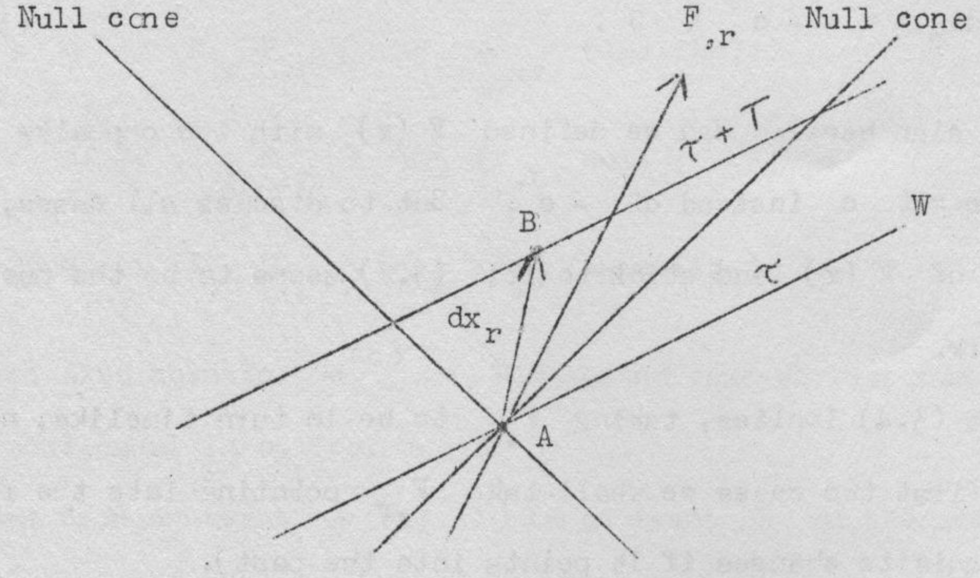

Fig. 2

Case where $F_{,r}$ is timelike (phase speed > c) and points into the future.

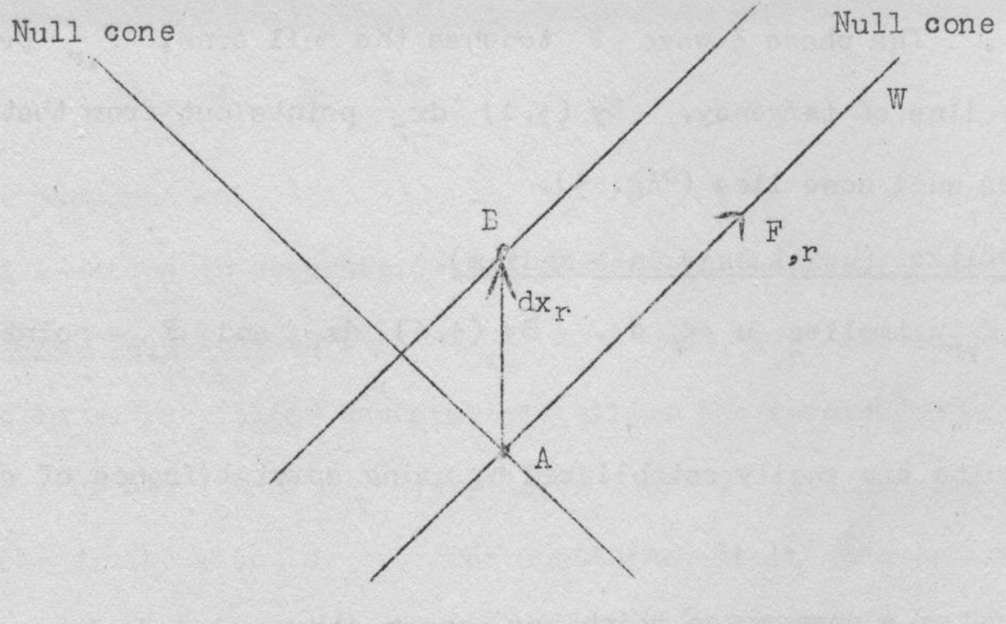

Fig. 3

Case where $F_{,r}$ is null (phase speed = c) and points into the future.

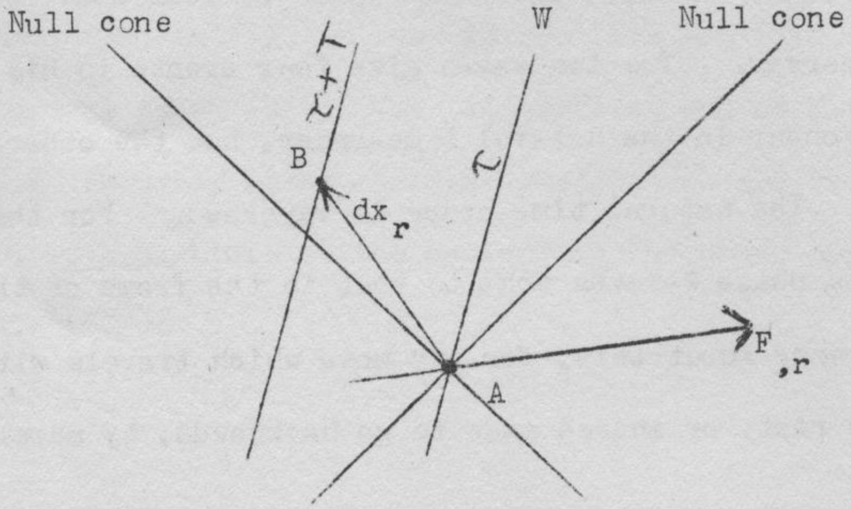

Fig. 4

Case where $F_{,r}$ is spacelike (phase speed $<$ c).

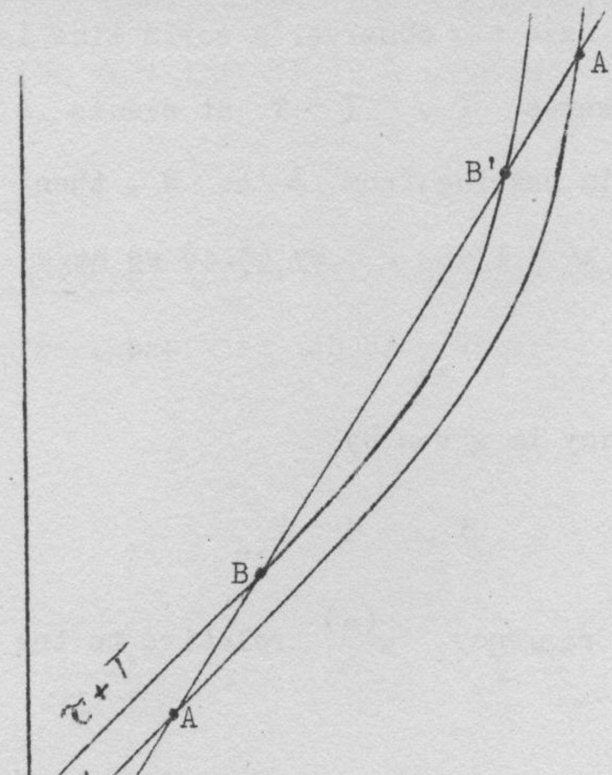

Fig. 5

Case where the natural time-order of phase waves is reversed.

passage through a medium, in which the phase speed is less than c . D is the world line of an observer. The two waves give four events in his experience. The first two (A , B) occur in the natural time-order, but the others (B' , A') in the opposite order. The natural time-order is reversed. For this to happen, it is necessary that the phase 2-waves come to rest in the frame of the observer; but there is nothing strange about this, for any wave which travels with speed less than c can be reduced to rest, or indeed made to go backwards, by merely changing the frame of reference.

Let us now consider the <u>frequency</u> of a set of phase 3-waves as measured by (a) any Galileian observer and (b) by an observer carried along by a particle of the medium.

In the first case the observer's world line is parallel to the t-axis. Let it cut the phase 3-waves τ , $\tau + T$ at events A , B respectively. If dt is the increment in t in passing from A to B , then dt is the period of the waves and the frequency is $\nu = 1/dt$. By (3.4) we have

$$F_{,4}\, ic\, dt = -c, \qquad (3.5)$$

and so the frequency is given by

$$i\nu = F_{,4}. \qquad (3.6)$$

To find the frequency $\nu^{(o)}$ relative to the medium we put $dx_r = \mu_r\, ds$ in (3.4), so that

$$F_{,r}\, \mu_r\, ds = -c \qquad (3.7)$$

for the passage from 3-wave τ to 3-wave $\tau + T$; hence

$$\nu^{(o)} = c/ds = -F_{,r}\, \mu_r. \qquad (3.8)$$

In the normal case (AB in Fig. 5) ds is positive and (3.8) gives a positive frequency; in the abnormal case (A'B') ds is negative and we get a <u>negative frequency</u>. A negative frequency means that the phase waves are received by the observer (in particular, by a particle of the medium) in the order opposite to that in which they were emitted.

4. The slowness-frequency 4-vector.

The partial derivatives $F_{,r}$ of the phase function form a 4-vector which is fundamental in the theory, and it is convenient to have another notation which shows it as a 4-vector without reference to the fact that it is a gradient. We write

$$\sigma_r = F_{,r} , \tag{4.1}$$

and call σ_r the <u>slowness-frequency 4-vector</u> (or briefly the slowness 4-vector) for reasons indicated by the following formulae. By (3.6) we have

$$\sigma_4 = i\nu , \tag{4.2}$$

and by (2.6), (2.7)

$$u_\rho = -ic\frac{\sigma_\rho\,\sigma_4}{\sigma_\pi\,\sigma_\pi} = c\nu\,\frac{\sigma_\rho}{\sigma_\pi\,\sigma_\pi}, \qquad u^2 = \frac{\nu^2\,c^2}{\sigma_\pi\,\sigma_\pi},$$
$$\sigma_\pi\,\sigma_\pi = \frac{\nu^2\,c^2}{u^2}, \qquad \sigma_\rho = c\nu\,\frac{u_\rho}{u^2}. \tag{4.3}$$

For ν positive, the 3-vectors σ_ρ and u_ρ have the same direction and the magnitude of σ_ρ is $\nu c/u$. Thus σ_ρ measures the slowness of the phase 2-waves.

The 4-vector σ_r is the relativistic generalization of Hamilton's slowness 3-vector; to within a constant factor, σ_ρ is the wave-number vector of modern physics, but Hamilton's terminology is more suggestive of its nature.

Let us write (2.8) and (3.8) in terms of the slowness 4-vector:

$$u^{(o)2} = c^2 \frac{(\mu_r \sigma_r)^2}{\sigma_n\sigma_n + (\mu_n \sigma_n)^2}, \qquad \nu^{(o)} = -\mu_r \sigma_r. \quad (4.4)$$

We note that a positive value of $\nu^{(o)}$ implies that, in the local rest frame, σ_ρ points in the direction of propagation of the phase 2-wave and not in the opposite direction; but further, since μ_r is necessarily timelike, a positive $\nu^{(o)}$ implies that σ_r points into the future (inside or outside the null cone) in the local rest frame (for which $\mu_\rho = 0$, $\mu_4 = i$).

5. The slowness-frequency equation.

So far our work has been purely kinematical and has involved no hypothesis as to the optical character of the medium. We now set up a very general hypothesis in the form of an equation

$$\Omega(\sigma, x) = 0, \quad (5.1)$$

connecting the 4-vector σ_r with the coordinates x_r. This slowness-frequency equation expresses (completely, as far as geometrical optics is concerned) the optical character of the medium in motion.

The slowness-frequency equation may be interpreted in several ways. If it is solved for σ_4 , so that we write

$$\begin{aligned} \sigma_4 &= i\, H(\sigma_1, \sigma_2, \sigma_3, x_1, x_2, x_3, x_4), \\ \nu &= H(\sigma_1, \sigma_2, \sigma_3, x_1, x_2, x_3, x_4), \end{aligned} \tag{5.2}$$

it expresses the frequency of phase waves in terms of their slowness 3-vector, position and time. On the other hand, if it is solved for $(\sigma_\pi \sigma_\pi)^{\frac{1}{2}}$, so that we write

$$(\sigma_\pi \sigma_\pi)^{\frac{1}{2}} = f(\sigma_1/\sigma_3, \sigma_2/\sigma_3, \sigma_4, x_1, x_2, x_3, x_4), \tag{5.3}$$

it expresses, by (4.3), the speed of the phase waves in terms of the direction of propagation, frequency, position and time.

For general mathematical arguments it is best to use the general form (5.1); (5.2) links the present theory with familiar Hamiltonian dynamics (H is essentially the Hamiltonian); (5.3) connects the theory with physical ideas, the speed of propagation being expressed in terms of direction of propagation and frequency.

In a vacuum the slowness-frequency equation is

$$2\,\Omega(\sigma, x) = \sigma_r \sigma_r = 0. \tag{5.4}$$

Here and later the factor 2 is introduced merely as a notational convenience. It is the <u>equation</u>, not the <u>function</u>, which is basic.

For an isotropic medium at rest we have

$$2\,\Omega(\sigma, x) = \sigma_\pi \sigma_\pi + n^2\sigma_4^2 = 0. \tag{5.5}$$

By (4.3) this is equivalent to $n = c/u$. The refractive index n is in general a function of σ_4 (to allow for dispersion) and a function of x_1 , x_2 , x_3 (to allow for heterogeneity).

For a crystalline medium at rest* we have

$$2\,\Omega(\sigma, x) = \sigma_1^2 A_2 A_3 + \sigma_2^2 A_3 A_1 + \sigma_3^2 A_1 A_2 = 0\,,$$

$$A_1 = a_1^2 \sigma_\pi \sigma_\pi + c^2\sigma_4^2\,, \quad A_2 = a_2^2 \sigma_\pi \sigma_\pi + c^2\sigma_4^2\,, \qquad (5.6)$$

$$A_3 = a_3^2 \sigma_\pi \sigma_\pi + c^2\sigma_4^2\,.$$

Here the axes are principal and a_ρ are the phase speeds in these principal directions, functions of σ_4 to allow for dispersion. The equation (5.6) reduces to (5.5) if we put $a_1 = a_2 = a_3 = c/n$.

6. Media in uniform motion.

Let there be any number of media, each in uniform motion; they may be separated by vacuum, or they may slide on one another. Each medium is, when viewed in its rest frame, either isotropic (but possibly heterogeneous) or crystalline (we consider only homogeneous crystalline media).

The history of each medium carves out a domain of space-time and for each of these domains we have, as in (5.5) and (5.6), the slowness-frequency equation in the rest frame of the domain. Our problem is a simple one: to pass from the local rest frames to a single Galileian frame from which all the media are viewed. This we do

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* cf. Hamilton (1931), p. 280; there is a slight change in notation since Hamilton's (σ, τ, υ) have the dimensions $[L^{-1}\,T]$ whereas our σ_r have the dimensions $[T^{-1}]$.

by constructing Lorentz-invariant equations which reduce to (5.5) and (5.6) in the local rest frames. For a vacuum we have of course as in (5.4)

$$2\,\Omega(\sigma, x) = \sigma_r \sigma_r = 0\,, \tag{6.1}$$

a Lorentz-invariant equation.

Let μ_r be the 4-velocity of a medium; this vector field is constant in the history of the medium, since the motion is uniform. For an isotropic medium we have

$$2\,\Omega(\sigma, x) = \sigma_r\sigma_r - (n^2 - 1)(\sigma_r\mu_r)^2 = 0 \tag{6.2}$$

(check by putting $\mu_\rho = 0$, $\mu_4 = i$). Here n is a function of $\sigma_r\mu_r$ (to allow for dispersion) and also of x_r (to allow for heterogeneity), subject to

$$\frac{\partial n}{\partial x_r}\mu_r = 0\,. \tag{6.3}$$

For a crystalline medium the principal axes appear in a general Galileian frame as three unit vectors forming an orthogonal tetrad with μ_r. We denote them by $\lambda_r^{(\rho)}$. In the rest frame of the medium we have (using the principal axes as axes of co-ordinates)

$$\lambda_1^{(1)} = \lambda_2^{(2)} = \lambda_3^{(3)} = 1\,, \tag{6.4}$$

all the other components vanishing. Then, for a general Galileian frame, we have

$$
\begin{aligned}
2\,\Omega(\sigma, x) &= \sigma_r \lambda_r^{(1)} A_2 A_3 + \sigma_r \lambda_r^{(2)} A_3 A_1 + \sigma_r \lambda_r^{(3)} A_1 A_2\,,\\
A_1 &= a_1^2\sigma_r\sigma_r + (a_1^2 - c^2)(\sigma_r\mu_r)^2\,,\\
A_2 &= a_2^2\sigma_r\sigma_r + (a_2^2 - c^2)(\sigma_r\mu_r)^2\,,\\
A_3 &= a_3^2\sigma_r\sigma_r + (a_3^2 - c^2)(\sigma_r\mu_r)^2\,.
\end{aligned}
\tag{6.5}
$$

Here $\lambda_r^{(\rho)}$ and μ_r are constants for the medium and a_ρ are functions of $\sigma_r \mu_r$ to allow for dispersion. In each medium Ω is independent of x_r, but for the whole of space-time (the totality of media and vacuum) it is to be regarded as a discontinuous function of x_r.

To follow the propagation of light through media as considered above we need the concept of rays and also the laws of reflection and refraction; these will be treated in Sections 8 and 12 respectively.

7. Isotropic fluid medium in general motion.

We cannot here attempt to set up a frequency-slowness equation for the accelerated motion of a crystalline medium or even of a solid isotropic medium, such as glass. But we can make a reasonable hypothesis for the case of an accelerated fluid medium.

We shall assume a basic relation between the phase speed $u^{(o)}$ and the frequency $\nu^{(o)}$, both measured in the local rest frame. Since we use the speed and not the 3-velocity, this amounts to an assumption of isotropy even under acceleration. It is convenient to introduce the refractive index n by the definition

$$n = c / u^{(o)} . \qquad (7.1)$$

Then by (4.4)

$$n^2 - 1 = \frac{\sigma_r \sigma_r}{(\sigma_n \mu_n)^2} , \qquad \nu^{(o)} = - \sigma_r \mu_r . \qquad (7.2)$$

The following equation then holds:

$$2\,\Omega(\sigma, x) = \sigma_r \sigma_r - (n^2 - 1)(\sigma_r \mu_r)^2 = 0 . \qquad (7.3)$$

This is the required slowness-frequency equation for a fluid in general motion provided we assume a knowledge of n as a function of $\nu^{(o)}$ and $\rho(x)$, the proper density of the fluid; we write

$$n = n(\sigma_r\mu_r , \rho) . \qquad (7.4)$$

Note that Ω involves σ_r , explicitly and also implicitly through n ; it involves x_r through $\mu_r(x)$ and also through $\rho(x)$, these functions being assumed given.

The equation (7.3) has the same form as (6.2); the difference is that we previously thought of μ_r as constant (uniform motion), and now we think of it as variable (accelerated motion).

As a particular case of (7.4) we may take the Sellmeier formula (cf. R. W. Wood (1936), p. 470)

$$n^2 - 1 = \sum_M \frac{D_M \lambda^2}{\lambda^2 - \lambda_M^2} = \sum_M \frac{D_M \nu_M^2}{\nu_M^2 - \nu^{(o)2}} , \qquad (7.5)$$

where D_M and ν_M (the absorption frequencies for the medium at rest) are to be regarded as known functions of the density ρ . Thus for a Sellmeier medium the slowness-frequency equation is

$$2\Omega(\sigma, x) = \sigma_r\sigma_r - (\sigma_r\mu_r)^2 \sum_M \frac{D_M \nu_M^2}{\nu_M^2 - (\sigma_n\mu_n)^2} . \qquad (7.6)$$

This formula becomes physically invalid as we approach the absorption frequencies; for relatively small frequencies the approximate form is

$$\left.\begin{aligned} 2\Omega(\sigma, x) &= \sigma_r\sigma_r - (n^2 - 1)(\sigma_r\mu_r)^2 , \\ n^2 - 1 &= A(\rho) + B(\rho)(\sigma_r\mu_r)^2 . \end{aligned}\right\} \qquad (7.7)$$

8. Hamilton's partial differential equation. Characteristics or rays.

When we substitute $F_{,r}$ for σ_r the slowness-frequency equation

$$\Omega(\sigma, x) = 0 \tag{8.1}$$

becomes a partial differential equation of the first order for the phase function $F(x)$. This is Hamilton's partial differential equation and we shall refer to it as HPDE; it is essentially the same as that equation which, in dynamics, is called the Hamilton-Jacobi equation.

We are concerned with the problem of solving the HPDE subject to certain initial conditions, viz. F assigned on a given 3-space. The method of solving this problem by means of characteristics is well known, but the argument will be presented here because it is desirable to have it before us in the present notation and in a form which shows us when the method breaks down.

If $\Omega(\sigma, x)$ is any function of the eight quantities σ_r, x_r (not necessarily satisfying (8.1)), the ordinary differential equations

$$\frac{dx_1}{\partial\Omega/\partial\sigma_1} = \dots = \frac{dx_4}{\partial\Omega/\partial\sigma_4} = \frac{d\sigma_1}{-\partial\Omega/\partial x_1} = \dots = \frac{d\sigma_4}{-\partial\Omega/\partial x_4} \tag{8.2}$$

define a congruence of curves in the 8-dimensional (σ, x)-space, one curve passing through any assigned point in that space. These equations may also be written

$$\frac{dx_r}{d\chi} = \frac{\partial\Omega}{\partial\sigma_r}, \qquad \frac{d\sigma_r}{d\chi} = -\frac{\partial\Omega}{\partial x_r}, \tag{8.3}$$

χ being a parameter suitably chosen along each curve. It is evident that (8.3) imply

$$\frac{d\Omega}{d\chi} = 0 . \tag{8.4}$$

Instead of thinking in terms of (σ, x)-space, let us use space-time. Then (8.2) or (8.3) gives us a set of world lines with associated vector fields σ_r along them; each such world line with the associated σ_r is a <u>characteristic</u> or <u>ray</u>, a characteristic or ray being determined by an initial event and an initial σ_r, provided the function $\Omega(\sigma, x)$ has been assigned. We get the same ray if we change Ω into any function of it, the only difference being in the parameter χ.

We note that the condition for a timelike or null ray is

$$\frac{\partial\Omega}{\partial\sigma_r}\frac{\partial\Omega}{\partial\sigma_r} \leqslant 0 \tag{8.5}$$

(This is an essential relativistic condition when we come to identify the rays with the world lines of photons, since otherwise we would have a signal speed greater than c.)

We now turn to the HPDE,

$$\Omega(\sigma, x) = 0, \qquad \sigma_r = F_{,r}, \tag{8.6}$$

and seek a solution F taking assigned values f on some assigned 3-space S. (For emission, as in Fig. 5, S would be a thin tube enclosing C with $F = f = -c\tau/T$ on it.)

The plan is to choose σ_r on S to satisfy

$$\sigma_r\,\delta x_r = \delta f, \qquad \Omega(\sigma, x) = 0, \tag{8.7}$$

for every displacement in S; since there are three degrees of freedom in this displacement, we have four equations here for four quantities. But although the number of equations is right, it may happen that no such vector σ_r can be found. To investigate this, we fix our attention on some event x_r on S and think in terms of a Minkowskian 4-space in which σ_r are coordinates. The first of (8.7) determines the orthogonal projection of σ_r on the tangent 3-flat of S (say Π), and the second places the extremity of σ_r on a certain 3-space $\Omega = 0$. We can solve (8.7) if, and only if, the orthogonal projection of $\Omega = 0$ on Π contains the assigned orthogonal projection of σ_r on Π. And if a solution exists, there may be more than one.

Having thus opened up the question of the possible insolubility of (8.7) and of multiple solutions, we shall assume that a solution exists and that it is unique.

With the initial values of σ_r on S given by (8.7), we proceed to draw the rays by (8.3). In general these rays form a congruence filling a portion of space-time, but there is an exceptional case. If, on S, we have

$$\frac{\partial \Omega}{\partial \sigma_r} S_{,r} = 0 \tag{8.8}$$

(we have taken the equation of S to be S = 0), then the rays do not leave S. Let us suppose that this does not occur.

We have then a congruence of rays emanating from S. At any event x_r in the domain filled by them we define F (x) by

$$F(x) = f(a) + \int_a^x \sigma_r \, dx_r , \tag{8.9}$$

where a_r is the event where the ray through x_r meets S, and the integral is taken along the ray. Here f (a) is the assigned value on S, and we see at once that F (x) satisfies the initial condition.

We have now to show that F (x) satisfies the HPDE, and to do this we vary x_r, obtaining

$$F_{,r}\,\delta x_r = \delta f(a) + \delta \int_a^x \sigma_r\, dx_r , \qquad (8.10)$$

the variation being from the ray to a neighbouring ray. Since we made $\Omega(\sigma, x) = 0$ on S, it follows from (8.4) that $\Omega = 0$ everywhere, and therefore $\delta\Omega = 0$. Hence, on integration by parts, the last term in (8.10) is

$$\begin{aligned} \delta \int_a^x \sigma_r\, dx_r &= [\sigma_r\, \delta x_r]_a^x + \int_a^x (\delta\sigma_r\, dx_r - \delta x_r\, d\sigma_r) , \\ &= [\sigma_r\, \delta x_r]_a^x + \int_a^x \delta\Omega\, dw = [\sigma_r\, \delta x_r]_a^x , \end{aligned} \qquad (8.11)$$

and so

$$F_{,r}\,\delta x_r = \delta f(a) + \sigma_r\,\delta x_r - (\sigma_r\,\delta x_r)_S . \qquad (8.12)$$

Using (8.7) we have then

$$F_{,r}\,\delta x_r = \sigma_r\,\delta x_r , \qquad F_{,r} = \sigma_r , \qquad (8.13)$$

and therefore F satisfies the HPDE since $\Omega(\sigma, x) = 0$.

Thus, to get the solution of the HPDE with F assigned on S, we first determine the rays by solving (8.3) with initial values given by (8.7), and then write down $F(x)$ as in (8.9).

We note that the rays satisfy the variational principle

$$\delta \int \sigma_r\, dx_r = 0 , \qquad \Omega(\sigma, x) = 0 , \qquad (8.14)$$

for fixed end events and for σ_r arbitrary except for the condition shown. This is evident since the variational equation with the side condition leads at once to (8.3).

9. Ray velocity and group speed.

The world line of a ray with equations (8.3) may be regarded as the history of a moving point, and the 3-velocity of this point, or ray velocity, is

$$v_\rho \quad = \quad ic \frac{dx_\rho}{dx_4} \quad = \quad ic \frac{\partial\Omega/\partial\sigma_\rho}{\partial\Omega/\partial\sigma_4} \ . \tag{9.1}$$

Let us connect this with group velocity, or, more precisely, group speed. We can use any form of Ω which yields the correct slowness-frequency equation, and it is convenient to take it in the form (5.3):

$$\Omega\,(\sigma, x) \quad = \quad (\sigma_\rho \sigma_\rho)^{\frac{1}{2}} - f\left(\frac{\sigma_1}{\sigma_3}, \frac{\sigma_2}{\sigma_3}, \sigma_4, x_1, x_2, x_3, x_4\right) = 0 . \tag{9.2}$$

Then, since f is homogeneous of degree zero in σ_ρ, we have

$$\sigma_\rho \frac{\partial\Omega}{\partial\sigma_\rho} = (\sigma_\rho \sigma_\rho)^{\frac{1}{2}}, \qquad \frac{\partial\Omega}{\partial\sigma_4} = -\frac{\partial f}{\partial\sigma_4} . \tag{9.3}$$

Then (9.1) gives

$$v_\rho \sigma_\rho \quad = \quad -ic \frac{(\sigma_\rho \sigma_\rho)^{\frac{1}{2}}}{\partial f/\partial\sigma_4} , \tag{9.4}$$

or, with N_ρ defined by

$$N_\rho \quad = \quad \frac{\sigma_\rho}{(\sigma_\pi \sigma_\pi)^{\frac{1}{2}}} , \tag{9.5}$$

we have

$$\frac{1}{v_\rho N_\rho} \quad = \quad -\frac{1}{ic}\frac{\partial f}{\partial\sigma_4} \quad = \quad \frac{1}{c}\frac{\partial f}{\partial\nu} . \tag{9.6}$$

But, by (4.3) and (9.2), we have $N_\rho = u_\rho / u$ (the direction cosines of the normal to the 2-wave) and

$$f = (\sigma_\rho \sigma_\rho)^{\frac{1}{2}} = \nu c / u . \tag{9.7}$$

Thus (9.6) reads

$$\frac{1}{v_\rho N_\rho} = \frac{\partial}{\partial \nu}\left(\frac{\nu}{u}\right) , \tag{9.8}$$

the partial differentiation being carried out holding N_ρ fixed. (Note that for fixed N_ρ and x_r (9.2) is an equation connecting u and ν .) But a general formula for the group speed g is

$$\frac{1}{g} = \frac{\partial}{\partial \nu}\left(\frac{\nu}{u}\right) , \tag{9.9}$$

where u is the phase speed corresponding to frequency ν , and so (9.8) tells us that <u>the component of the ray velocity normal to the phase 2-wave is equal to the group speed</u>. The above proof of this fundamental result is a variant of the method given elsewhere (Synge (1954), p.33).

Although the isotropic fluid in general motion is covered by the above argument it is interesting to examine it directly. By (7.3) we have

$$\begin{aligned} \frac{\partial \Omega}{\partial \sigma_\rho} &= \sigma_\rho - (n^2 - 1)\mu_\rho \sigma_n \mu_n - n n' \mu_\rho (\sigma_n \mu_n)^2 , \\ \frac{\partial \Omega}{\partial \sigma_4} &= \sigma_4 - (n^2 - 1)\mu_4 \sigma_n \mu_n - n n' \mu_4 (\sigma_n \mu_n)^2 , \end{aligned} \tag{9.10}$$

where n' is the derivative with respect to $\sigma_r \mu_r$. Substitution in (9.1) gives the ray velocity. In the local rest frame we have $\mu_\rho = 0$, $\mu_4 = i$, and so by (4.3)

$$v_\rho^{(o)} = i\,c\,\frac{\sigma_\rho^{(o)}}{\sigma_4^{(o)}\,(n^2 + i\,n\,n'\,\sigma_4^{(o)})}$$

$$= c^2\,\frac{u_\rho^{(o)}}{u^{(o)2}\,(n^2 - n\,n'\,\nu^{(o)})} \qquad (9.11)$$

$$= u_\rho^{(o)}\,\frac{n^2}{n^2 - n\,n'\,\nu^{(o)}}$$

We observe that the ray velocity and the phase velocity have the same direction (or possibly opposite directions) in the local rest frame, and we have

$$\frac{c}{v^{(o)}} = \frac{n^2 - n\,n'\,\nu^{(o)}}{n} = n + \nu^{(o)}\,\frac{\partial n}{\partial \nu^{(o)}} = \frac{\partial}{\partial \nu^{(o)}}\,(n\nu^{(o)}) \qquad (9.12)$$

The right hand side is a well known form for c/g , equivalent to (9.9), and thus we verify in this particular case the more general result established above.

This brief discussion of group velocity has been included here in order to link the ideas of the present paper to those current in physics. But, having seen this connection, one may very well forget about group velocity and think instead of ray velocity, to which it appears to be equivalent. In the theory of media at rest group velocity seems an artificial addendum; variation of the frequency plays no role in ordinary geometrical optics. But in the present theory variation of frequency is inherent, since in the stationary principle (8.14) for rays, frequency is one of the quantities to be varied. Thus the essential idea of group velocity is built into the present theory.

The inequality (8.5) is the condition that the group velocity should not exceed c .

10. Photons.

We may introduce photons into geometrical optics by the following statement: Each ray is a possible world line of a photon, and the momentum-energy 4-vector of the photon is

$$p_r = h\,\sigma_r \,, \qquad (10.1)$$

where h is Planck's constant and σ_r the slowness-frequency 4-vector associated with the ray.

This statement gives to the photon a role in geometrical optics analogous to the role of the particle in geometrical mechanics. It is a moving point endowed with momentum and energy. It is a scalar photon which does not exhibit polarization much as the particle of geometrical mechanics does not exhibit spin.

The components p_r in (10.1) have the dimensions of energy. We may write

$$p_\rho = c\,M_\rho \,, \qquad p_4 = i\,E \,, \qquad (10.2)$$

where M_ρ is the 3-momentum of the photon and E its energy. Then by (4.3) and (10.1) we have

$$M_\rho = \frac{p_\rho}{c} = \frac{h\sigma_\rho}{c} = h\nu\frac{u_\rho}{u^2}\,, \qquad E = \frac{p_4}{i} = \frac{h\sigma_4}{i} = h\nu\,, \qquad (10.3)$$

where u_ρ is the phase velocity.

The stationary principle (8.14), satisfied by a ray, may be written

$$\delta\int c^{-1}\,p_r\,dx_r = 0\,, \qquad \Omega\,(p/h\,,\,x) = 0\,. \qquad (10.4)$$

The integral occurring here may be written (it is convenient to change the sign)

$$A = -\int c^{-1}\,p_r\,dx_r = -\int (M_\rho\,dx_\rho - E\,dt)\,, \qquad (10.5)$$

so that we recognise it as an action integral. For integration along any world line (not necessarily a ray) we have

$$A = -c^{-1} h \int \sigma_r \, dx_r = -c^{-1} h \, [F] , \qquad (10.6)$$

where $F(x)$ is the phase function. If the world line leads from an event on a certain phase 3-wave to an event on the next phase 3-wave with the same phase, then by (3.2) we have $[F] = -c$ and therefore

$$A = h ; \qquad (10.7)$$

in words, the action between adjacent phase 3-waves of the same phase is equal to Planck's constant. (Cf. the process of primitive quantisation, Synge (1954), p.113.)

One may raise the question: Has a photon a proper mass? Such a question is of course meaningless without a definition of proper mass. It is natural, perhaps, to take the definition appropriate to a material particle, for which the proper mass m is given in terms of energy E and 3-momentum M_ρ by

$$m^2 c^4 = E - c^2 M_\rho M_\rho . \qquad (10.8)$$

When we substitute on the right from (10.3) we get

$$m^2 c^4 = h^2 \nu^2 (1 - c^2 / u^2) = -h^2 \sigma_r \sigma_r . \qquad (10.9)$$

For a vacuum this gives $m = 0$ [cf. (6.1)]. But in the ordinary case of a transparent medium, for which $u < c$, or equivalently σ_r is spacelike, it makes m^2 negative and m imaginary.

It may not be useful to speak of the proper mass of a photon at all. But should we wish for a definition which makes $m = 0$ for a vacuum and m real for a medium in which $u < c$, we have merely to change the sign in our definition, so that

$$m = \frac{h\nu}{c^2}\left(\frac{c^2}{u^2} - 1\right)^{\frac{1}{2}} = \frac{h}{c^2}\left(\sigma_r \sigma_r\right)^{\frac{1}{2}}. \qquad (10.10)$$

11. Emission of a photon by an atom. Source-event.

Let us try to form a picture of the emission of a photon by an atom in terms of geometrical optics.

First we recall the argument of Section 8. Given a 3-space S with F assigned on it, we sought to draw rays, and for that we had to solve (8.7), viz.

$$\sigma_r \delta x_r = \delta f, \qquad \Omega(\sigma, x) = 0, \qquad (11.1)$$

where $F = f$ on S. If these equations can be solved and the exceptional case (8.8) is avoided, we get rays emanating from S and filling a 4-dimensional region of space-time.

Suppose now that S is a 2-space instead of a 3-space. Then the equations (11.1) are effectively three in number (instead of four), and so (if we can solve these equations) we get ∞^1 values of σ_r at each event on S. Thus we have ∞^3 rays altogether, enough to fill a 4-dimensional region of space-time as before.

If S is a world line (1-space), then we have only two equations in (11.1) and hence ∞^2 values of σ_r at each event on S. There are ∞^1 events on S , and so again we get rays adequate to fill a 4-dimensional region of space-time. If this world line S is regarded as the world line of an emitting atom (the frequency of emission defining F on it), we see space-time filled with rays. We do not think of a photon on each of these rays, but only one photon for the whole process. The rays constitute the possible histories of this one photon, and the associated values of $h\sigma_r$ its possible momentum-energy 4-vectors. One can see in this picture a semblance of Heisenberg's uncertainty principle. The initial frequency is assigned and the initial time completely undetermined, and the later position, momentum and energy of the photon have that degree of indeterminacy exhibited by the fact that, although we have a definite pattern of rays in space-time, we do not know which one the photon has chosen to traverse until it shows its presence by colliding with matter.

If λ_r is the 4-velocity of the source or atom and ν_0 the assigned frequency of emission, then we have as in (11.1) the two equations

$$\sigma_r \lambda_r = -\nu_0, \qquad \Omega(\sigma, x) = 0, \qquad (11.2)$$

to be satisfied by σ_r on the world line of the source, and the totality of rays is to be found by solving (8.3) subject to these initial conditions. The phase function F is then given by (8.9) with $f(a) = -c\tau/T = s\nu_0$.

Interference phenomena can be treated only crudely in geometrical optics. If, by means of two pin-holes or otherwise, we offer two paths to the rays, we get brightness if the actions for the two rays differ by N h and darkness if they differ by $(N + \frac{1}{2})h$, N being an integer, these two cases corresponding to difference of phase angles of amounts $2\pi N$ and $2\pi N + \pi$, respectively.

In terms of probability, we may say that, in the first case, it is very probable that the photon will choose one or other of the two rays and in the second case that it is very improbable that it will choose either.

Suppose we know of a photon only one event in its history. This we may call a source event. The possible values of σ_r are subject only to $\Omega(\sigma, x) = 0$. (In fact, we have (11.1) with the first equation deleted.) This means that there are ∞^3 initial values of σ_r and we would expect, as in the cases considered above, a set of rays filling a 4-dimensional region of space-time. Let us examine the question, taking an isotropic medium with $\Omega(\sigma, x)$ as in (7.3).

Writing $\xi_r = dx_r / d\chi$, $q = n^2 - 1$, $q' = \partial q / \partial(\sigma_r \mu_r)$, we have

$$\xi_r = \frac{\partial \Omega}{\partial \sigma_r} = \sigma_r - q\mu_r \sigma_n \mu_n - \tfrac{1}{2} q' \mu_r (\sigma_n \mu_n)^2 . \tag{11.3}$$

Let us use the local rest frame, so that (7.3) becomes

$$2\Omega(\sigma, x) = \sigma_\rho \sigma_\rho + (q+1)\sigma_4^2 = 0 , \tag{11.4}$$

and (11.3) becomes

$$\xi_\rho = \sigma_\rho , \qquad \xi_4 = (1+q)\sigma_4 + \tfrac{1}{2} i q' \sigma_4^2 . \tag{11.5}$$

Then (11.4) gives

$$\xi_\rho \xi_\rho + (q+1)\sigma_4^2 = 0 , \tag{11.6}$$

wherein σ_4 is to be regarded as a function of ξ_4 determined by the second of (11.5), in which σ_4 occurs explicitly and also implicitly in q and q'.

In general the rays fill a 4-dimensional region of space-time, obtained by joining the origin of ξ_r-space to the points on the 3-space (11.6). But if the

the medium is <u>non-dispersive</u>, then $q' = 0$, q being a function of ρ only and hence a function of position only; the second of (11.5) reduces to

$$\xi_4 = (1 + q)\,\sigma_4 , \qquad (11.7)$$

and (11.6) reads

$$\xi_\rho \xi_\rho + n^{-2} \xi_4^2 = 0 . \qquad (11.8)$$

This is a cone, confined within the null cone (as it should be to make the speed of the photon less than c) if $n > 1$. The fact that it is a cone means that <u>the rays from a source-event in a non-dispersive isotropic medium fill only a 3-dimensional region in space-time.</u> This is of course true in particular for a vacuum, the cone being then the null cone ($\xi_r \xi_r = 0$).

12. <u>The laws of reflection and refraction.</u>

Let S be the history of a moving surface separating media M' , M . When light passes from M' into M (refraction) or is reflected back into M' , a discontinuity occurs, and the variational equation (8.14) leads at once to the equation

$$(\sigma_r - \sigma'_r)\,\delta x_r = 0 \qquad (12.1)$$

for all displacements δx_r in S , σ'_r and σ_r being the slowness-frequency 4-vectors in M' and M respectively at the event of reflection or refraction. Equivalently,

$$\sigma_r - \sigma'_r = k N_r , \qquad (12.2)$$

where N_r is the unit normal to S , pointing from M' into M , and k is an undetermined factor. This is the law of reflection and refraction; it tells us that the increment in the slowness-frequency 4-vector is normal to the history of the surface of separation.

Regarding σ'_r as given, and also the event at which the reflection or refraction occurs, we have in (12.2) four equations for the five quantities σ_r , k ; for refraction a fifth equation is supplied by the slowness-frequency equation of M ,

$$\Omega(\sigma, x) = 0 , \qquad (12.3)$$

while for reflection we are to use the slowness-frequency equation for M' , say

$$\Omega'(\sigma, x) = 0 , \qquad (12.4)$$

which must be satisfied not only by σ'_r but also by σ_r .

To investigate refraction, we must see whether (12.2) and (12.3) possess solutions; if they do, we have still to investigate whether the refracted ray actually passes into M . To investigate reflection, we must do the same, using (12.2) and (12.4).

In work of this sort we try to simplify the algebra as far as possible by choice of the frame of reference. Supposing, as is natural, that the speed of the surface of separation is less than c , the 4-vector N_r is spacelike. Let us take the x_1-axis in its direction, so that $N_1 = 1$, $N_2 = N_3 = N_4 = 0$. Then (12.2) gives

$$\sigma_1 = \sigma'_1 + k , \quad \sigma_2 = \sigma'_2 , \quad \sigma_3 = \sigma'_3 , \quad \sigma_4 = \sigma'_4 . \qquad (12.5)$$

Note that now we are dealing with a fixed surface; frequency is not changed. Suppressing the dependence on x_r (we are at a fixed event on S), (12.3) and (12.4) may be written

$$\begin{aligned} \Omega(\sigma'_1 + k, \sigma'_2, \sigma'_3, \sigma'_4) &= 0, \\ \Omega'(\sigma'_1 + k, \sigma'_2, \sigma'_3, \sigma'_4) &= 0; \end{aligned} \qquad (12.6)$$

these equations are to be solved for k, the first in the case of refraction and the second in the case of reflection.

Let us examine the case where the media are isotropic, so that by (7.3)

$$\begin{aligned} 2\Omega &= \sigma_r\sigma_r - (n^2 - 1)(\sigma_r\mu_r)^2 = 0, \\ 2\Omega' &= \sigma'_r\sigma'_r - (n'^2 - 1)(\sigma'_r\mu'_r)^2 = 0, \end{aligned} \qquad (12.7)$$

where n', n are the refractive indices of the media and μ'_r, μ_r their 4-velocities. But these 4-velocities lie in S, and therefore

$$\mu'_1 = \mu_1 = 0. \qquad (12.8)$$

The surviving components of μ'_r, μ_r form two timelike vectors in a 3-dimensional Minkowskian space-time, and we may complete the specification of frame (of which so far only the x_1-axis is fixed) by demanding that

$$\mu'_2 = \mu_2 = 0, \quad \mu'_3 = -\mu_3, \quad \mu'_4 = \mu_4. \qquad (12.9)$$

This may be called the <u>standard frame of reference for a pair of media</u> (Fig. 6); in it the two media slide past one another on the plane x_1 = const. with equal and opposite velocities. Denoting the common speed by w, we have

$$\begin{aligned} \mu'_1 = \mu_1 = \mu'_2 = \mu_2 = 0, \quad \mu'_3 = -\mu_3 = -\gamma w/c, \\ \mu'_4 = \mu_4 = i\gamma, \quad \gamma = (1 - w^2/c^2)^{\frac{1}{2}}, \end{aligned} \qquad (12.10)$$

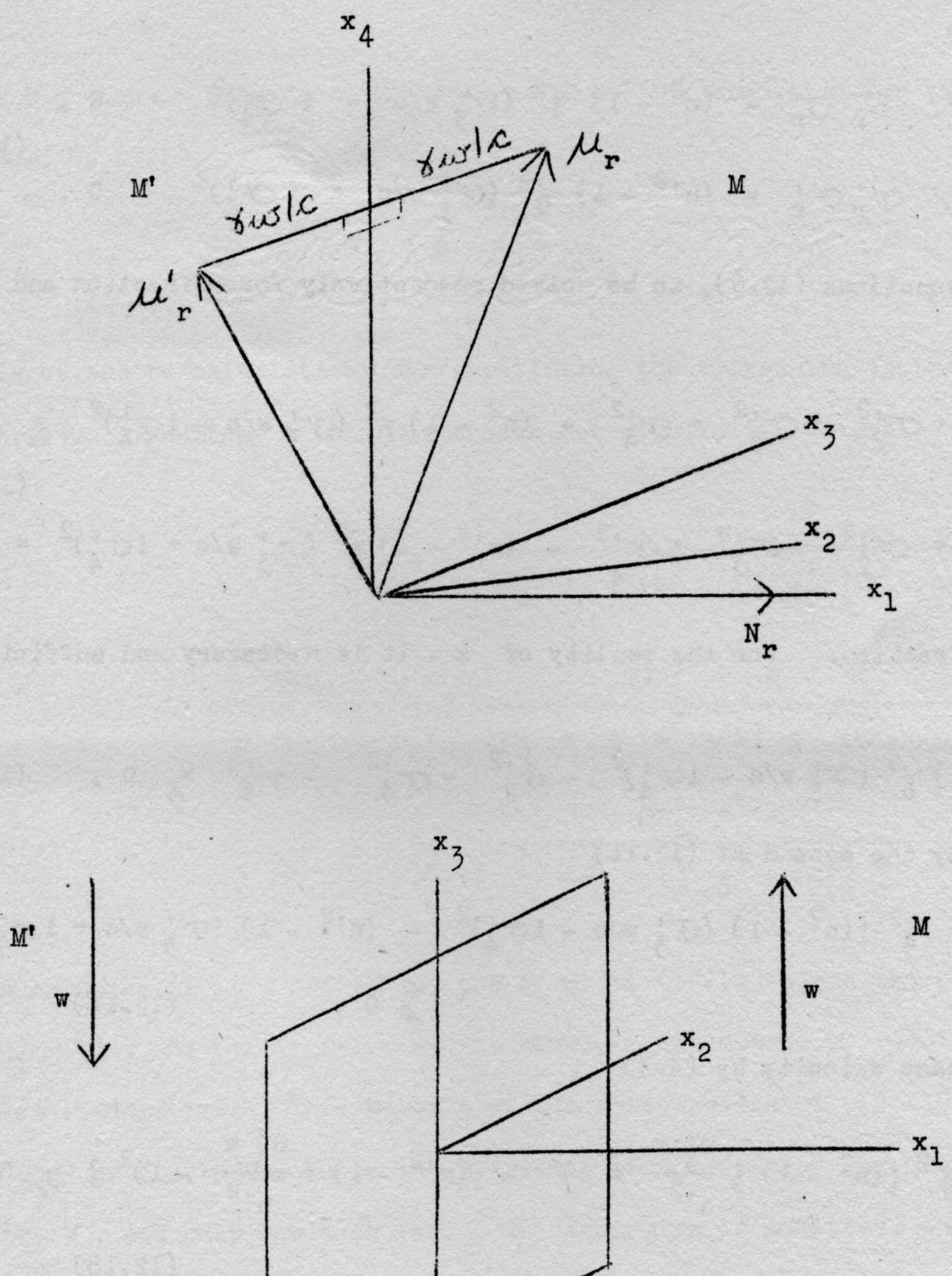

Fig. 6

Standard frame of reference for a pair of media.

Upper diagram: space-time.

Lower diagram: space.

and (12.7) become

$$2\Omega = \sigma_r \sigma_r - (n^2 - 1)\gamma^2 (\sigma_3 w/c - i\sigma_4)^2 = 0 ,$$
$$2\Omega' = \sigma'_r \sigma'_r - (n'^2 - 1)\gamma^2 (\sigma'_3 w/c + i\sigma'_4)^2 = 0 . \tag{12.11}$$

Accordingly the equations (12.6), to be solved respectively for refraction and reflection, read

$$(\sigma'_1 + k)^2 + \sigma'^2_2 + \sigma'^2_3 + \sigma'^2_4 - (n^2 - 1)\gamma^2 (\sigma'_3 w/c - i\sigma'_4)^2 = 0 ,$$
$$(\sigma'_1 + k)^2 + \sigma'^2_2 + \sigma'^2_3 + \sigma'^2_4 - (n'^2 - 1)\gamma^2 (\sigma'_3 w/c + i\sigma'_4)^2 = 0 . \tag{12.12}$$

Consider refraction. For the reality of k, it is necessary and sufficient that

$$(n^2 - 1)\gamma^2 (\sigma'_3 w/c - i\sigma'_4)^2 - \sigma'^2_2 - \sigma'^2_3 - \sigma'^2_4 \geqslant 0 , \tag{12.13}$$

or equivalently by the second of (12.11)

$$\sigma'^2_1 + \gamma^2 [(n^2 - 1)(\sigma'_3 w/c - i\sigma'_4)^2 - (n'^2 - 1)(\sigma'_3 w/c + i\sigma'_4)^2] \geqslant 0 , \tag{12.14}$$

or in terms of phase velocity by (4.3)

$$\frac{c^2 u'^2_1}{u'^4} + \gamma^2 \left[(n^2 - 1)\left(\frac{u'_3 w}{u'^2} + 1\right)^2 - (n'^2 - 1)\left(\frac{u'_3 w}{u'^2} - 1\right)^2\right] \geqslant 0 \tag{12.15}$$

the frequency ν' disappearing as a common factor (ν'^2), although still present implicitly in n' (dispersion).

Unless the above condition is satisfied, refraction from M' into M cannot take place. But there is a further condition for refraction. It must be possible for a photon to pass from M' into M, and the condition for this is $v_1 > 0$ where, by (9.1),

$$v_1 = ic \frac{\partial\Omega / \partial\sigma_1}{\partial\Omega / \partial\sigma_4} . \tag{12.16}$$

This we are to calculate by differentiating the expression in the first line of (12.11). Assuming M non-dispersive for simplicity, we get

$$\begin{aligned} v_1 &= ic \frac{\sigma_1}{\sigma_4 + i(n^2-1)\gamma^2(\sigma_3 w/c - i\sigma_4)} \\ &= ic \frac{\sigma_1' + k}{\sigma_4' + i(n^2-1)\gamma^2(\sigma_3' w/c - i\sigma_4')} \\ &= \frac{c}{\nu'} \frac{\sigma_1' + k}{1 + (n^2-1)\gamma^2\left(\frac{c\,u_3'}{u'^2}\frac{w}{c} + 1\right)} . \end{aligned} \tag{12.17}$$

Now assuming (12.13) satisfied, the first of (12.12) gives two real values of $(\sigma_1' + k)$, one the negative of the other. One makes v_1 positive and the other makes it negative. This means that the satisfaction of (12.13) (or equivalently (12.14) or (12.15)) ensures the existence of a refracted ray proceeding from M' into M, and only one such ray. If dispersion is admitted, there is no such simple result in general.

Actually, the case of equality in (12.13) should be ruled out, as then we have $v_1 = 0$. We may say that the condition for total reflection is

$$\frac{c^2 u_1'^2}{u'^4} + \gamma^2 \left[(n^2-1)\left(\frac{u_3' w}{u'^2} + 1\right)^2 - (n'^2-1)\left(\frac{u_3' w}{u'^2} - 1\right)^2 \right] < 0 \tag{12.18}$$

We recall that u_1' is the component of incident phase velocity normal to the surface of separation (fixed in our standard frame) and u_3' is the component in the direction of motion of the final medium M ; w is the speed of either medium.

We can check (12.18) by reducing both media to rest, so that $w = 0$, $\gamma = 1$, $n' = c/u'$, $\cos i = u_1' / u'$, where i is the angle of incidence. Then (12.18) becomes

$$n'^2 \cos^2 i + n^2 - n'^2 < 0 , \qquad (12.19)$$

which is immediately recognised as the elementary formula for total reflection.

To discuss reflection we use the second of (12.6) instead of the first, or, for isotropic media with the standard frame of reference, the second of (12.12) instead of the first. But σ_r' satisfy the second of (12.11), and so the equation simplifies to

$$(\sigma_1' + k)^2 - \sigma_1'^2 = 0 . \qquad (12.20)$$

This gives the two solutions

$$k = 0 , \quad k = -2\sigma_1' . \qquad (12.21)$$

The first is ruled out (it merely gives the incident ray); from the second we get for the reflected slowness-frequency 4-vector σ_r

$$\sigma_1 = -\sigma_1' , \quad \sigma_2 = \sigma_2' , \quad \sigma_3 = \sigma_3' , \quad \sigma_4 = \sigma_4' . \qquad (12.22)$$

This is the very simple law of reflection when the standard frame of reference is used.

Since the momentum-energy 4-vector of a photon is $p_r = h\,\sigma_r$, and this is changed abruptly by reflection or refraction, we conclude that a 4-impulse acts on the photon and (in view of the conservation of momentum and energy) that an equal and opposite 4-impulse acts on the mirror or medium. By (12.2) the 4-impulse on the photon is

$$I_r = h k N_r , \tag{12.23}$$

k being determined by (12.2), (12.3) for refraction and by (12.2), (12.4) for reflection. For a fixed mirror this gives an impulse of magnitude $2h\nu/c$ with no fourth component. For a pair of isotropic media referred to their standard frame the energy of a photon is unchanged by reflection or refraction, and the only component of momentum to be changed is that along the normal to the (fixed) surface of separation.

13. Connection with standard Hamiltonian dynamics.

The theory of this paper has been developed by methods very close to those used by Hamilton in his geometrical optics, and if I did not translate the results into the much more familiar language of Hamiltonian dynamics the reader might fail to see that what is here developed is in fact the Hamiltonian dynamics of a photon. This is not a modification of Hamiltonian dynamics (e.g. with replacement of ordinary time by proper time); it is precisely Hamiltonian dynamics of the form familiar in Newtonian theory.

The best way to make the translation is by means of a dictionary of notation, one column containing symbols used in the present paper and another those universally recognised in dynamics, with the linkages indicated. For simplicity of translation we use those units for which c = h = 1.

	Present paper.		Standard dynamics.
Coordinates	x_ρ	$(x_\rho = q_\rho)$	q_ρ
Time	x_4	$(x_4 = i\,t)$	t
Momentum	σ_ρ	$(\sigma_\rho = p_\rho)$	p_ρ
Energy	σ_4	$(\sigma_4 = i\,H)$	H
Hamiltonian*	$\Omega(\sigma, x) = \sigma_4 - iH(\sigma_\rho, x_r) = 0$		$H = H(p, q, t)$

Equations of motion (8.3)

$$i\,\frac{dx_\rho}{dx_4} = i\,\frac{\partial\Omega/\partial\sigma_\rho}{\partial\Omega/\partial\sigma_4} = \frac{\partial H}{\partial\sigma_\rho}, \qquad \frac{dq_\rho}{dt} = \frac{\partial H}{\partial p_\rho},$$

$$i\,\frac{d\sigma_\rho}{dx_4} = -i\,\frac{\partial\Omega/\partial x_\rho}{\partial\Omega/\partial\sigma_4} = -\frac{\partial H}{\partial x_\rho}, \qquad \frac{dp_\rho}{dt} = -\frac{\partial H}{\partial q_\rho}. \tag{13.1}$$

Variational principle

$$\delta\int \sigma_r\,dx_r = 0, \qquad \delta\int (p_\rho\,dq_\rho - H\,dt) = 0. \tag{13.2}$$

Note that in (13.1) and (13.2) the familiar Hamiltonian equations of motion and action principle are the exact transcriptions of the corresponding formulae in the present theory.

In standard dynamical theory (Newtonian) we are used to starting with a Lagrangian $L = L(q, q', t)$ where $q' = dq/dt$ and Hamilton's principle

---

* We solve for σ_4 as in (5.2)

$$\delta \int L \, dt = 0 . \tag{13.3}$$

We define the generalised momenta by

$$p_\rho = \partial L / \partial q'_\rho , \tag{13.4}$$

and solve these equations for q'_ρ . We define the Hamiltonian by

$$H (p , q , t) = p_\rho q'_\rho - L (q , q' , t) . \tag{13.5}$$

Then

$$L \, dt = p_\rho \, dq_\rho - H \, dt , \tag{13.6}$$

and we recognise the equivalence of Hamilton's principle in the form (13.3) and the action principle written second in (13.2).

On the other hand we might start with $H (p , q , t)$, define q'_ρ by

$$q'_\rho = \partial H / \partial p_\rho , \tag{13.7}$$

solve for p_ρ , and define L by

$$L (q , q' , t) = p_\rho q'_\rho - H (p , q , t) , \tag{13.8}$$

which is simply (13.6) written a little differently.

This second plan is not the usual one in classical dynamics, but it is essentially the plan we must follow here if we are to obtain a Lagrangian for our photon dynamics. In view of the dictionary, we can do this by the method of (13.7), (13.8), but it is better (having seen the connection with standard dynamics) to go back to our general method, which takes $\Omega (\sigma , x)$ in a general form and not in the special form $\sigma_4 - i H$.

The technique is contained in the equations (cf. Synge (1954), p. 14; I here write θ instead of $1/\theta$)

$$\theta \alpha_r = \frac{\partial \Omega}{\partial \sigma_r}, \qquad \Omega(\sigma, x) = 0, \qquad \theta f = -\sigma_r \frac{\partial \Omega}{\partial \sigma_r}, \tag{13.9}$$

with

$$\alpha_r \alpha_r = -1. \tag{13.10}$$

The plan is to solve the first of (13.9) (four equations) for σ_r in terms of $(\theta \alpha_r, x_r)$, substitute these values in the second equation and solve for θ as a function of (α_r, x_r); hence we obtain σ_r in terms of (α_r, x_r). Substitution in the last of (13.9) gives

$$f = f(x, \alpha). \tag{13.11}$$

Taking any timelike world line with unit tangent $\alpha_r = dx_r / ds$, we have

$$f(x, \alpha)\, ds = -\theta^{-1} \sigma_r \frac{\partial \Omega}{\partial \sigma_r} ds = -\sigma_r \alpha_r\, ds = -\sigma_r\, dx_r, \tag{13.12}$$

and so we reconcile the action principle $\delta \int \sigma_r\, dx_r = 0$ with the Lagrangian principle

$$\delta \int f(x, \alpha)\, ds = 0. \tag{13.13}$$

In fact, $f(x, \alpha)$ is a <u>Lagrangian</u> or, if we prefer Hamilton's term, the <u>medium function</u> or its reciprocal.

The variational principle (13.13) gives the Euler-Lagrange equations for the rays,

$$\frac{d}{ds} \frac{\partial f}{\partial \alpha_r} - \frac{\partial f}{\partial x_r} = 0, \tag{13.14}$$

provided f is homogeneous of degree unity in the α's. As a matter of fact, if we find f as described above without using (13.10), it will come out automatically with the requisite homogeneity. But if in the process we use (13.10) in order to simplify the algebra, we can always restore homogeneity by means of (13.10).

14. Determination of the medium function (Lagrangian) for an isotropic medium in general motion.

For an isotropic medium in general motion we have the slowness-frequency equation (7.3)

$$2\Omega(\sigma, x) = (\sigma\sigma) - q(\sigma\mu)^2 = 0, \qquad (14.1)$$

where for the scalar product of any two 4-vectors we write

$$A_r B_r = (AB), \qquad (14.2)$$

and where

$$n^2 - 1 = q = q[(\sigma\mu), \rho]. \qquad (14.3)$$

We shall denote by q' the partial derivative of q with respect to $(\sigma\mu)$. We recall that n is the refractive index, computed in the ordinary way for the local rest frame, and that in that frame the frequency is $\nu^{(o)} = -(\sigma\mu)$.

If we are not given the explicit form of the dependence of q on $(\sigma\mu)$, we cannot hope to carry out explicitly the determination of the medium function $f(x, \alpha)$ as described in the preceding section. Even if we are given that form, the algebra may be very complicated. But we can at least analyze the problem with a view to explicit calculations in the next section for a medium which is only slightly dispersive. Here we proceed without approximation.

The basic equations are (13.9) with (13.10). We have then

$$\theta\alpha_r = \frac{\partial\Omega}{\partial\sigma_r} = \sigma_r - q\mu_r(\sigma\mu) - \tfrac{1}{2}q'\mu_r(\sigma\mu)^2, \qquad (14.4)$$

to be solved for σ_r. Multiplication by μ_r gives, since $\mu_r\mu_r = -1$,

$$\theta(\alpha\mu) = (\sigma\mu)(1 + q) + \tfrac{1}{2}q'(\sigma\mu)^2. \qquad (14.5)$$

This is to be regarded as an equation for $(\sigma\mu)$, which appears not only explicitly, but also in q and q' . Let the solution be

$$(\sigma\mu) = G[\theta(\alpha\mu), \rho], \tag{14.6}$$

a function of two variables, now supposed known. Then by (14.4)

$$\sigma_r = \theta\alpha_r + \mu_r(qG + \tfrac{1}{2}q'G^2); \tag{14.7}$$

but by (14.5)

$$qG + \tfrac{1}{2}q'G^2 = \theta(\alpha\mu) - G, \tag{14.8}$$

and so

$$\sigma_r = \theta\alpha_r + \mu_r[\theta(\alpha\mu) - G]. \tag{14.9}$$

Hence

$$(\sigma\sigma) = -\theta^2 + 2\theta(\alpha\mu)[\theta(\alpha\mu) - G] - [\theta(\alpha\mu) - G]^2$$

$$= \theta^2[(\alpha\mu)^2 - 1] - G^2. \tag{14.10}$$

Our instructions are to substitute for σ_r in (14.1), which is the second of (13.9), and we do this by substituting (14.6) and (14.10); we get

$$\theta^2[(\alpha\mu)^2 - 1] - (1 + q)G^2 = 0. \tag{14.11}$$

Now q is a function of G and ρ, and G is a function of θ, $(\alpha\mu)$ and ρ ; thus (14.11) is an equation to determine θ as a function of $(\alpha\mu)$ and ρ ; let the solution be

$$\theta = K[(\alpha\mu), \rho]. \tag{14.12}$$

By the last of (13.9) combined with (14.4) we have

$$\theta f = -\sigma_r \frac{\partial\Omega}{\partial\sigma_r} = -[(\sigma\sigma) - q(\sigma\mu)^2 - \tfrac{1}{2}q'(\sigma\mu)^3], \tag{14.13}$$

or, by (14.1),

$$f = f(x, \alpha) = \tfrac{1}{2}\theta^{-1} q' (\sigma\mu)^3$$
$$= \tfrac{1}{2}\theta^{-1} q' G^3 . \qquad (14.14)$$

It is easy to see that this is in fact a function of x_r and α_r, known when the functions G and K are known.

The two key equations in the above work are (14.5), to be solved for $(\sigma\mu)$, and (14.11), to be solved for θ; it is here that we are likely to meet practical difficulties in any explicit calculation.

We can however concentrate the algebraic difficulties in a single equation by eliminating θ between (14.5) and (14.11). By (14.5) we have

$$\theta(\alpha\mu) = \tfrac{1}{2} q' G^2 + (1 + q) G . \qquad (14.15)$$

Substituting in (14.11) and dividing by G^2, we get

$$\frac{(\alpha\mu)^2 - 1}{(\alpha\mu)^2} (\tfrac{1}{2} q' G + 1 + q)^2 - (1 + q) = 0 . \qquad (14.16)$$

This equation has to be solved for G (we remember that q and q' are known functions of G and ρ). When G has been found, θ is given by (14.15). By (14.14) the Lagrangian or medium function is

$$f(x, \alpha) = (\alpha\mu) \frac{\tfrac{1}{2} q' G^2}{\tfrac{1}{2} q' G + 1 + q} . \qquad (14.17)$$

The case of a non-dispersive medium is singular. For such a medium $q' = 0$ and G disappears from (14.16), which reduces to

$$q(\alpha\mu)^2 = 1 + q , \qquad (14.18)$$

in which q is a function of ρ only. In fact, the method breaks down for a non-dispersive medium, and the reason is not hard to see. A Lagrangian principle of the form $\delta \int f\, ds = 0$ can be expected to hold only if two arbitrary events (or at least events arbitrary between limits) can be joined by an extremal, or equivalently if the rays from an event fill a 4-dimensional region in space-time. That is not the case for a non-dispersive medium. Only a cone is filled, as indicated in (11.8). It is easy to reconcile (11.8), which uses the local rest frame, with the more general formula (14.18); in the local rest frame we have $\mu_\rho = 0$, $\mu_4 = i$, and (14.18) may be written in the following equivalent forms:

$$
\begin{aligned}
-q\alpha_4^2 &= 1 + q, \\
q\alpha_4^2 &= (1+q)\,\alpha_r\alpha_r, \qquad (14.19)\\
(1+q)\,\alpha_\rho\alpha_\rho + \alpha_4^2 &= 0, \\
n^2\alpha_\rho\alpha_\rho + \alpha_4^2 &= 0,
\end{aligned}
$$

the last of which is the same as (11.8) in a different notation.

We see that the Lagrangian or medium function does not exist for a non-dispersive medium, and even in the case of a dispersive medium its determination in explicit form is in general not possible. It is in fact better to discuss rays by means of the Hamiltonian equations (8.3) instead of trying to make use of the Euler-Lagrange equations (13.14); the equations (8.3) can be used for non-dispersive media as well as for dispersive media.

15. Medium function (Lagrangian) for a special class of media.

In the Sellmeier formula (7.5), let ν_M be large; expanding and retaining only the leading terms, we get [cf. (7.7)]

$$n^2 - 1 = A(\rho) + B(\rho)\,\nu^{(0)2}, \qquad (15.1)$$

where

$$A(\rho) = \sum_M D_M, \qquad B(\rho) = \sum_M D_M/\nu_M^2. \qquad (15.2)$$

Here $B(\rho)$ is small. $B = 0$ gives a non-dispersive medium, and we may think of the formula (15.1) as referring to a slightly dispersive medium.

Let us apply the method of Section 14 to find the medium function (Lagrangian) of a medium for which (15.1) holds. Our work applies in particular to a slightly dispersive medium (B small), but the approximations are tricky, and it is clearer to accept (15.1) as an exact formula and make no approximations based on the smallness of B. Accordingly in the notation of Section 14 we have

$$q = A + B\,\nu^{(0)2} = A + B(\sigma\mu)^2, \qquad q' = 2B(\sigma\mu). \qquad (15.3)$$

The key equation is (14.16), and for the values (15.3) it is quadratic in G. Accordingly the calculations can be carried out explicitly.

To simplify the writing, we note that $(\alpha\mu)$ is an invariant, and its value may be obtained by using the local rest frame; we have

$$(\alpha\mu) = i\,\alpha_4 = -(1-\beta^2)^{-\frac{1}{2}}, \qquad (15.4)$$

where

$$\beta = v^{(0)}/c, \qquad (15.5)$$

$v^{(o)}$ being the ray speed in the local rest frame (or the group speed). Then

$$\frac{(\alpha\mu)^2 - 1}{(\alpha\mu)^2} = \beta^2 . \qquad (15.6)$$

We have

$$\begin{aligned} \tfrac{1}{2} q' G + 1 + q &= 1 + A + 2 B G^2 , \\ 1 + q &= 1 + A + B G^2 , \end{aligned} \qquad (15.7)$$

and if we define X by

$$X = 1 + A + 2 B G^2 , \qquad (15.8)$$

(14.16) gives the following quadratic equation for X :

$$2 \beta^2 X^2 - X - (1 + A) = 0 . \qquad (15.9)$$

Hence, selecting the positive root,

$$X = \frac{1}{4 \beta^2} [1 + \sqrt{1 + 8 \beta^2 (1 + A)}] . \qquad (15.10)$$

Then, taking B to be positive, we have

$$\begin{aligned} \tfrac{1}{2} q' G + 1 + q &= X , \\ \tfrac{1}{2} q' G^2 &= B G^3 = B^{-\frac{1}{2}} (B G^2)^{3/2} \\ &= (8 B)^{-\frac{1}{2}} (X - 1 - A)^{3/2} , \end{aligned} \qquad (15.11)$$

and (14.17) gives

$$f (x , \alpha) = \frac{(\alpha\mu)}{(8 B)^{\frac{1}{2}}} \frac{(X - 1 - A)^{3/2}}{X} . \qquad (15.12)$$

Remembering that X is given in terms of $(\alpha\mu)$ by (15.10), we have here the required medium function (Lagrangian). To sum up, for a moving isotropic dispersive medium, with refractive index as in (15.1), the rays satisfy the stationary principle $\delta \int f(x, \alpha)\, ds = 0$ with f as in (15.12), X being given by (15.10) with β as in (15.6).

The stationary principle may also be written

$$\delta \int \frac{(X - 1 - A)^{3/2}}{X B^{\frac{1}{2}}} \mu_r \, dx_r = 0 . \tag{15.13}$$

If the medium is in uniform motion, we may reduce it to rest, making $\mu_\rho = 0$, $\mu_4 = i$ everywhere; then the stationary principle becomes

$$\delta \int \frac{(X - 1 - A)^{3/2}}{X B^{\frac{1}{2}}} \, dt = 0 . \tag{15.14}$$

If, further, the medium is homogeneous, so that A and B are constants, we have

$$\delta \int \frac{(X - 1 - A)^{3/2}}{X} \, dt = 0 , \tag{15.15}$$

no matter how small B may be, provided it does not vanish. This stationary principle is not to be confused with Fermat's principle for which the frequency is held fixed and also the end points (but not the end events). In fact, (15.15) is a Hamiltonian principle ($\delta \int L\, dt = 0$) , whereas Fermat's principle is the analogue of Jacobi's stationary principle, usually written in dynamical theory as $\delta \int (E - V)^{\frac{1}{2}} ds = 0$.

If we wish to use the Euler-Lagrange equations (13.14) for the rays, we must be careful to make $f(x, \alpha)$ homogeneous of degree unity in the α's before taking the partial derivative $\partial f / \partial \alpha_r$. This homogeneity is obtained in (15.12) if, in substituting for X from (15.10), we make it homogeneous of degree zero by using for β^2 the following expression, equivalent to (15.6):

$$\beta^2 = \frac{(\alpha\mu)^2 + (\alpha\alpha)}{(\alpha\mu)^2} . \tag{15.16}$$

16. Rays in a rotating medium.

On account of the stresses set up by rotation, we cannot discuss the passage of light through a cylinder of glass spinning about its axis by supposing it isotropic in the sense of Section 7. But we may reasonably assume that a rotating fluid remains isotropic in the local rest frame, so that the slowness-frequency equation is of the form (7.3).

Consider, then, a fluid (e.g. water) spinning about the x_3-axis with constant angular velocity ω, so that its 3-velocity is

$$\begin{aligned} w_1 &= -\omega x_2 , \quad w_2 = \omega x_1 , \quad w_3 = 0 , \\ w^2 &= \omega^2 r^2 , \quad r^2 = x_1^2 + x_2^2 . \end{aligned} \tag{16.1}$$

Writing

$$\begin{aligned} \gamma_w &= (1 - w^2/c^2)^{-\frac{1}{2}} = (1 - \omega^2 r^2 / c^2)^{-\frac{1}{2}} , \\ k(r) &= \omega \gamma_w / c , \end{aligned} \tag{16.2}$$

we have for the 4-velocity μ_r of the medium

$$\mu_1 = \gamma_w w_1 / c = - k x_2 , \quad \mu_2 = \gamma_w w_2 / c = k x_1 ,$$
$$\mu_3 = 0 , \qquad \mu_4 = i \gamma_w . \tag{16.3}$$

We shall assume the medium to be non-dispersive and of constant proper density; otherwise our calculations would be much more complicated and less explicit. As in (7.3), the slowness-frequency equation is

$$2 \Omega(\sigma , x) = (\sigma\sigma) - q (\sigma\mu)^2 = 0 , \tag{16.4}$$

where $q \; (= n^2 - 1)$ is a constant. For the rays we have the equations (8.3):

$$\frac{dx_r}{d\chi} = \frac{\partial \Omega}{\partial \sigma_r} = \sigma_r - q \mu_r (\sigma\mu) ,$$
$$\frac{d\sigma_r}{d\chi} = - \frac{\partial \Omega}{\partial x_r} = q (\sigma\mu) \sigma_t \mu_{t,r} , \tag{16.5}$$

the comma denoting a partial derivative $(\mu_{t,r} = \partial\mu_t / \partial x_r)$. Explicitly

$$\frac{dx_1}{d\chi} = \sigma_1 - q \mu_1 (\sigma\mu) , \qquad \frac{d\sigma_1}{d\chi} = q (\sigma\mu) (\sigma_1 \mu_{1,1} + \sigma_2 \mu_{2,1}) ,$$
$$\frac{dx_2}{d\chi} = \sigma_2 - q \mu_2 (\sigma\mu) , \qquad \frac{d\sigma_2}{d\chi} = q (\sigma\mu) (\sigma_1 \mu_{1,2} + \sigma_2 \mu_{2,2}) ,$$
$$\frac{dx_3}{d\chi} = \sigma_3 , \qquad \frac{d\sigma_3}{d\chi} = 0 ,$$
$$\frac{dx_4}{d\chi} = \sigma_4 - q \mu_4 (\sigma\mu) , \qquad \frac{d\sigma_4}{d\chi} = 0 . \tag{16.6}$$

Here (since $\mu_3 = 0$)

$$(\sigma\mu) = \sigma_1 \mu_1 + \sigma_2 \mu_2 + \sigma_4 \mu_4 . \tag{16.7}$$

We note the first integrals

$$\sigma_3 = \text{constant}, \qquad \sigma_4 = \text{constant}, \tag{16.8}$$

on account of which we may say that a <u>photon conserves its axial component of momentum and its energy</u> [cf. (10.1)].

Let us define P and Q by

$$P = \sigma_1 x_1 + \sigma_2 x_2 , \qquad Q = \sigma_2 x_1 - \sigma_1 x_2 ; \tag{16.9}$$

then, by (16.3),

$$\sigma_1 \mu_1 + \sigma_2 \mu_2 = k Q , \qquad \sigma_2 \mu_1 - \sigma_1 \mu_2 = - k P ,$$
$$(\sigma \mu) = k Q + \sigma_4 \mu_4 . \tag{16.10}$$

Also by (16.3) with $k' = dk/dr$

$$\mu_{1,1} = - k' x_1 x_2 / r , \qquad \mu_{1,2} = - k - k' x_2^2 / r ,$$
$$\mu_{2,1} = k + k' x_1^2 / r , \qquad \mu_{2,2} = k' x_1 x_2 / r , \tag{16.11}$$

so that

$$x_1 \mu_{1,1} + x_2 \mu_{1,2} = - (k + k' r) x_2 ,$$
$$x_1 \mu_{2,1} + x_2 \mu_{2,2} = (k + k' r) x_1 , \tag{16.12}$$

and

$$x_1 \mu_{1,2} - x_2 \mu_{1,1} = - k x_1 ,$$
$$x_1 \mu_{2,2} - x_2 \mu_{2,1} = - k x_2 . \tag{16.13}$$

We now differentiate P and Q as given in (16.9), substitute for the derivatives of x_1 , x_2 , σ_1 , σ_2 from (16.6), and make use of (16.10), (16.12), (16.13). This gives

$$\frac{dP}{d\chi} = \sigma_1^2 + \sigma_2^2 + k' r q (\sigma\mu) Q , \qquad (16.14)$$

$$\frac{dQ}{d\chi} = 0 . \qquad (16.15)$$

Thus

$$Q = \sigma_2 x_1 - \sigma_1 x_2 = \text{constant} , \qquad (16.16)$$

which tells us a <u>photon conserves its angular momentum about the axis of rotation</u>.

By (16.4) we have

$$\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2 - q(\sigma\mu)^2 = 0 , \qquad (16.17)$$

and so (16.14) may be written

$$\begin{aligned} \frac{dP}{d\chi} &= -\sigma_3^2 - \sigma_4^2 + q(\sigma\mu)^2 + k' r q (\sigma\mu) Q \\ &= -\sigma_3^2 - \sigma_4^2 + q(kQ + \sigma_4\mu_4)[(k + k'r)Q + \sigma_4\mu_4] . \end{aligned} \qquad (16.18)$$

Note that the right hand side consists of constants and known functions of r.

We have also by (16.6)

$$\frac{d}{d\chi}\left(\tfrac{1}{2} r^2\right) = x_1 \frac{dx_1}{d\chi} + x_2 \frac{dx_2}{d\chi} = P , \qquad (16.19)$$

and so (16.18) gives a differential equation of the second order to determine r as a function of χ :

$$\frac{d^2}{d\chi^2}\left(\tfrac{1}{2} r^2\right) = -\sigma_3^2 - \sigma_4^2 + q(kQ + \sigma_4\mu_4)[(k + k' r)Q + \sigma_4\mu_4] , \qquad (16.20)$$

in which

$$k\,(r) = \omega \gamma_w / c\,, \qquad \mu_4 = i\,\gamma_w\,, \qquad \gamma_w = (1 - \omega^2 r^2 / c^2)^{-\frac{1}{2}}\,. \tag{16.21}$$

Since, by (16.6), we have

$$dx_3 = \sigma_3\, d\chi \tag{16.22}$$

and σ_3 is a constant, we can at once change the independent variable from χ to x_3 if we want to.

Let us write (16.20) more explicitly. From (16.2) we find

$$k + k'\, r = \omega \gamma_w^3 / c\,, \tag{16.23}$$

and if we define a real dimensionless constant K by

$$K = -\frac{\omega\, Q}{c\, i\, \sigma_4}\,, \tag{16.24}$$

our equation for r becomes

$$\frac{d^2}{d\chi^2}(\tfrac{1}{2} r^2) = -\sigma_3^2 - \sigma_4^2 - \frac{q\sigma_4^2\,(1-K)}{1 - \omega^2\, r^2 / c^2}\left[1 - \frac{K}{1 - \omega^2\, r^2 / c^2}\right], \tag{16.25}$$

or, by (16.22),

$$\frac{d^2}{dx_3^2}(\tfrac{1}{2} r^2) = -1 - \frac{\sigma_4^2}{\sigma_3^2}\left\{1 + \frac{q\,(1-K)}{1 - \omega^2\, r^2 / c^2}\left[1 - \frac{K}{1 - \omega^2\, r^2 / c^2}\right]\right\}. \tag{16.26}$$

Everything on the right is a constant except r.

We now introduce the azimuthal angle ϕ by

$$x_1 = r\cos\phi\,, \qquad x_2 = r\sin\phi\,, \qquad \tan\phi = x_2 / x_1\,, \tag{16.27}$$

and obtain by (16.6)

$$r^2\frac{d\phi}{d\chi} = x_1\frac{dx_2}{d\chi} - x_2\frac{dx_1}{d\chi} = Q - q\,(\sigma\mu)\, k\, r^2\,, \tag{16.28}$$

or

$$\frac{d\phi}{d\chi} = \frac{Q}{r^2} - q\,k\,(k\,Q + \sigma_4\,\mu_4)$$

$$= \frac{Q}{r^2} - q\,\frac{\omega}{c}\,\gamma_w^2\,i\,\sigma_4\,(1 - K)$$

$$= \frac{Q}{r^2} - \frac{\omega}{c}\,\frac{q\,i\,\sigma_4\,(1 - K)}{1 - \omega^2 r^2 / c^2}\,. \tag{16.29}$$

When r has been found as a function of χ from (16.25), then (16.29) gives ϕ as a function of χ. Thus the path of the photon is determined. As for time-dependence, we have by (16.6)

$$\frac{dx_4}{d\chi} = \sigma_4 - q\,\mu_4\,(\sigma\mu)$$

$$= \sigma_4\,[\,1 + \frac{q\,(1 - K)}{1 - \omega^2 r^2 / c^2}\,]\,. \tag{16.30}$$

Since it is easy to get a first integral of (16.26) (multiply by $d\,(r^2) / dx_3$), the complete determination of the rays is essentially reduced to quadratures.

The formulae are much simplified if the ray starts from the axis of rotation, the angular momentum of the photon about that axis being then zero. Let us take as initial conditions, for $\chi = 0$,

$$x_1 = x_2 = x_3 = x_4 = 0\,,$$
$$r = 0\,, \qquad dr / dx_3 = \tan\theta_0\,, \tag{16.31}$$

θ_0 being the inclination of the initial ray to the axis of rotation. By (16.9), (16.24) we have

$$Q = 0\,, \qquad K = 0\,, \tag{16.32}$$

and the equation (16.26) for r simplifies to

$$\frac{d^2}{dx_3^2}\left(\tfrac{1}{2} r^2\right) = -1 - \frac{\sigma_4^2}{\sigma_3^2}\left\{1 + \frac{q}{1 - \omega^2 r^2 / c^2}\right\}. \quad (16.33)$$

Applying (16.31), we get

$$\tan^2 \theta_o = -1 - n^2 \frac{\sigma_4^2}{\sigma_3^2}, \quad \frac{i\sigma_4}{\sigma_3} = -\frac{1}{n}\sec\theta_o, \quad (16.34)$$

and so (16.33) may be written

$$\frac{d^2}{dx_3^2}\left(\tfrac{1}{2} r^2\right) = \frac{1}{n^2}\sec^2\theta_o - 1 + \frac{q \sec^2\theta_o}{n^2(1 - \omega^2 r^2 / c^2)}. \quad (16.35)$$

Multiplying by $d(r^2)/dx_3$ and integrating, we get

$$\left[\frac{d}{dx_3}\left(\tfrac{1}{2} r^2\right)\right]^2 = r^2\left(\frac{1}{n^2}\sec^2\theta_o - 1\right) - \frac{c^2 q \sec^2\theta_o}{n^2 \omega^2}\log(1 - \omega^2 r^2 / c^2), \quad (16.36)$$

which, if we define θ by

$$\tan\theta = dr / dx_3, \quad (16.37)$$

may be expressed as

$$\tan^2\theta = \frac{1}{n^2}\sec^2\theta_o - 1 - \frac{c^2 q \sec^2\theta_o}{n^2 \omega^2 r^2}\log(1 - \omega^2 r^2 / c^2). \quad (16.38)$$

As for ϕ and x_4, we substitute from (16.32) and (16.34) in (16.29) and (16.30), obtaining

$$\frac{d\phi}{dx_3} = \frac{\omega}{c}\frac{q}{n}\frac{\sec\theta_o}{1 - \omega^2 r^2 / c^2}, \quad (16.39)$$

$$\frac{1}{i}\frac{dx_4}{dx_3} = \frac{1}{n}\sec\theta_o\left[1 + \frac{q}{1 - \omega^2 r^2 / c^2}\right]. \quad (16.40)$$

Hence

$$c \frac{dt}{dx_3} = \frac{1}{n} \sec \theta_0 + \frac{c}{\omega} \frac{d\phi}{dx_3} , \tag{16.41}$$

so that

$$\omega t - \phi = \frac{\omega}{n c} \sec \theta_0 \cdot x_3 , \tag{16.42}$$

if we choose the axes so that $\phi = 0$ initially for the ray under discussion. This gives us the angular lag of the ray or photon behind the medium; note that it is proportional to x_3.

So far all calculations are exact. Let us now approximate taking $\omega r / c$ small. Expanding the logarithm in (16.38), we get

$$\left(\frac{dr}{dx_3} \right)^2 = \tan^2 \theta_0 + L^2 r^2 , \tag{16.43}$$

where

$$L = \frac{\omega}{c} \sec \theta_0 \sqrt{\frac{q}{2 n^2}} = \frac{\omega}{c} \sec \theta_0 \sqrt{\frac{n^2 - 1}{2 n^2}} . \tag{16.44}$$

Hence

$$r = L^{-1} \tan \theta_0 \sinh L x_3 , \tag{16.45}$$

or

$$r = \tan \theta_0 \cdot x_3 + \frac{1}{6} \tan \theta_0 \cdot L^2 x_3^3 , \tag{16.46}$$

to the order ω^2 / c^2 inclusive. The first term corresponds to rectilinear propagation; the second term indicates a curvature of the ray away from the axis of rotation. To the same order (16.39) gives for the rate of twisting of the ray

$$\frac{d\phi}{dx_3} = \frac{\omega}{c} \frac{q}{n} \sec \theta_0 , \tag{16.47}$$

and hence

$$\phi = \frac{\omega}{c} \frac{q}{n} \sec \theta_0 \cdot x_3 , \tag{16.48}$$

or, by (16.46),

$$\phi = \frac{\omega}{c} \cdot \frac{q}{n} \operatorname{cosec} \theta_0 \cdot r . \tag{16.49}$$

We recall that the argument of the present Section deals with the propagation of light in an isotropic medium rotating about an axis with constant angular velocity, the medium being non-dispersive and of constant proper density so that the refractive index is a constant. The theory was general to (16.30) inclusive; then we considered rays emanating from a point on the axis of rotation; and finally we approximated in this case by assuming $\omega r / c$ small.

17. A photon as a dynamical system with four degrees of freedom.

In the dictionary of Section 13, a photon appears as a dynamical system with three degrees of freedom; coordinates x_ρ, momenta σ_ρ, x_4 imaginary time, and σ_4 imaginary energy (if we put $c = h = 1$). But there is another way of looking at this dynamical system.*

Let $\Omega(\sigma, x)$ be any given function of the eight variables σ_r, x_r. Consider the equations

$$\frac{dx_r}{d\chi} = \frac{\partial \Omega}{\partial \sigma_r}, \qquad \frac{d\sigma_r}{d\chi} = -\frac{\partial \Omega}{\partial x_r} . \tag{17.1}$$

We recognize these as Hamiltonian equations of motion in classical form for a dynamical system with four degrees of freedom; the coordinates are x_r, the

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* Suggested by Professor C. Lanczos at a seminar.

momenta are σ_r , the "time" is χ , the usual Hamiltonian function is Ω ; this Hamiltonian is independent of the "time", and so (17.1) possess the first integral

$$\Omega(\sigma, x) = \text{const.} \qquad (17.2)$$

This constant does not have to vanish.

If $\Omega(\sigma, x) = 0$ happens to be the slowness-frequency equation of some optical problem, then the solutions of (17.1) provide us with the rays, as previously discussed. But these rays are included in the wider class of solutions of (17.1), because we can start with arbitrary initial values of x_r , σ_r , not subject to $\Omega(\sigma, x) = 0$. This wider viewpoint may be of advantage in discussing systems of rays, because it brings us more closely in touch with classical theory; for example, we may think in terms of a phase-space of ten dimensions, in which the coordinates are x_r , σ_r , χ , Ω , and then we recognize

$$\oint (\sigma_r \, dx_r - \Omega \, d\chi) \qquad (17.3)$$

as that relative integral invariant which is usually written

$$\oint (p_r \, dq_r - H \, dt) . \qquad (17.4)$$

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