ERRATA to

GRAVITATIONAL RADIATION

by

Julian McCrea

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p.35, Eqn.(3.2.35) should read $\underset{4}{T}^{44} = -\underset{4}{\tau}(\underline{x}) - i\int_{0}^{x_2} \underset{4}{E}^{4}\, dt$.

p.65, Eqn.(4.3.35), line 2, 3rd term in bracket

should read: $-ik_r\, E_n\, \tilde{F}_{,(n+1)}$.

Eqn.(4.3.36) should read: $J_r = -\int (D_r + i\, k_r\, E)\, E_n\, \tilde{F}_{,(n)}\, d_3x'$.

p.67, Eqn.(4.3.45) last term in square brackets on right-hand side

should read: $+\overset{-2}{E}_1\, \tilde{E}_{uvwf,(1)}$.

Eqn.(4.3.47), right-hand side should read: $k_b\, T_{cd}$.

p.74, expression (5.1.23) should read: $u \sim \frac{\log y}{y}$ as $y \to \infty$.

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GRAVITATIONAL RADIATION

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[illegible]

GRAVITATIONAL RADIATION *)

Julian McCrea

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INTRODUCTION

The purpose of the present work is to approach the problem of gravitational radiation by a method of successive approximations. The first chapter consists of a brief historical survey of the problem. In Chapter II, the method of Das, Florides and Synge for determining static and stationary fields due to a single body is described and in Chapter III this method is extended to the non-stationary case. Chapter IV is devoted to a discussion of the properties of the non-stationary model in the first approximation and in Chapter V we examine higher approximations. In the final chapter, the question of loss of mass due to gravitational radiation is discussed.

Particular points of notation will be explained as we proceed, but the general notation, which we use throughout the whole work, is as follows. Latin suffixes take the values 1,2,3,4 and Greek suffixes the values 1,2,3. A comma denotes a partial derivative and a stroke a covariant derivative. We take the signature of space-time to be +2 and time to be purely imaginary, so that the flat space metric, in an appropriate coordinate system, is the Kronecker delta, δ_{ij}. We denote the Christoffel symbols of the first kind by $\Gamma_{ij,k} = \frac{1}{2}(g_{jk,i} + g_{ki,j} - g_{ij,k})$ and those of the second kind by $\Gamma^k_{ij} = g^{kl}\,\Gamma_{ij,l}$.

CHAPTER I

The problem of gravitational radiation.

The excellent review articles by Pirani[1,2] on gravitational radiation render superfluous any detailed survey of the problem in the present work. In this chapter, therefore, we shall merely give a brief introduction to the subject, sufficient to pose the problem in its general outlines.

The question of gravitational radiation was first raised by Einstein in a paper[3] published in the same year as his "Grundlage der allgemeinen Relativitätstheorie". Two years later he published a second paper[4] in which he used essentially the same ideas as in his previous paper but presented in what he considered to be a more satisfactory manner. He considers a weak gravitational field i.e. one which deviates only slightly from flat space-time. Using a coordinate system x_i where x_μ are rectangular Cartesian coordinates in 3-space and $x_4 = it$, the metric tensor is then given by

$$g_{ij} = \delta_{ij} + \gamma_{ij} , \qquad (1.1)$$

where γ_{ij} (together with their first and second partial derivatives) are small quantities of the first order.

Substituting (1.1) into the field equations

$$G_{ij} = -\kappa T_{ij}, \qquad \kappa = 8\pi \tag{1.2}$$

where T_{ij} is the energy tensor of the matter distribution, and neglecting terms of the second order in γ_{ij} he obtains, by a now familiar process,

$$\gamma^{*}_{ij,aa} = -2\kappa T_{ij}, \tag{1.3}$$

as the equations to be satisfied by γ_{ij}, where

$$\gamma^{*}_{ij} = \gamma_{ij} - \tfrac{1}{2}\delta_{ij}\gamma_{aa}, \tag{1.4}$$

and γ^{*}_{ij} satisfy the "coordinate conditions",*

$$\gamma^{*}_{ij,j} = 0. \tag{1.5}$$

One recognizes that (1.3) is simply the usual wave equation and, taking the retarded potential solution, one obtains

$$\gamma^{*}_{ij} = 4\int \frac{T_{ij}(\underset{\sim}{x}', it')}{|\underset{\sim}{x}-\underset{\sim}{x}'|}\, d_3x', \tag{1.6}$$

$$t' = t - |\underset{\sim}{x}-\underset{\sim}{x}'|,$$

where $\underset{\sim}{x}$ and $\underset{\sim}{x}'$ are the 3-dimensional radius vectors from the origin of coordinates to the field point and source point respectively.

Einstein considers the matter distribution, or source, to be of finite dimensions, with centre of mass at the origin of spatial

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* The notation and terminology used here is not that of Einstein.

coordinates. He then proceeds to calculate the loss of energy per unit time by the source by considering the field at a distance from the origin large compared to the dimensions of the source. He takes the stress, energy and momentum of the gravitational field to be given by his pseudotensor, $(-g)^{\frac{1}{2}} t_i^{\ j}$, where this is defined by requiring it to satisfy

$$(\sqrt{g}\, T_i^{\ j} + \sqrt{g}\, t_i^{\ j})_{,j} = 0 . \qquad (1.7)$$

From (1.7) he finds an expression for the components of the energy pseudotensor in terms of the derivatives of the γ_{ij}. This expression is quadratic and hence of the second order. It is clearly not unique and several alternative pseudotensors have been subsequently proposed. On the basis of this expression he then calculates the total flux of energy through a sphere of large radius compared with the dimensions of the source and concludes that this represents the loss of energy from the source per unit time. Similar work on the linear approximation by other authors[5,6,7] adds nothing essential to Einstein's results.

Despite the elegance of these results, serious doubts may be raised as to their validity. In effect, the interpretation of the pseudotensor as representing the gravitational field energy is questionable. A great variety of such pseudotensors has been found (cf. Trautman 1962[8]), but they are either not covariant under general coordinate transformations or they depend on arbitrary vector fields. Hence, there is a high degree of arbitvariness in the concept of gravitational energy as defined by such methods. Furthermore, as pointed out by Synge[9], the approximate g_{ij} obtained by linearization may be considered as the exact metric of some distribution of matter. If one calculates the energy tensor for this g_{ij}, using the full (non-linearized) Einstein equations $T_{ij} = -\kappa^{-1} G_{ij}$, it will be found that the T_{ij} are not zero outside the body which acts as source, but are small quantities of the second order. In other words, they are of the same order of magnitude as the pseudotensor components. It would seem therefore that in order to make physically meaningful statements about the flux of energy outside the body (assuming that one can define "energy") one must go beyond the first approximation.

In recent work, Sachs[10] considers solutions of the linearized field equations which are singular along a timelike world-line which represents the history of a point multipole source. He then examines the structure of the Riemann tensor at large distances from the source in terms of the Petrov classification[11] of the Riemann tensor in vacuo. His results are vitiated, in the view of the present author, by the rather unreal character of his source and could only be regarded as provisional until proved true for the case in which the source is an extended body. This proof is carried out in Chapter IV of the present work.

A number of authors have tried to go beyond the linear approximation, using the method of Einstein, Infeld and Hoffmann[12] (the EIH method). They arrive at highly divergent conclusions. Infeld[13] concludes that "we can always find a reasonable coordinate system in which there is no gravitational radiation", and is supported in his opinion by Scheidegger[14]. Others, such as Peres[15], Havas[16] and Goldberg[17], claim to have shown that the EIH method does not exclude the possibility of radiation. The divergence of these results would lead one to suspect that perhaps the EIH method is not suited to the study of gravitational radiation.

Another method has been developed by Bonnor[18], in which he envisages the source as a pair of equal particles connected by a spring and expands the metric and other quantities in a double series involving two parameters, a characteristic mass and a characteristic dimension of the source. He works through the second approximation and finally concludes that the system loses mass equal to the amount of pseudotensor energy which flows through a large sphere surrounding it. The use of the pseudotensor raises doubts as to the validity of this very interesting work. Also, one would like to see the problem tackled with a more realistic source.

A different approach has been elaborated by Bondi and his co-workers[19]. They consider an axi-symmetric material system confined within a closed surface, the rest of space being empty and tending to flatness at spatial infinity. They then study solutions of Einstein's equations only in vacuo and at a large distance from the system. A feature of their method is the use of a coordinate system specially suited to a radiation field. The azimuth ϕ is invariantly defined by the axial symmetry; coordinates u, θ are chosen such that u = const., θ = const., ϕ = const., is the equation of an outgoing light ray, u being timelike; a coordinate r is chosen so that the element of area of the 2-surface, u = const., r = const., is $r^2 \sin\theta \, d\theta \, d\phi$. With these coordinates they can write the line-element in a form which involves four functions of u, r, θ. The field equations for empty space then yield a set of partial differential equations for these four functions. Moreover, the assumption is made that each of the four functions has the form of a polynomial in negative powers of r, together with a remainder decreasing (with its derivatives) more rapidly than the lowest power of r occurring, as r tends to infinity. They then show that the behaviour of the gravitational field far from the source is fully determined by a single function of u and θ, called the news function. Their main result is that the mass of a system is constant if and only if there is no news, and if there is news the mass decreases monotonically so long as it (the news) continues. Sachs[20] has generalised the work of Bondi by dropping the assumption of axial symmetry and obtains similar results. In the more general case there are two new functions instead of one. The latter author also investigates the asymptotic behaviour of the Riemann tensor in this case and finds it to be analogous to that which is obtained in the linear case. Unti and Newman[21] have obtained similar results using the tetrad-spinor method of Newman and Penrose[22].

The foregoing work of Bondi, Newman and others has opened up some very interesting avenues of research in general relativity theory. However, they are still far from solving the problem of gravitational radiation. As has been pointed out, they study the gravitational field only in vacuo and at a large distance from the (supposed) source. In no case do they prove the possibility of continuing the field inwards and joining them to a physically realistic source - in particular, the energy tensor inside the matter would have to be such as to give a positive density. Recent attempts[23,24] to gain information concerning the supposed source by examining the asymptotic field seem unlikely to succeed.

What, then, must we look for if we are to find a satisfactory solution to the problem of gravitational radiation? We must construct a model universe given by a certain metric tensor g_{ij}. This metric tensor will have to satisfy the following conditions:

(i) It must certainly yield a non-stationary field - perhaps other conditions will also prove necessary;

(ii) The energy tensor, calculated by equations $T^{ij} = -\kappa^{-1} G^{ij}$, must vanish outside a certain timelike world-tube.

(iii) Inside the world-tube, the energy tensor must yield a physically realistic distribution of matter. It is difficult to give a comprehensive definition of what is meant by "physically realistic", but we should certainly demand that the density be positive.

Having obtained such a model, one would then have to extract physically meaningful information from it. In all the work which we have described above, this involved making statements about the transfer of energy by radiation and the consequent loss of the mass of the source. However, we are confronted here with a real difficulty. There is no such concept in general relativity as "energy of the gravitational field" or "mass of an extended body", or at least no one has yet proposed a generally acceptable invariant definition of such quantities, applicable to all types of field. Definitions based on the various pseudotensors cannot possibly be considered as satisfactory. One must therefore continue the search for invariant quantities of the gravitational field, capable of physical interpretation, which may possibly play analogous roles in Einstein's theory to those of energy and mass in the Newtonian theory.

In the following chapters, the "ideal solution" which has just been outlined is not obtained, but the author hopes to have gone a certain amount of the way towards achieving it, in a manner which will be explained in Chapters II and III. In Chapter IV the linear approximation is discussed and certain results of Sachs are put on what is believed to be a firmer basis. The questions of energy and mass are discussed in Chapters IV and V, although the conclusions arrived at are, for the most part, of a rather negative character.

CHAPTER II

Stationary gravitational fields due to single bodies.

2.1 Model universes.

The present chapter will be concerned with model universes, and in particular with models constructed by a certain method of successive approximations. It would be as well at the outset to explain the spirit in which such problems are approached in this work. The problem is to find 20 functions g_{ij}, T^{ij} to satisfy the Einstein equations

$$G^{ij} = -\kappa T^{ij}, \quad \kappa = 8\pi. \qquad (2.1.1)$$

The g_{ij} tell us about the geometry of the universe in a given coordinate system, while from the T^{ij} information can be derived concerning the physics of the universe (cf. Synge[25]). The equation (2.1.1) connects the two sets of quantities.

The ideal would be to have an exact solution of the Einstein equations for whatever physical situation is being envisaged, a solution which is, moreover, valid both in vacuo and in the interior of material bodies - in particular, the energy tensor inside the matter must be such as to yield a positive density. However, so far, the only example of such a complete (interior and exterior) exact solution having physical significance is the spherically symmetric Schwarzschild solution.* Confronted with this problem, there are two courses which one may follow. The first is to concentrate on the left-hand side of equation (2.1.1) and look for exact solutions of the vacuum equations,

$$G^{ij} = 0 , \tag{2.1.2}$$

in the hope of finding some physically realistic solution which can then be continued with the requisite junction conditions into the interior of a material source. Although an increasing number of exact vacuum solutions exist, in no case has it been possible to tie them in with a physically realistic extended source, apart from the Schwarzschild case already mentioned.*

In the second approach, which is that adopted in the present work, the functions T^{ij} are placed on an equal footing with the g_{ij}. We try to solve equations (2.1.1) by introducing a dimensionless parameter k, small but not infinitesimal, and expanding g_{ij} and T^{ij} in a power series in k. Using a coordinate system x_i, where x_μ are rectangular Cartesian coordinates in 3-space and $x_4 = it$, the power series take the form

$$g_{ij} = \delta_{ij} + \underset{1}{g}_{ij} + \underset{2}{g}_{ij} + \dots , \tag{2.1.3}$$

$$T^{ij} = \underset{1}{T}^{ij} + \underset{2}{T}^{ij} + \dots , \tag{2.1.4}$$

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* This statement should be modified in the light of a recent paper by W.C. Hernandez in Phys. Rev. 153 (1967), 1359, in which he presents an interior, static, axially symmetric solution.

the numerical subscripts indicating the powers of k contained as a factor in the various terms. The reason for starting with k^2 will be clarified later. The T^{ij} will be zero outside a 3-cylinder in space-time, representing the history of a material body and must be physically reasonable, certainly giving positive density. The model is built up according to a well defined pattern of mutual interdependence between the metric and energy tensors at each stage of approximation. In this method there are no terms thrown away as in the approximate methods of Einstein[4] and Eddington[5,6].

Let us suppose the method to be stopped at a certain stage of the approximation, say k^N. We shall then have

$$g_{ij} = \delta_{ij} + \underset{2}{g}_{ij} + \dots + \underset{N}{g}_{ij} , \qquad (2.1.5)$$

$$T^{ij} = \underset{2}{T}^{ij} + \underset{3}{T}^{ij} + \dots + \underset{N}{T}^{ij} , \qquad (2.1.6)$$

with

$$\underset{P}{G}^{ij} = -\kappa \underset{P}{T}^{ij} \quad (P = 2, 3, \dots N) \qquad (2.1.7)$$

where $\underset{P}{G}^{ij}$ is the term of order k^P in the expansion of the Einstein tensor G^{ij} for the metric tensor (2.1.5).

It is true that the 20 functions given by (2.1.5) and (2.1.6) do not satisfy the equations (2.1.1) exactly. What we construct is a model universe with a certain energy tensor inside the history of the body, and outside, instead of a vacuum, a residual energy tensor of order k^{N+1}. In other words, if we take the metric tensor (2.1.5) as the exact tensor of our universe and calculate T^{ij} from the equations, $T^{ij} = -\kappa^{-1} G^{ij}$, we shall have an energy tensor inside the body, the first N terms of which are given by (2.1.6), and outside, a residual energy tensor of order k^{N+1}. By proceeding far enough in the approximation method we can make this residual energy tensor as small as we please.

The model which we intend to construct is that of a radiation field due to a finite extended source. In other words, we shall consider a lump of matter such as, for instance, the earth or the sun or a star, and some sort of disturbance taking place in the interior of the matter giving rise to a radiative field. The method used will be an extension of that due to Das, Florides and Synge[26], as modified by Florides and Synge[27], which gives the field of a single body at rest or in uniform rotation about an axis of symmetry. These two papers will henceforth be referred to as DFS and FS respectively. In order to gain a better insight into the extended method, it will be of some advantage to consider first the static and stationary case as treated in DFS and FS.

2.2 Static and stationary models.

Consider a single body at rest or in uniform rotation about an axis of symmetry (stationary system). The Newtonian equations of such a system in rectangular Cartesian coordinates are

$$(\rho u_\alpha u_\beta - S_{\alpha\beta})_{,\beta} = \rho V_{,\alpha} \; ,$$
$$(\rho u_\beta)_{,\beta} = 0, \qquad (2.2.1)$$

where ρ is the density, u_α the velocity, $S_{\alpha\beta}$ the stress and V is the potential. Commas, as usual, indicate partial derivatives. These equations are invariant under the following transformations:

$$\rho \to k^2 \rho, \quad u_\alpha \to k u_\alpha, \quad S_{\alpha\beta} \to k^4 S_{\alpha\beta} \; , \qquad (2.2.2)$$

where k is an arbitrary constant. Thus, given one Newtonian model, we have an infinite sequence of such models. This suggests that a relativistic model should have the same property and that the metric tensor should contain an arbitrary dimensionless parameter k. Furthermore, from (2.2.2) we conclude that in order to avoid fractional powers, the density should be represented by k^2. We seek, therefore, a power series in k for the metric tensor, where k^2 is a dimensionless parameter of the order of magnitude of ρa^2 or $m a^{-1}$, m being the mass and a being a typical radius of the body. In the case of the sun and the earth $m a^{-1}$ is of order 10^{-6} and 10^{-9} respectively. Since the principal contribution to the field comes from the mass, the metric tensor will have the form given by (2.1.3).

Let us agree to consider a finite number of terms, so that

$$g_{ij} = \delta_{ij} + \underset{1}{g}_{ij} + \dots + \underset{N}{g}_{ij} . \qquad (2.2.3)$$

Explicit calculation of the contravariant components of the Einstein tensor for this metric is similar to that of DFS for the covariant components. It is found that G^{ij} may be expressed as an infinite power series

$$G^{ij} = \underset{1}{G}^{ij} + \underset{2}{G}^{ij} + \dots + \underset{N}{G}^{ij} + \underset{N+1}{\hat{G}}^{ij} + \dots \qquad (2.2.4)$$

where the circumflex accent indicates the term of order k^{N+1} in the expression for the metric (2.2.3) which ends with $\underset{N}{g}_{ij}$.

Defining the star-conjugate of a tensor by

$$P^{*ij} = P^{ij} - \tfrac{1}{2}\,\delta_{ij}\,P^{aa} , \qquad (2.2.5)$$

it is easy to show that

$$P^{**ij} = P^{ij} . \qquad (2.2.6)$$

From DFS we take the result

$$\underset{N}{G}^{ij} = \underset{N}{L}^{*}_{ij} + \underset{N}{\bar{G}}^{ij} , \qquad (2.2.7)$$

where

$$\underset{N}{L}^{*}_{ij} = \tfrac{1}{2}\left(\underset{N}{g}_{ij,aa} + \underset{N}{g}_{aa,ij} - \underset{N}{g}_{ia,aj} - \underset{N}{g}_{ja,ai}\right) \qquad (2.2.8)$$

and $\underset{N}{\bar{G}}^{ij}$ is the term of order k^{N} in the expansion of the Einstein tensor for the metric

$$g_{ij} = \delta_{ij} + \underset{1}{g}_{ij} + \dots + \underset{N-1}{g}_{ij} . \qquad (2.2.9)$$

From (2.2.8) we get the useful identity

$$\underset{N}{L}_{ij,j} = 0 . \qquad (2.2.10)$$

If $\underset{N}{g}_{ij}$ satisfy the coordinate conditions

$$\underset{N}{g}^{*}_{ij,j} = 0 , \qquad (2.2.11)$$

one may easily verify that

$$\underset{N}{L}^{*}_{ij} = \tfrac{1}{2}\, \underset{N}{g}_{ij,aa} \; ; \qquad \underset{N}{L}_{ij} = \tfrac{1}{2}\, \underset{N}{g}^{*}_{ij,aa} . \qquad (2.2.12)$$

With these preliminary remarks made, we shall now proceed to give the essential steps in the FG method following the exposition of Synge[28]. Consider a timelike 3-cylinder B dividing space-time into two regions, an interior I and an exterior E. The interior will represent the history of the body. We seek 20 functions g_{ij}, T^{ij} of the coordinates to satisfy the following conditions:

(i) $\quad G^{ij} = -\kappa T^{ij}$ in $E + I$ (2.2.13)

(ii) $\quad T^{ij} = 0$ in E, (2.2.14)

(iii) $\quad T^{ij}$ is reasonable physically in I, certainly giving positive density, (2.2.15)

(iv) $\quad T^{ij} n_j = 0$ on B, (2.2.16)

where n_j is the unit outward normal to B.

In the present method, condition (ii) is relaxed. We shall be satisfied if we can make T^{ij} as small as we please in E, smaller than, say, the energy tensor of radiation in the solar system. The plan is to build up metric and energy tensors of the forms (2.1.3) and (2.1.4) in successive steps by a process of induction.

Suppose we have found

$$\underset{2}{g}_{ij}, \underset{3}{g}_{ij}, \ldots, \underset{N-1}{g}_{ij}; \quad \underset{2}{T}^{ij}, \underset{3}{T}^{ij}, \ldots \underset{N-1}{T}^{ij}, \qquad (2.2.17)$$

such that for the metric

$$g_{ij} = \delta_{ij} + \underset{2}{g}_{ij} + \cdots + \underset{N-1}{g}_{ij} \qquad (2.2.18)$$

we have

$$\underset{P}{G}^{ij} = -\kappa \underset{P}{T}^{ij} \quad \text{in } E + I \quad (P = 2, \ldots N-1), \qquad (2.2.19)$$

$$\underset{P}{T}^{ij} = 0 \quad \text{in } E \quad (P = 2, \ldots N-1), \qquad (2.2.20)$$

with conditions (iii) and (iv) satisfied. Let us assume that $\underset{P}{g}_{ij}$ (P = 2,..., N-1) are of order x^{-1} ($x = (x_\mu x_\mu)^{\frac{1}{2}}$) as x goes to infinity, and of class C^1, piecewise C^3. We now have a universe which has in E an energy tensor (called the residual energy tensor) of order k^N. In order to reduce the residual energy tensor to order k^{N+1} we seek functions $\underset{N}{g}_{ij}$ and $\underset{N}{T}^{ij}$ such that for

$$g_{ij} = \delta_{ij} + \underset{2}{g}_{ij} + \cdots + \underset{N-1}{g}_{ij} + \underset{N}{g}_{ij}, \qquad (2.2.21)$$

we shall have

$$\underset{N}{G}^{ij} = -\kappa \underset{N}{T}^{ij} \quad \text{in} \quad E + I , \tag{2.2.22}$$

$$\underset{N}{T}^{ij} = 0 \quad \text{in} \quad E, \quad \underset{N}{T}^{ij} n_j = 0 \quad \text{on } B. \tag{2.2.23}$$

We choose $\underset{N}{T}^{ij}$ first and then deduce $\underset{N}{g}_{ij}$.

Our choice of $\underset{N}{T}^{ij}$ is rather severely restricted by the previous steps. Using the divergence identity

$$G^{ij}{}_{|j} = 0 \tag{2.2.24}$$

we derive

$$G^{ij}{}_{,j} - \kappa K^i = 0 , \tag{2.2.25}$$

where

$$K^i = -\kappa^{-1}(\Gamma^i_{aj} G^{aj} + \Gamma^j_{aj}) G^{ia} . \tag{2.2.26}$$

Substituting the metric (2.2.21) in (2.2.25) and taking the terms of order k^N we obtain

$$\underset{N}{G}^{ij}{}_{,j} = \kappa \underset{N}{K}^i \tag{2.2.27}$$

where, by (2.2.19) and the fact that $\underset{N}{K}^i$ involves only $\underset{P}{g}_{ij}$ and $\underset{P}{G}^{ij}$ of order P less than N,

$$\underset{N}{K}^i = (\Gamma^i_{aj} T^{aj} + \Gamma^j_{aj} T^{ia})_N . \tag{2.2.28}$$

The right-hand side of (2.2.28) is the term of order k^N in the expansion of the expression inside the brackets. It is clear that it only contains terms of g_{ij} and T^{ij} up to and including order κ^{N-2}.

From (2.2.27), we see that in order to satisfy (2.2.22) $\underset{N}{T}^{ij}$ must satisfy partial differential equations

$$\underset{N}{T}^{ij}{}_{;j} = -\underset{N}{K}^{i} , \qquad (2.2.29)$$

in which the expression on the right-hand side has already been determined by the earlier choices. Note that $\underset{N}{K}^{i} = 0$ in E, so that the problem of finding $\underset{N}{T}^{ij}$ resolves itself into an interior problem in the domain I.

It may be remarked at this stage that what has been said from equation (2.2.3) up to now, with the exception of the manner in which g_{ij} behaves as r goes to infinity, is equally valid for the non-stationary case and will be used when we come to study that case. For the present, however, we consider a stationary field so that B is given by an equation $f(x_1,x_2,x_3) = 0$ and all the functions occurring above are independent of x_4. Equation (2.2.29) then breaks into two parts:

(i) Solve in I: $\underset{N}{T}^{\mu\nu}{}_{;\nu} = -\underset{N}{K}^{\mu}$, with $\underset{N}{T}^{\mu\nu} n_{\nu} = 0$ on B, (2.2.30)

(ii) Solve in I: $\underset{N}{T}^{4\nu}{}_{;\nu} = -\underset{N}{K}^{4}$, with $\underset{N}{T}^{4\nu} n_{\nu} = 0$ on B. (2.2.31)

The necessary and sufficient conditions for the existence of a solution of (2.2.30) and (2.2.31) (a solution which will be highly indeterminate as there are only 4 equations for 9 unknowns) are that for an arbitrary Euclidean Killing vector field ξ_{μ} satisfying

$$\xi_{\mu,\nu} + \xi_{\nu,\mu} = 0 \qquad (2.2.32)$$

and for

$$\xi_4 = 1 , \qquad (2.2.33)$$

we should have

$$\underset{N}{J} = \int_\Sigma \underset{N}{E}^i \xi_i \, d_3x = 0 , \qquad (2.2.34)$$

where $d_3x = dx_1dx_2dx_3$. Since the $\underset{N}{E}^i$ have already been determined, the method would break down if (2.2.34) were not satisfied. However, it is an essential feature of the method that (2.2.34) is, in fact, satisfied. To prove this, we substitute the metric tensor (2.2.18) into the identity (2.2.26). Picking out the term of order κ^N and using (2.2.28) and (2.2.19) we obtain

$$\underset{N}{\hat{G}}^{ij}{}_{,j} = \kappa \underset{N}{E}^i . \qquad (2.2.35)$$

Hence

$$\kappa \underset{N}{J} = \int_\Sigma \underset{N}{\hat{G}}^{ij}{}_{,j} \xi_i \, d_3x . \qquad (2.2.36)$$

Integrating by parts, this integral may be carried onto B and through B, by virtue of the continuity of $\underset{N}{\hat{G}}^{ij} n_j$ (which is easily proved) to the infinite sphere. Furthermore, since the g's are of order r^{-1} at infinity and each differentiation increases the order by unity, $\underset{N}{\hat{G}}^{ij}$ will be of order r^{-4} at infinity and so the integral over the infinite sphere vanishes and $\underset{N}{J} = 0$.

We have therefore proved that quantities $\underset{N}{T}^{0\nu}$ and $\underset{N}{T}^{i\nu}$ exist, satisfying (2.2.30) and (2.2.31). Note that the component $\underset{N}{T}^{44}$ does not occur in these equations. We can therefore choose this component arbitrarily in so far as the mathematics of the problem is concerned. Normally the choice of $\underset{N}{T}^{44}$ will be guided by the physical situation envisaged.

Assuming then that we have found $\underset{N}{T}^{ij}$ satisfying (2.2.29) and the conditions (2.2.14) and (2.2.15), we define

$$\underset{N}{g}_{ij}(x) = 4\int_{D+I} \frac{\underset{N}{T}^{*ij}(x') + \kappa^{-1}\,\underset{N}{\bar{G}}^{*ij}(x')}{|x - x'|}\, d_3x' \,. \tag{2.2.37}$$

It is not difficult to prove that (2.2.37) implies

$$\underset{N}{g}^{*}_{ij,j} = 0 \,, \tag{2.2.38}$$

which are the so-called coordinate conditions. In proving this, a little care is required with regard to the boundary B (cf. DPS appendix).

From (2.2.37) we obtain

$$\frac{1}{2}\Delta\, \underset{N}{g}^{*}_{ij} = -\kappa\, \underset{N}{T}^{ij} - \underset{N}{\bar{G}}^{ij} \,, \tag{2.2.39}$$

where Δ is the Laplacian operator. By (2.2.38) and (2.2.12) this may be written

$$\underset{N}{h}_{ij} = -\kappa\, \underset{N}{T}^{ij} - \underset{N}{\bar{G}}^{ij} \,. \tag{2.2.40}$$

Hence, by (2.2.7), the requirement (2.2.22), namely

$$\underset{N}{G}^{ij} = -k \underset{N}{T}^{ij} \quad \text{in} \quad E + I$$

is satisfied.

Since $\underset{N}{G}^{ij}$ is of order x^{-4} at infinity, the integral (2.2.37) converges and furthermore $\underset{N}{g}_{ij}$ is of order x^{-1} at infinity, as is also required for the inductive process.

We have thus found a set of functions $\underset{N}{g}_{ij}$ and $\underset{N}{T}^{ij}$ to satisfy (2.2.22) and (2.2.23) and established an inductive process which can be carried out to any order. The residual energy tensor outside the body may therefore be reduced to any order in the parameter k. In the case of a static field only even powers of k will occur, whereas in the stationary case one will have both even and odd powers.

We conclude this section with some remarks concerning the conditions imposed on T^{ij} at the boundary B. In the above work, we have demanded that $T^{ij} n_j = 0$ on B. A stronger condition would be to require T^{ij} to vanish smoothly on the boundary, that is, that T^{ij} together with its partial derivatives to some order should be zero on B. With this condition all of the preceding work is, a fortiori, valid. The only difficulty which might arise is the following:

Let us suppose that $\underset{P}{T}^{ij}$ $(P = 2,\ldots,N-1)$ vanish, together with their partial derivatives up to the Mth order. We write this as

$$\underset{P}{T}^{ij} \underset{M}{=} 0 \quad \text{on } B \qquad (P = 2,\dots,N-1). \tag{2.2.41}$$

It follows that

$$\underset{N}{K}^{i} \underset{M}{=} 0 \quad \text{on } B \, . \tag{2.2.42}$$

Instead of (2.2.30) and (2.2.31) we now have the problem of finding in I a solution of

$$\begin{aligned} \underset{N}{T}^{\mu\nu}{}_{,\nu} &= -\underset{N}{K}^{\mu} , \\ \underset{N}{T}^{4\nu}{}_{,\nu} &= -\underset{N}{K}^{4} , \end{aligned} \tag{2.2.43}$$

with $\underset{N}{K}^{\mu}$, $\underset{N}{K}^{4}$ satisfying (2.2.42) and (2.2.34), such that

$$\begin{aligned} \underset{N}{T}^{\mu\nu} &\underset{M}{=} 0 \quad \text{on } B , \\ \underset{N}{T}^{4\nu} &\underset{M}{=} 0 \quad \text{on } B . \end{aligned} \tag{2.2.44}$$

The existence of such a solution is proved by a theorem of McCrea and Synge[29]. As a matter of fact, it is proved in this theorem that there exist, under the above conditions on $\underset{N}{K}^{i}$, solutions $\underset{N}{T}^{\mu\nu}$, $\underset{N}{T}^{4\nu}$ satisfying

$$\begin{aligned} \underset{N}{T}^{\mu\nu} &\underset{M+1}{=} 0 \quad \text{on } B , \\ \underset{N}{T}^{4\nu} &\underset{M+1}{=} 0 \quad \text{on } B . \end{aligned} \tag{2.2.45}$$

Hence, the condition of strong vanishing on the boundary can be carried right through every stage of the approximation method.

This modification of the boundary conditions will be of some advantage when we go on to consider the non-stationary case. It is justified physically by the fact that in the sun, say, or in a star, the density and pressure tend to zero at the surface.

CHAPTER III.

Gravitational radiation from a single extended source.

3.1 Introductory remarks.

The physical situation envisaged is that of a body at rest up to a certain time $t = 0$. The interior and exterior fields are therefore static up to that instant. Then, at time $t = 0$, some disturbance occurs which changes the energy tensor in the body. The disturbance, or at least its first-order part, lasts for a finite time, after which the field becomes static once again (that is, to the first order of approximation - for higher orders it is found that the time-dependence must continue). Thus to the first order of approximation, one has a region of space-time which is radiative sandwiched between two static regions. However, when we consider the higher orders the picture is not quite so simple, as we shall see.

The method used in this chapter is an extension of the DFS method. The essential inductive procedure is set up in paragraph C of section (3.2) and the first step in the induction is described in paragraph A of the same section. Strictly speaking, these two paragraphs are sufficient to give the essentials of the method but, in order to introduce with greater clarity the distinctive features of the non-stationary model, the second stage of the approximation is described in detail in paragraph 3. It is well known that in other approximation methods, such as that of EIH, difficulties of convergence arise at higher approximations. <u>In the present method, the quantities describing the field are expressed in the form of integrals which converge at any given event of space-time.</u> The behaviour of these integrals as one allows the field event to go to infinity along various types of world-lines (to be specified) is examined in Chapter 7.

Before entering into any detail, it is as well to describe the rather obvious formal extension of the work of the previous chapter to the non-stationary case. As we have already pointed out, what has been said in the previous chapter from (2.2.3) to (2.2.29) applies equally well to the non-stationary case. The completion of the work for this case depends on the existence of a time-dependent $\underset{N}{T}^{ij}$ which satisfies (2.2.29) and on the convergence of any integrals met with in the ensuing work. These questions will be dealt with in the next section, but for the present let us assume that we have found $\underset{N}{T}^{ij}$ satisfying (2.2.29) and the requisite boundary conditions. We may then define

$$\underset{N}{g}_{ij}(\underline{x},it) = 4\int_{\mathrm{Nul}}\left(\underset{N}{T}^{*ij}(\underline{x}',it') + x^{-1}\underset{N}{\dot{T}}^{*ij}(\underline{x}',it')\right)d\omega \qquad (3.1.1)$$

where $d\omega = |\underline{x}-\underline{x}'|^{-1}d_3x'$, $t' = t-|\underline{x}-\underline{x}'|$; $d\omega$ is the absolute 3-content of a 3-cell on the null-cone drawn into the past from the field point $(\underline{x}, it)$ (cf. Synge[30]).

The coordinate conditions (2.2.30) are again satisfied and (3.1.1) implies that

$$\tfrac{1}{2}\,\Box\,\underset{N}{g}^{*}{}_{ij} \;=\; -\kappa\,\underset{N}{T}^{ij} - \underset{N}{G}^{ij}\,, \qquad (3.1.2)$$

where $\Box$ is the D'Alembertian operator. By (2.2.30) and (2.2.12), this may be written

$$\underset{N}{L}^{ij} \;=\; -\kappa\,\underset{N}{T}^{ij} - \underset{N}{G}^{ij}\,, \qquad (3.1.3)$$

and hence, by (2.2.7),

$$\underset{N}{G}^{ij} \;=\; -\kappa\,\underset{N}{T}^{ij}\,. \qquad (3.1.4)$$

Finally, we emphasize again that the validity of these formal calculations will depend on the existence of $\underset{N}{T}^{ij}$ satisfying the required conditions and on the convergence of the integral (3.1.1).

3.2 Non-stationary model universe.

The model will be described in terms of a background Minkowskian space-time. The set of functions g_{ij} is then a tensor field over this space-time.

Let S be a 3-cylinder, with generators parallel to the t-axis, dividing space-time into two domains, an exterior E and an interior I. The interior will represent the history of a body. In the static case no special assumptions of symmetry were required, but we shall find it necessary in the present case, when we go beyond the first approximation, to choose the various scalar, vector and tensor quantities occurring in the work in such a way that they possess reflectional symmetry in three mutually orthogonal planes in the background Euclidean 3-space. In anticipation of this, the surface of the body is taken to have the same reflectional symmetry. Let $t = 0$, $t = \tau$ be two hyperplanes enclosing a region I_1' of I. Let S_1 and S_2 be the lower and upper boundaries respectively of the region of space-time traversed by the null-cones drawn into the future from all events in I_1'. S_1 and S_2 will divide I into three regions, I_0, I_1 and I_2, with $I_1 = I_1' + I_1''$. Likewise, E is divided into three regions, E_0, E_1 and E_2 (cf. Fig.1).

The model will be built up step by step in accordance with the method of the preceding chapter as completed in section 1 of the present chapter. The field in $I_0 + E_0$ will be taken to be static.

A. The first approximation.

We begin the method at the first order, that is at $O(k^2)$. In accordance with (2.2.14) and (2.2.29), and since the term $\underset{2}{K}^{i}$ does not appear, we must choose $\underset{2}{\Gamma}^{ij}$ to satisfy

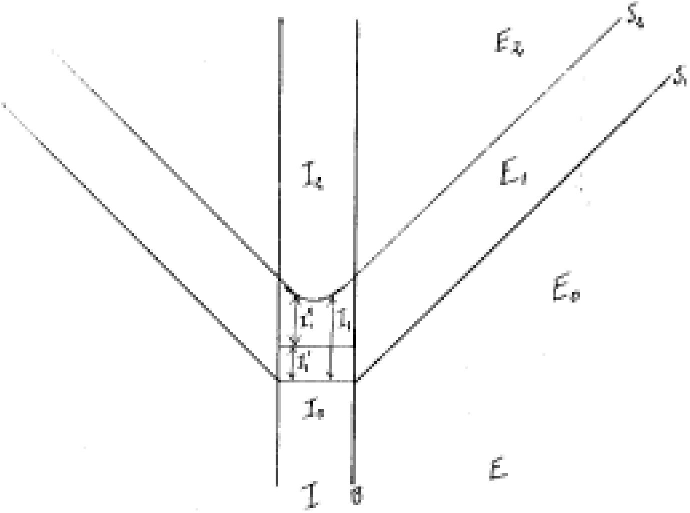

Figure 1. Non-stationary model universe.

$$\underset{2}{T}^{ij} = 0 \quad \text{in } E , \tag{3.2.1}$$

$$\underset{2}{T}^{ij}{}_{,j} = 0 \quad \text{in } I + E . \tag{3.2.2}$$

Furthermore, guided by PS, we require that

$$\begin{aligned} \underset{2}{T}^{44}{}_{,4} &= 0 \quad \text{in } I_0 + I_1^* + I_2 , \\ \underset{2}{T}^{\mu\nu} &= 0 \quad \text{in } I_0 + I_1^* + I_2 , \\ \underset{2}{T}^{\mu 4} &= 0 \quad \text{in } I_0 + I_1^* + I_2 . \end{aligned} \tag{3.2.3}$$

Also, we demand that $\underset{2}{T}^{ij}$ shall be of class C^N on B and on $t = 0$, $t = \tau$, where N is some sufficiently large integer.

To construct $\underset{2}{T}^{ij}$ in accordance with these requirements, we start with

$$\underset{2}{T}^{\mu\nu} = \underset{2}{A}^{\mu\nu}(\underset{\sim}{x}) \; F(t) , \tag{3.2.4}$$

$\underset{2}{A}^{\mu\nu}(\underset{\sim}{x}) = 0$ in E and both it and $F(t)$ are of class C^N, with $F(t)$ satisfying

$$\begin{aligned} F(t) &= 0 \quad \text{for } t < 0 \text{ and } t > \tau , \\ \int_0^\tau F(t)dt &= 0 . \end{aligned} \tag{3.2.5}$$

The first of (3.2.5) ensures that the second of (3.2.3) is fulfilled. We demand further, for the reason already given, that $\underset{2}{A}^{\mu\nu}(\underset{\sim}{x})$ possess reflectional symmetry in the same three mutually orthogonal planes as mentioned with regard to the surface of the body.

Next, in accordance with the third of (3.2.3) and (3.2.2), we define $\underset{2}{T}^{\mu 4}$ by

$$\underset{2}{T}^{\mu 4} = 0 \quad \text{for} \quad t < 0 \, ,$$
$$\underset{2}{T}^{\mu\nu}{}_{,\nu} + \underset{2}{T}^{\mu 4}{}_{,4} = 0 \quad \text{in} \quad E + I \, , \tag{3.2.6}$$

and satisfy the latter by taking

$$\underset{2}{T}^{\mu 4} = -i \int_0^t \underset{2}{T}^{\mu\nu}{}_{,\nu} \, dt \, , \quad t > 0 \, . \tag{3.2.7}$$

Thus we obtain

$$\underset{2}{T}^{\mu 4} = -i \, \underset{2}{A}^{\mu\nu}{}_{,\nu} \int_0^t F(t) \, dt \, , \quad t > 0 \, . \tag{3.2.8}$$

On account of (3.2.5) it is clear that

$$\underset{2}{T}^{\mu 4} = 0 \, , \qquad t > \tau \, . \tag{3.2.9}$$

We have thus satisfied the third of (3.2.3).

Finally, in accordance with the first of (3.2.3) and (3.2.2), we define $\underset{2}{T}^{44}$ by

$$\underset{2}{T}^{44} = -\rho_2(\underset{\sim}{x}) \, , \quad t < 0 \tag{3.2.10}$$

(where $\rho_2(\underset{\sim}{x})$ is positive and possesses the same reflectional symmetry as the surface of the body), and

$$\underset{2}{T}^{4\nu}{}_{,\nu} + \underset{2}{T}^{44}{}_{,4} = 0 \quad \text{everywhere,} \tag{3.2.11}$$

so that

$$\underset{2}{T}^{44} = -\rho_2(\underset{\sim}{x}) - i\int_0^t \underset{2}{T}^{4\nu}{}_{,\nu}\, dt\,, \quad t > 0\,. \qquad (3.2.12)$$

Hence,

$$\underset{2}{T}^{44} = -\rho_2(\underset{\sim}{x})\,, \quad t < 0$$

$$\underset{2}{T}^{44} = -\rho_2(\underset{\sim}{x}) - \underset{2}{A}^{\mu\nu}{}_{,\mu\nu}\int_0^t d\eta \int_0^\eta F(\xi)d\xi\,, \quad t > 0\,. \qquad (3.2.13)$$

Because of (3.2.5) it follows that

$$\underset{2}{T}^{44} = -\rho_2(\underset{\sim}{x}) - \underset{2}{A}^{\mu\nu}{}_{,\mu\nu}\int_0^\tau d\eta \int_0^\eta F(\xi)\, d\xi$$

$$= -\underset{2}{\sigma}(\underset{\sim}{x}) \quad (\text{say})\,, \quad t > \tau\,. \qquad (3.2.14)$$

Note that we have $A^{\mu\nu}$, $F(t)$ and τ at our disposal to ensure that $\underset{2}{\sigma}(\underset{\sim}{x})$ is positive.

We have thus obtained $\underset{2}{T}^{ij}$ to satisfy all the requirements (3.2.1) to (3.2.3). Note that, to this order, we have a time-dependent energy tensor sandwiched between two static ones. In effect, referring to Fig.1,

$$\underset{2}{T}^{\mu\nu} = 0\,, \ \underset{2}{T}^{\mu 4} = 0\,, \ \underset{2}{T}^{44} = -\rho_2(\underset{\sim}{x}) \quad \text{in } I_0\,,$$

$$\underset{2}{T}^{ij} \text{ is variable in } I_1'\,, \qquad (3.2.15)$$

$$\underset{2}{T}^{\mu\nu} = 0\,, \ \underset{2}{T}^{\mu 4} = 0\,, \ \underset{2}{T}^{44} = -\underset{2}{\sigma}(\underset{\sim}{x}) \quad \text{in } I_1'' + I_2\,.$$

From $\underset{2}{T}^{ij}$ we derive $\underset{2}{g}_{ij}$ by the retarded potential formula as in DFS,

$$\underset{2}{g}_{ij} = 4\int \underset{2}{T}^{*ij}\, d\sigma , \qquad (3.2.16)$$

the integral being taken over the null-cone drawn into the past from the field point (x, it). Since $\underset{2}{T}^{ij} = 0$ in E, the domain of integration is, in fact, finite.

From (3.2.15) and (3.2.16) we derive the following information:

$$\underset{2}{g}_{\mu 4} = 0 , \quad \underset{2}{g}_{ij,4} = 0 \quad \text{in } I_0 + E_0 + I_2 + E_2 , \qquad (3.2.17)$$

$$\underset{2}{g}_{ij} \quad \text{is variable in } I_1 + E_1 .$$

B. The second approximation.

We now have $\underset{2}{g}_{ij}$ and a residual energy tensor of order k^4 in E. Since we want the field in $I_0 + E_0$ to be static, we shall use only even powers of k. To reduce the residual energy tensor to order k^6, we begin by choosing $\underset{4}{T}^{ij}$ to satisfy, in accordance with (2.2.25),

$$\underset{4}{T}^{ij} = 0 \quad \text{in } E ,$$

$$\underset{4}{T}^{ij}{}_{,j} = -\underset{4}{K}^{i} \quad \text{in } I + E , \qquad (3.2.18)$$

$$\underset{4}{T}^{ij} \quad \text{is of class } C^{N} \text{ on } B ,$$

where, by (2.2.28),

$$\underset{4}{K}^{i} = (\underset{2}{\Gamma}^{i}{}_{\alpha j}\, \underset{2}{T}^{\alpha j} + \underset{2}{\Gamma}^{j}{}_{\alpha j}\, \underset{2}{T}^{i\alpha}) . \qquad (3.2.19)$$

Since here and later T^{ij} is to vanish in Σ, we can express the last condition in (3.2.18) by saying that $\underset{4}{T}^{ij}$ vanishes strongly to order N on B, and writing

$$\underset{4}{T}^{ij} \underset{N}{=} 0 \quad \text{on } B . \tag{3.2.20}$$

It is clear from (3.2.19) that $K^i \underset{N}{=} 0$ on B.

It is at this stage that we use the condition of reflectional symmetry in three mutually orthogonal planes which were imposed on the surface of the body and on $\underset{2}{A}^{\mu\nu}(\underset{\sim}{x})$, $p_2(\underset{\sim}{x})$. This condition implies that we shall have the same symmetry in all quantities constructed from these, and, in particular, $\underset{4}{K}^{\mu}$ has this symmetry. Hence, it follows from considerations analogous to those of Florides, Synge and Yukawa[35] that

$$\int_{\Sigma^4} K^{\mu} \xi_{\mu} \, d_3 x = 0 , \tag{3.2.21}$$

for all t, ξ_{μ} being an arbitrary Killing 3-vector, that is, satisfying

$$\xi_{\mu,\nu} + \xi_{\nu,\mu} = 0 . \tag{3.2.22}$$

Consider now

$$\underset{4}{K}^{4} = (\underset{2}{\Gamma}^{4}_{\alpha j} \underset{2}{T}^{\alpha j} + \underset{2}{\Gamma}^{j}_{\alpha j} \underset{2}{T}^{4\alpha}) . \tag{3.2.23}$$

In $\Sigma_0 + \Sigma_2$, we know, from (3.2.15) and (3.2.17), that

$$\underset{2}{T}^{\mu\nu} = \underset{2}{T}^{\mu 4} = 0 , \tag{3.2.24}$$

and

$$\underset{2}{g}_{\mu 4} = \underset{2}{g}_{ij,4} = 0 \,. \tag{3.2.25}$$

Hence, from (3.2.23), we obtain

$$\underset{4}{K}^{4} = 0 \quad \text{in} \quad I_{o} + I_{2} \,. \tag{3.2.26}$$

With this information concerning $\underset{4}{K}^{4}$ we now return to equation (3.2.18). Our task is to find $\underset{4}{T}^{ij}$ in I to satisfy

$$\underset{4}{T}^{\mu\nu}{}_{,\nu} + \underset{4}{T}^{\mu 4}{}_{,4} = - \underset{4}{K}^{\mu} \tag{3.2.27}$$

$$\underset{4}{T}^{4\nu}{}_{,\nu} + \underset{4}{T}^{44}{}_{,4} = - \underset{4}{K}^{4} \tag{3.2.28}$$

and such that $\underset{4}{T}^{ij} \underset{N}{=} 0$ on B. Since $\int \underset{4}{K}^{\mu} \xi_{\mu} \, d_{3}x = 0$ for all t, and since by our choice of $\underset{2}{T}^{ij}$ we have ensured that $\underset{4}{K}^{\mu} \underset{N}{=} 0$ on B, then, by the theorem of McCrea and Synge[28], there exists $\underset{4}{T}^{\mu\nu}$ satisfying

$$\underset{4}{T}^{\mu\nu}{}_{,\nu} = - \underset{4}{K}^{\mu} \quad \text{for all} \quad t \,, \tag{3.2.29}$$

and such that $\underset{4}{T}^{\mu\nu} \underset{N}{=} 0$ on B. Thus to satisfy (3.2.27) we choose

$$\underset{4}{T}^{\mu 4} = 0 \quad \text{everywhere.} \tag{3.2.30}$$

It remains to satisfy (3.2.28), which reduces to

$$\underset{4}{T}^{44}{}_{,4} = - \underset{4}{K}^{4} \,, \tag{3.2.31}$$

where $\underset{4}{K}^{4}$ is subject to the condition (3.2.26). We satisfy these by taking

$$\underset{4}{T}^{44} = -\rho_4(\underline{x}) - i\int_0^t \underset{4}{K}^{4}\,dt\ , \qquad (3.2.32)$$

where $\rho_4(x)$ is the time-independent value of $\underset{4}{T}^{44}$ in I_0. For the value of $\underset{4}{T}^{44}$ in I_2, we shall have

$$\underset{4}{T}^{44} = -\rho_4(\underline{x}) - i\int_0^t \underset{4}{K}^{4}\,dt \qquad (3.2.33)$$

which again is time-independent.

To summarise then, we have the following situation as regards the part of order κ^4 of the energy tensor:

In I_0,

$\underset{4}{T}^{\mu 4} = 0$, $\underset{4}{T}^{44}$ and $\underset{4}{T}^{\mu\nu}$ are independent of time;

(3.2.34)

in I_1,

$\underset{4}{T}^{\mu 4} = 0$, $\underset{4}{T}^{\mu\nu}$ and $\underset{4}{T}^{44}$ are time-dependent; (3.2.35)

in I_2,

$\underset{4}{T}^{\mu 4} = 0$, $\underset{4}{T}^{\mu\nu}$ and $\underset{4}{T}^{44}$ are independent of time.

(3.2.36)

In accordance with the general method, we then calculate for the metric

$$g_{ij} = \delta_{ij} + \underset{2}{g}_{ij} \qquad (3.2.37)$$

the part of order k^4 in G^{ij}, viz. $\underset{4}{G}^{ij}$, and define

$$\underset{4}{g}_{ij}(\underset{\sim}{x}, it) = 4 \int [\underset{4}{T}^{*ij}(\underset{\sim}{x}', it') + \kappa^{-1} \underset{4}{\dot{G}}^{*ij}(\underset{\sim}{x}', it')] \, d\omega \, , \qquad (3.2.38)$$

$$t' = t - |\underset{\sim}{x} - \underset{\sim}{x}'|$$

For $(\underset{\sim}{x}, it)$ in $I_0 + E_0$, this is a static DFS metric tensor and therefore the integral over the null-cone converges everywhere in this region. At any event in $I_1 + E_1 + I_2 + E_2$, for which t is finite, the above integral also converges. In effect, it is the sum of two integrals – one over an infinite domain (the intersection of the null-cone with E) and another over a finite domain (the rest of the null-cone). The first integral converges because the values of the integrand are those of the static FS case and the second is finite because the domain of integration is finite. Therefore the whole integral has a finite value. The question of how $\underset{4}{g}_{ij}(\underset{\sim}{x}, it)$ behaves as the field event "goes to infinity" in $I_1 + E_1 + I_2 + E_2$ is deferred to a later chapter. It will be seen that one may remove the restriction of having t finite.

From (3.2.38) and (3.2.34) – (3.2.36) we obtain

$$\underset{4}{g}_{ij,4} = 0 , \qquad \underset{4}{g}_{\mu 4} = 0 \quad \text{in } I_0 + E_0 , \qquad (3.2.39)$$

$$\underset{4}{g}_{ij} \quad \text{is time-dependent in } I_1 + E_1 + I_2 + E_2 . \qquad (3.2.40)$$

For the second equation of (3.2.39) we use the fact that

$$\underset{4}{G}^{\mu 4} = 0 \quad \text{in } I_0 + E_0 , \qquad (3.2.41)$$

as may be verified by straightforward calculation.

We now have a metric

$$g_{ij} = \delta_{ij} + \underset{2}{g}_{ij} + \underset{4}{g}_{ij} \qquad (3.2.42)$$

such that when we calculate from it the energy tensor by

$$T^{ij} = -\kappa^{-1} G^{ij} \qquad (3.2.43)$$

we obtain

$$T^{ij} = \underset{2}{T}^{ij} + \underset{4}{T}^{ij} + O(k^6) \qquad (3.2.44)$$

and so, in E, there is a residual energy tensor of order k^6.

C. Induction from $O(k^{2N-2})$ to $O(k^{2N})$.

Suppose we have found

$$\underset{2}{g}_{ij}, \underset{4}{g}_{ij}, \ldots, \underset{2N-2}{g}_{ij} ; \quad \underset{2}{T}^{ij}, \underset{4}{T}^{ij}, \ldots, \underset{2N-2}{T}^{ij} , \qquad (3.2.45)$$

such that for the metric

$$g_{ij} = \delta_{ij} + \underset{2}{g}_{ij} + \cdots + \underset{2N-2}{g}_{ij} \qquad (3.2.46)$$

we have

$$\underset{2P}{G}^{ij} = -\kappa \underset{2P}{T}^{ij} \quad \text{in } I + E \quad (P = 1, \ldots, N-1), \qquad (3.2.47)$$

$$\underset{2P}{T}^{ij} = 0 \quad \text{in } E \quad (P = 1, \ldots, N-1) , \qquad (3.2.48)$$

and

$$\underset{2P}{T}^{ij} \underset{M}{=} 0 \quad \text{on } B \quad (P = 1, \ldots, N-1) . \qquad (3.2.49)$$

We then have a universe which has in E a residual energy tensor of order k^{2N}. To reduce the residual energy tensor to order k^{2N+2} we seek $\underset{2N}{g}_{ij}$ and $\underset{2N}{T}^{ij}$ such that for

$$g_{ij} = \delta_{ij} + \underset{2}{g}_{ij} + \dots + \underset{2N}{g}_{ij} \qquad (3.2.50)$$

we shall have

$$\underset{2N}{G}^{ij} = -\kappa \underset{2N}{T}^{ij} \quad \text{in } I + E , \qquad (3.2.51)$$

$$\underset{2N}{T}^{ij} = 0 \quad \text{in } E, \qquad \underset{2N}{T}^{ij}\, n_j = 0 \quad \text{on } S . \qquad (3.2.52)$$

As before, we must find $\underset{2N}{T}^{ij}$ to satisfy

$$\underset{2N}{T}^{ij}{}_{,j} = -\underset{2N}{K}^{i} \quad \text{in } I \qquad (3.2.53)$$

where

$$\underset{2N}{K}^{i} = (\Gamma^{i}_{jk} T^{jk} + \Gamma^{j}_{jk} T^{ik})_{2N} . \qquad (3.2.54)$$

Equations (3.2.53) may be split in the usual way into

$$\underset{2N}{T}^{\mu\nu}{}_{,\nu} + \underset{2N}{T}^{\mu 4}{}_{,4} = -\underset{2N}{K}^{\mu} , \qquad (3.2.55)$$

$$\underset{2N}{T}^{4\nu}{}_{,\nu} + \underset{2N}{T}^{44}{}_{,4} = -\underset{2N}{K}^{4} . \qquad (3.2.56)$$

To satisfy these equations, we take $\underset{2N}{T}^{\mu 4} = 0$. Equation (3.2.55) then becomes

$$\underset{2N}{T}^{\mu\nu}{}_{,\nu} = -\underset{2N}{K}^{\mu} . \qquad (3.2.57)$$

Because of the reflectional symmetry we again have

$$\int_{\Sigma} \underset{2N}{K^{\mu}} \xi_{\mu} \, d_3 x = 0 \qquad (3.2.58)$$

for all t, where ξ_{μ} is a Killing 3-vector in the background Euclidean 3-space. Furthermore, by (3.2.49),

$$\underset{2N}{K^{\mu}} \underset{W}{=} 0 \quad \text{on } B . \qquad (3.2.59)$$

Therefore, by the theorem of McCrea and Synge already mentioned, there exist solutions of (3.2.57) such that

$$\underset{2N}{T^{\mu\nu}} \underset{W}{=} 0 \quad \text{on } B . \qquad (3.2.60)$$

From (3.2.56) and from the fact that $\underset{2N}{K^4} = 0$ in I_o, we can satisfy (3.2.56) by taking

$$\underset{2N}{T^{44}} = -\underset{2N}{\rho}(\underset{\sim}{x}) - i \int_o^t \underset{2N}{K^4} \, dt \quad \text{in } I , \qquad (3.2.61)$$

where $-\underset{2N}{\rho}(\underset{\sim}{x})$ is the arbitrary time-independent value of $\underset{2N}{T^{44}}$ in I_o. Since t is finite in I_2, so is $\underset{2N}{T^{44}}$ and we can always adjust the function $\underset{2N}{\rho}(\underset{\sim}{x})$ to ensure that

$$\underset{2N}{T^{44}} < 0 .$$

We then define for any event in $I + B$

$$\underset{2N}{G_{ij}}(\underset{\sim}{x}, it) = 4 \int (\underset{2N}{T^{*ij}}(\underset{\sim}{x}', it') + \kappa^{-1} \underset{2N}{\hat{g}^{*ij}}(\underset{\sim}{x}', it')) \, dw , \qquad (3.2.62)$$

$$t' = t - |\underset{\sim}{x} - \underset{\sim}{x}'| .$$

For $(\underset{\sim}{x}, it)$ in $I_o + B_o$, the integral (3.2.62) is typical of the kind which occurs in the static PS case and therefore converges. At an event in $I_1 + E_1 + I_2 + E_2$, for which t is finite, it is the sum of two integrals (cf. Fig.2) - one over an infinite domain (the intersection of the null-cone with E_2) and another over a finite domain (the rest of the null-cone). The first integral converges because the values of the integrand are those of the static FS case and the second integral is finite because the domain of integration is finite. Hence the whole integral has a finite value. The question of how $\underset{2N}{g}_{ij}(\underset{\sim}{x}, it)$ behaves as one "goes to infinity" (in a sense which will be defined) is deferred to Chapter V. It will be shown there that one may remove the restriction of having t finite. One may then show in the usual manner that

$$\underset{2N}{G}^{ij} = -\kappa \underset{2N}{T}^{ij} \quad \text{in } I + E . \qquad (3.2.63)$$

We have thus established an inductive process in $I + E$ which can be carried out to any order and which exhibits the interior and exterior gravitational field of a single radiating body to any desired degree of accuracy. The residual energy tensor outside the body at the Nth stage of approximation is of order k^{2N+2} .

CHAPTER IV.

The radiation field in the first approximation.

4.1 The asymptotic field in first approximation.

Having constructed a model universe representing a radiation field due to a finite extended source, we now proceed to investigate some of the properties of such a field. To do this, we shall consider the principal part of the field (i.e. the first approximation) at distances from the source large compared to the dimensions of the latter and investigate what is observed by a distant observer equipped with some kind of apparatus to measure the components of the Riemann tensor. As was first suggested by Pirani[32], the key to such an investigation is contained in the invariant classification of gravitational fields due to Petrov[11]. Before tackling the main problem it would be as well to present the salient features of the Petrov classification.

A. The Petrov classification.

The brief exposition given here follows that of Synge[33]. Besides the usual convention for small Latin and small Greek letters, in this section we use capital Latin letters having the range (1,2,3,4,5,6).

Let us take a particular event in space-time and let us suppose that we have transformed the coordinates so that at this event the metric has the values

$$g_{ab} = \delta_{ab} . \qquad (4.1.1)$$

The components of the Riemann tensor in this coordinate system will be denoted by R_{abcd}. We suppose, further, that the event is in empty space-time, that is, outside any material bodies and, therefore, that

$$R_{ab} = 0 . \qquad (4.1.2)$$

What we have to say concerning the Riemann tensor in vacuo applies equally to the Weyl conformal tensor in the general case.

Because of its symmetries, the Riemann tensor may be regarded as a mapping of the 6-dimensional vector space of bivectors into itself and so we are led to consider the problem of finding the eigenvectors and eigenvalues of a given mapping, i.e. of finding λ and F_{ab} such that

$$R_{abcd} F_{cd} = \lambda F_{ab} . \qquad (4.1.3)$$

If we correlate number pairs in the range 1,2,3,4 to single numbers in the range 1,2,3,4,5,6 according to the scheme

$$(23) \leftrightarrow 1,\ (31) \leftrightarrow 2,\ (12) \leftrightarrow 3,\ (14) \leftrightarrow 4,\ (24) \leftrightarrow 5,\ (34) \leftrightarrow 6. \qquad (4.1.4)$$

we may exhibit the whole set of components of R_{abcd} (except for reversals of sign) in the form of a 6x6 matrix R_{AB} and the components of F_{ab} in the form of a column 6-vector F_A, so that (4.1.3) becomes

$$R_{AB} F_B = \lambda F_A . \qquad (4.1.5)$$

Furthermore, given (4.1.2), it may be shown that R_{AB} may be written in the matrix form

$$R_{AB} = \begin{pmatrix} M & N \\ N & M \end{pmatrix} , \qquad (4.1.6)$$

where M is a real, symmetric, trace-free 3x3 matrix and N is a symmetric, trace-free 3x3 matrix, all of the components of which are pure imaginaries. It is easy to check that there are thus 10 independent components, as required for the Riemann tensor in vacuo. Correspondingly, we may write F_A in the form

$$\begin{pmatrix} F_A \end{pmatrix} = \begin{pmatrix} G \\ H \end{pmatrix} , \qquad (4.1.7)$$

where G is a real 3-vector and H is a pure imaginary 3-vector. In this way, (4.1.5) becomes

$$\begin{pmatrix} M & N \\ N & M \end{pmatrix} \begin{pmatrix} G \\ H \end{pmatrix} = \lambda \begin{pmatrix} G \\ H \end{pmatrix} . \qquad (4.1.8)$$

Then, defining

$$K = M + N, \qquad J = G + H , \qquad (4.1.9)$$

so that K is a symmetric complex 3x3 matrix with zero trace and $\mathcal{J}$ is a complex column 3-vector, we find that (4.1.8) is equivalent to

$$K\mathcal{J} = \lambda\mathcal{J}. \qquad (4.1.10)$$

The problem is therefore reduced to that of an eigenvalue problem in a 3-dimensional complex Euclidean space. One may then proceed to find the eigenvalues and eigenvectors and to write the matrix K in canonical form, that is, in the form which it assumes in a basis defined by the eigenvectors. The details may be found in the work of Synge already mentioned. We shall merely give the main results here.

We may exhibit the complete classification as follows:

Class 1:

eigenvalues: $\lambda_1, \lambda_2, \lambda_3$ distinct $(\lambda_1 + \lambda_2 + \lambda_3 = 0)$;

eigenvectors: 3 non-null, mutually orthogonal, with lines uniquely determined;

canonical form of matrix:

$$K = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}; \qquad (4.1.11)$$

(The terms "null" and "orthogonal" are used in the sense of the complex Euclidean space).

Class 2a:

Eigenvalues: $\lambda_1 \neq \lambda_2 = \lambda_3 \quad (\lambda_1 + 2\lambda_2 = 0)$;

eigenvectors: $\mathcal{J}_1$ non-null with line uniquely determined, and all vectors in plane orthogonal to $\mathcal{J}_1$;

canonical form of matrix:

$$K = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix} ; \qquad (4.1.12)$$

Class 2b:

eigenvalues: $\lambda_1 \neq \lambda_2 = \lambda_3 \quad (\lambda_1 + 2\lambda_2 = 0)$;

eigenvectors: ℓ_1 non-null with line uniquely determined;

ℓ_2 null with line uniquely determined, orthogonal to ℓ_1 ;

canonical form of matrix:

$$K = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2+1 & 1 \\ 0 & 1 & \lambda_2-1 \end{pmatrix} ; \qquad (4.1.13)$$

Class 3a:

eigenvalues: $\lambda_1 = \lambda_2 = \lambda_3 = 0$;

eigenvectors: one null vector N with line uniquely determined, and all vectors orthogonal to N;

canonical form of matrix:

$$K = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix} ; \qquad (4.1.14)$$

Class 3b:

eigenvalues: $\lambda_1 = \lambda_2 = \lambda_3 = 0$;

eigenvectors: one null vector with line uniquely determined;

canonical form of matrix:

$$K = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & i \\ 0 & i & 0 \end{pmatrix} ; \qquad (4.1.15)$$

Class 3c $(K = 0)$:

eigenvalues: zero ;

eigenvectors: all vectors .

Other ways of characterising the Petrov types have been developed by Debever[34] and Penrose[35] based on the multiplicity of distinct principal null directions of the Riemann tensor at an event. The relation between their characterisation and that given above is resumed in Table 1 .

B. Petrov type of the first order part of the Riemann tensor at large distances.

In this section, we consider the field given by the metric tensor

$$g_{ij} = \delta_{ij} + \underset{2}{g}_{ij} \qquad (4.1.16)$$

where, by (3.2.16),

$$\underset{2}{g}_{ij}(\underset{\sim}{x}, it) = 4 \int \frac{\underset{2}{T}^{ij}(\underset{\sim}{x}', it')}{|\underset{\sim}{x} - \underset{\sim}{x}'|} d_3x' , \qquad t' = t - |\underset{\sim}{x} - \underset{\sim}{x}'| . \qquad (4.1.17)$$

<u>Table 1.</u> The first column gives the several classes as defined in the text. The last three columns are taken from Pirani (Brandeis lectures, 1964). The second column exhibits the degree of multiplicity of the principal null directions, e.g. 1111 means that all four principal null directions are distinct, 211 that two of them coincide, etc. In the fourth k_a is a principal null direction – in the second and subsequent rows it is the multiple principal null direction. Type D is also referred to as degenerate type I, and type N as degenerate type II. Square brackets in the fourth column denote antisymmetrization.

Class	Partition of principal null direction	Symbol for Petrov type	Equation satisfied by R_{abcd}
1	[1111]	I	$k_{[a} R_{b]cd[e} k_{f]} k^c k^d = 0$
2b	[211]	II	$R_{bcd[e} k_{f]} k^c k^d = 0$
2a	[22]	D (or I_d)	$R_{bcd[e} k_{f]} k^c k^d = 0$
3b	[31]	III	$R_{bcd[e} k_{f]} k^d = 0$
3a	[4]	N (or II_d)	$R_{bcde} k^e = 0$
3c	–	0	$R_{bcde} = 0$

In other words, we are dealing with the linearized field which constitutes the principal part of the total field. Sachs[10] has treated of solutions of the linearized field equations which have a singularity along a timelike world-line and the asymptotic behaviour of the Riemann tensor of such a field. We have already indicated in Chapter I why Sachs' treatment must be considered as unsatisfactory. The advantage of the present treatment is that instead of a line singularity we consider a physically realistic source i.e. one which is extended, with an internal structure given by the energy tensor. Secondly, the components of the Riemann tensor are obtained explicitly in terms of the energy tensor of the source.

The part of $O(k^2)$ of the Riemann tensor derived from the metric (4.1.16) is

$$\underset{2}{R}_{ijkl} = \tfrac{1}{2}\left(\underset{2}{g}_{il,jk} + \underset{2}{g}_{jk,il} - \underset{2}{g}_{ik,jl} - \underset{2}{g}_{jl,ik}\right) . \qquad (4.1.18)$$

In order to determine how $\underset{2}{R}_{ijkl}$ behaves asymptotically, we expand the denominator in the integrand of (4.1.17) in the form

$$|\mathbf{x} - \mathbf{x}'|^{-1} = |\mathbf{x}|^{-1} + O(|\mathbf{x}|^{-2}) . \qquad (4.1.19)$$

Hence, we may write

$$\underset{2}{g}_{ij}(\mathbf{x},it) = 4|\mathbf{x}|^{-1} \int \underset{2}{T}^{*ij}(\mathbf{x}',it')\, d_3x' + O(|\mathbf{x}|^{-2}) . \qquad (4.1.20)$$

The integral on the right-hand side will be a function of $\underset{\sim}{x}$ and t. From (3.2.15) and (3.2.17), it is clear that at this stage we must distinguish between the region E_1, where $\underset{2}{g}_{ij}$ is time-dependent, and the regions E_0 and E_2 where $\underset{2}{g}_{ij}$ is static. Let us consider first the region E_1.

Since $\underset{2}{T}^{ij}$ vanishes smoothly on B we may certainly differentiate twice with respect to x_μ and t under the integral sign. From

$$t' = t - |\underset{\sim}{x} - \underset{\sim}{x}'| \qquad (4.1.21)$$

we obtain

$$\frac{\partial t'}{\partial x_\mu} = -\frac{x_\mu - x'_\mu}{|\underset{\sim}{x} - \underset{\sim}{x}'|} = -\frac{x_\mu}{|\underset{\sim}{x}|} + O(|\underset{\sim}{x}|^{-1}) . \qquad (4.1.22)$$

Hence, for any function $f(\underset{\sim}{x}', it')$, we have

$$\frac{\partial f}{\partial x_\mu} = -i\frac{x_\mu}{|\underset{\sim}{x}|} \cdot \frac{\partial f}{\partial x'_4} + O(|\underset{\sim}{x}|^{-1}) \qquad (4.1.23)$$

and

$$\frac{\partial f}{\partial x_4} = \frac{\partial f}{\partial x'_4} . \qquad (4.1.24)$$

We thus obtain from (4.1.20)

$$\underset{2}{g}_{ij,\mu\nu} = -4\, x_\mu x_\nu |\underline{x}|^{-3} \int \underset{2}{T}^{*ij}{}_{,44}\, d_3x' + O(|\underline{x}|^{-2}),$$

$$\underset{2}{g}_{ij,\mu 4} = -4\, i\, x_\mu\, |\underline{x}|^{-2} \int \underset{2}{T}^{*ij}{}_{,44}\, d_3x' + O(|\underline{x}|^{-2}),$$

$$\underset{2}{g}_{ij,44} = 4\, |\underline{x}|^{-1} \int \underset{2}{T}^{*ij}{}_{,44}\, d_3x' + O(|\underline{x}|^{-2}) . \tag{4.1.25}$$

For the sake of brevity we have omitted the independent variables. In order to simplify matters, the Riemann tensor will be calculated at a point $(x_1, 0, 0, it)$. In so doing there is no loss of generality as the spatial axes may be rotated so that, for any event considered, $x_2 = x_3 = 0$. Under such a rotation all the formulae used up to now remain unchanged.

Let us define

$$t^{ij} = 2 \int \underset{2}{T}^{*ij}{}_{,44}\, d_3x' , \tag{4.1.26}$$

the integral being taken over the intersection with Σ of the null-cone from the (fixed) distant point (x, it). Then, denoting the $O(|\underline{x}|^{-1})$ term of the Riemann tensor by $\overset{-1}{\underset{2}{R}}_{ijkl}$ and writing this as a 6x6 matrix according to the scheme (4.1.4), we obtain by straightforward calculation

$$\overset{-1}{(\underset{2}{R}_{1jkl})} = |\underset{\sim}{x}|^{-1}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & t^{22} & -t^{23} & it^{12}-t^{24} & it^{23} & it^{22} \\ 0 & -t^{23} & t^{22} & -it^{12}-t^{24} & -it^{22} & -it^{23} \\ 0 & -it^{12}-t^{24} & -it^{12}-t^{34} & -2it^{14}-t^{11}+t^{44} & -it^{24}-t^{13} & -it^{34}-t^{12} \\ 0 & it^{23} & -it^{22} & -it^{24}-t^{12} & -t^{22} & -t^{23} \\ 0 & it^{22} & -it^{23} & -it^{34}-t^{13} & -t^{23} & -t^{22} \end{pmatrix}. \qquad (4.1.27)$$

This is further simplified by the fact that (cf. 2.2.30)

$$\overset{*}{\underset{2}{g}}_{1j,j4} = 0 . \qquad (4.1.28)$$

Substituting for $\overset{*}{\underset{2}{g}}_{1j}$ from (4.1.20), using (4.1.25) and equating coefficients of $|\underset{\sim}{x}|^{-1}$ with zero, the four equations of (4.1.28) become

$$\begin{aligned} t^{22} + t^{33} &= 0 , \\ -t^{11} - 2it^{14} + t^{44} &= 0 , \\ -it^{12} + t^{24} &= 0 , \\ -it^{13} + t^{34} &= 0 . \end{aligned} \qquad (4.1.29)$$

Hence, $\overset{-1}{\underset{2}{R}}_{1jkl}$ may be written in the form (4.1.6) with

$$M = |\underset{\sim}{x}|^{-1}\begin{pmatrix} 0 & 0 & 0 \\ 0 & t^{22} & -t^{23} \\ 0 & -t^{23} & -t^{22} \end{pmatrix} , \qquad (4.1.30)$$

$$N = |\underset{\sim}{x}|^{-1}\begin{pmatrix} 0 & 0 & 0 \\ 0 & it^{23} & it^{22} \\ 0 & it^{22} & -it^{23} \end{pmatrix} . \qquad (4.1.31)$$

Hence, by (4.1.9),

$$R = b\,|\underset{\sim}{x}|^{-1}\begin{pmatrix}0 & 0 & 0\\ 0 & 1 & i\\ 0 & i & -1\end{pmatrix}, \quad b = t^{22} + it^{23} . \quad (4.1.32)$$

Comparing (4.1.32) with (4.1.14), we recognize that $\overset{-1}{\underset{2}{R}}{}_{i\beta kl}$ is of class 3a or Petrov degenerate type II. Hence the result of Sachs is also true for an extended source, at least with regard to the $|\underset{\sim}{x}|^{-1}$ term. The complete Peeling Theorem of Sachs for the linearized field, which deals also with the succeeding inverse powers of $|\underset{\sim}{x}|$, will be considered in a later section. In the general context of the present work, the physical significance of the succeeding powers is doubtful because when we go to the next order of approximation there is a $|\underset{\sim}{x}|^{-1}$ term of order k^4 and, no matter how small k is, this term will dominate the $|\underset{\sim}{x}|^{-2}$ term of order k^2 for sufficiently large $|\underset{\sim}{x}|$.

By using the tensor approach to the Petrov classification (cf. col.4 of Table 1), as we shall do when considering the Peeling theorem, we might have obtained this result in a more direct way. However, the matrix method as used above yields an explicit expression for $\overset{-1}{\underset{2}{R}}{}_{i\beta kl}$ which is particularly simple and will enable us to bring out further interesting features of the asymptotic field and the way in which this is related to the source. Before considering these in detail, we shall deal with the Petrov classification in regions R_0 and R_2.

To fix our ideas, let us take the $O(\kappa^2)$ part of the Riemann tensor at large distances from the source in R_0. Since $\underset{2}{T}^{ij}$ is independent of time in I_0, (4.1.20) will be slightly modified to become

$$\underset{2}{g}_{ij}(\underset{\sim}{x}) = 4\,|\underset{\sim}{x}|^{-1} \int \underset{2}{T}^{*ij}(\underset{\sim}{x}')\, d_3x' + O(|\underset{\sim}{x}|^{-2}). \qquad (4.1.33)$$

The integral in this case is an ordinary volume integral in 3-space instead of being over the null-cone as in the time-dependent case. Differentiating with respect to a spatial coordinate of any term involving an inverse of $|\underset{\sim}{x}|$ now increases the order of the inverse power by unity. Hence

$$\underset{2}{g}_{ij,\mu\nu} = 4(3x_\mu x_\nu - \delta_{\mu\nu}|\underset{\sim}{x}|^2)|\underset{\sim}{x}|^{-5}\int \underset{2}{T}^{*ij}(\underset{\sim}{x}')d_3x' + O(|\underset{\sim}{x}|^{-4}). \qquad (4.1.34)$$

The dominant part of the Riemann tensor for large $|\underset{\sim}{x}|$ is therefore of order $|\underset{\sim}{x}|^{-3}$ and will be denoted by $\overset{-3}{\underset{2}{R}}_{ijkl}$. Calculating the components of this part at a point $(x_1, 0, 0, it)$ as before, and using the values $\underset{2}{T}^{ij}$ as given by the first of (3.2.15), we obtain

$$E = m\,|\underset{\sim}{x}|^{-3}\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad H = 0, \qquad (4.1.35)$$

where

$$m = \int_{\Sigma} \underset{2}{\rho}(\underset{\sim}{x}')\, d_3x' \, . \qquad (4.1.36)$$

Hence, by (4.1.12), $\overset{-3}{\underset{2}{R}}{}_{ijkl}$ is of class Ia or Petrov type D in E_0. This is as one would expect since, to this order of approximation, the field at large distances from the source is a Schwarzschild field.

In E_2, $\overset{-3}{\underset{2}{R}}{}_{ijkl}$ is given by the same expression as (4.1.35) except that instead of m we shall have

$$m' = \int \underset{2}{\sigma}(\underset{\sim}{x}')\, d_3x' \, , \qquad (4.1.37)$$

where, by (3.2.14),

$$\underset{2}{\sigma}(\underset{\sim}{x}') = \underset{2}{\rho}(\underset{\sim}{x}') + q\, \underset{2}{A}{}^{\mu\nu}{}_{,\mu\nu}(\underset{\sim}{x}') \, , \qquad (4.1.38)$$

$$q = \int_0^{\tau} d\tau \int_0^{\eta} F(\xi)\, d\xi \quad \text{(a constant).} \qquad (4.1.39)$$

Substituting from (4.1.34) in (4.1.33), applying Gauss's theorem and using the fact that $\underset{2}{A}{}^{\mu\nu}$ vanishes strongly on B, we obtain

$$m' = m \, . \qquad (4.1.40)$$

Hence the dominant term in the $O(k^2)$ part of the Riemann tensor is the same in E_0 and E_2. Equation (4.1.36) states that, <u>to this order of approximation</u>, there is no loss of mass due to the gravitational radiation in the model under consideration.

4.2 Transfer of energy.

In order to gain a more intuitive, physical interpretation of the results of the preceding section we shall use the equation of geodesic deviation (cf. Synge and Schild[36])

$$\frac{\delta^2 \eta^i}{\delta u^2} + R^i{}_{jkl}\, U^j \eta^k U^l = 0 , \qquad (4.2.1)$$

which describes the relative acceleration of two neighbouring test-particles. In (4.2.1) u is a special parameter along the geodesic world-line C of one of the particles, $U^i = dx^i/du$ is a vector tangent to C, η^i is the orthogonal connecting vector between C and the geodesic world-line C' of the other particle.

Let $\lambda^i_{(a)}$ be an orthonormal tetrad propagated parallelly along C. Suppose that u is the proper time on C, so that $U^i = dx^i/du$ is a unit vector, and let $U^i = \lambda^i_{(4)}$. Multiplying (4.2.1) by $\lambda^i_{(a)}$, we obtain

$$\frac{d^2 \eta_{(a)}}{du^2} + \lambda^i_{(a)}\, R_{ijkm}\, \lambda^j_{(4)}\, \eta^k \lambda^m_{(4)} = 0 . \qquad (4.2.2)$$

Since

$$\eta^k = \lambda^k_{(b)}\, \eta^{(b)} , \qquad (4.2.3)$$

we derive from (4.1.38) that (cf. Synge[37])

$$\frac{d^2 \eta_{(\alpha)}}{du^2} + R_{(\alpha 4 \beta 4)}\, \eta^{(\beta)} = 0 . \qquad (4.2.4)$$

This, then, is the equation of geodesic deviation in terms of the components on an orthonormal tetrad, propagated parallelly along one of the (neighbouring) geodesic lines and such that the timelike vector of the tetrad is always tangent to the world-line.

Going back to our model universe, let us consider an observer moving freely at a large distance from the body. His world-line will be a timelike geodesic. Since we are considering only terms of order k^2 and the Riemann tensor is already of that order, we may take the world-line of the observer to be a geodesic in the flat background space for any interval of proper time which is not large like k^{-1}. Any higher approximation to the exact geodesic will merely give terms of order greater than k^2 in the components of the Riemann tensor referred to a tetrad such as that specified above. The geodesic will therefore be simply a straight line. We take this line parallel to the t-axis, and so the proper time along the geodesic will be t. In other words, to the first approximation, the observer is at rest relative to the source. Let us suppose that the observer (who himself may be treated as a test-particle) throws out test-particles in various directions and measures their distances and accelerations relative to himself. He may then calculate $R_{(\alpha 4)(\beta 4)}$ in some frame of three orthogonal spacelike axes. If these happened to be parallel to the coordinate axes which we have used, then for $|\underset{\sim}{x}| >> |\underset{\sim}{x}'|$ he would obtain $\overset{-3}{E}_{2(\alpha 4\beta 4)}$ in E_0 and $\overset{-1}{E}_{2(\alpha 4\beta 4)}$ in E_2 as given by the matrix M in (4.1.30) and (4.1.35) respectively. In general, of course, he will have chosen a different frame, but the eigenvalues and eigenvectors of his matrix M will be the same as those of the M obtained above. Let us therefore consider M as a real symmetric, trace-free 3x3 matrix in its own right. Define

$$c = t^{22}, \qquad d = t^{23} . \tag{4.2.5}$$

Then, (4.1.30) may be written

$$M = |\underset{\sim}{x}|^{-1} M' , \qquad M' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & c & -d \\ 0 & -d & -c \end{pmatrix} . \tag{4.2.6}$$

M' has three distinct eigenvalues, 0, $(c^2+d^2)^{\frac{1}{2}}$, $-(c^2+d^2)^{\frac{1}{2}}$, with eigenvectors $(1,0,0)$, $(0,-c-(c^2+d^2)^{\frac{1}{2}},d)$ $(0,-c+(c^2+d^2)^{\frac{1}{2}},d)$, respectively (the last two must be normalized to obtain unit vectors). Hence, by a rotation in the 23-plane we can get M in the form

$$M = |\underset{\sim}{x}|^{-1} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sqrt{c^2+d^2} & 0 \\ 0 & 0 & -\sqrt{c^2+d^2} \end{pmatrix} \tag{4.2.7}$$

which gives the components of $\overset{-1}{E}_{2(\alpha 4\beta 4)}$ in a frame consisting of the three eigenvectors of M. By following the same procedure our observer would arrive at the same (invariant) result. The same argument is valid for the region E_0.

Now, since the tangent vector to the observer's world-line is given by

$$\lambda^{i}_{(4)} = i\,\delta^{i}_{4} \ , \qquad (4.2.8)$$

the equation (4.1.40) becomes

$$\frac{d^2 v_{(\alpha)}}{dt^2} - \frac{1}{2}(C_{\alpha 4\beta 4})\, v_{(\beta)} = 0 \ . \qquad (4.2.9)$$

In E_2 we shall have, by (4.1.31),

$$\frac{d^2 v_{(1)}}{dt^2} = 2m\,|\underset{\sim}{x}|^{-3} v_{(1)} \ , \quad \frac{d^2 v_{(2)}}{dt^2} = -m\,|\underset{\sim}{x}|^{-3} v_{(2)}$$

$$\frac{d^2 v_{(3)}}{dt^2} = -m\,|\underset{\sim}{x}|^{-3} v_{(3)} \ , \qquad (4.2.10)$$

so that a particle in the 1-direction is accelerated away from the observer and particles in the 2- and 3-directions are accelerated towards the observer. In E_3 the relative accelerations to $O(|\underset{\sim}{x}|^{-1})$ will be given by

$$\frac{d^2 v_{(1)}}{dt^2} = 0 \ , \quad \frac{d^2 v_{(2)}}{dt^2} = \sqrt{a^2 + c^2}\; v_{(2)} \ ,$$

$$\frac{d^2 v_{(3)}}{dt^2} = -\sqrt{a^2 + c^2}\; v_{(3)} \ . \qquad (4.2.11)$$

Here the 1-direction is parallel to the x_1-axis of our original coordinate system but the 2- and 3-directions have been rotated to point in the directions of the remaining two eigenvectors of

M. Hence, when the observer passes from R_0 to R_1, he will notice a new pattern of relative accelerations polarised in the 23-plane. Particles in the 2-direction are accelerated away from the observer and those in the 3-direction towards him. When his world-line enters R_2 the relative accelerations become the same as in R_0. The observer is thus able to "measure" the effect of the changes taking place in the source. From an intuitive Newtonian point of view, one might conclude that the added pattern of accelerations in R indicates the presence of an energy flux in that region. However, as long as the concept of energy is not defined in an invariant manner in general relativity, such a conclusion is of doubtful validity.

4.3 The Peeling Theorem.

As already mentioned, the asymptotic behaviour of the Riemann tensor in the linearized case has been treated in a thorough manner by Sachs[10]. As the source of the gravitational field he takes an unaccelerated point multipole with world-line $z^a(s)$, where s is the proper time, and with world-velocity $v^a = dz^a/ds$. Taking the retarded potential solution of the linearized field equations he found that the (linearized) Riemann tensor could be expanded in the form

$$R_{abcd} = \frac{N_{abcd}}{r} + \frac{III_{abcd}}{r^2} + \frac{II_{abcd}}{r^3} + \frac{I_{abcd}}{r^4} + \frac{I'_{abcd}}{r^5} + O(r^{-6}), \qquad (4.3.1)$$

where

$$r = -e_a(x^a - z^a) , \qquad (4.3.2)$$

x^a being the field event at which the Riemann tensor is calculated. This has been generalised in subsequent work of Sachs[20] and of Newman and Penrose[22], but in their work there is no explicit introduction of a source - it is merely assumed that a physically realistic source exists which is compatible with the asymptotic fields which they consider. However, remaining within the context of the linearised theory, we have already indicated why the use of a point source is to be considered unsatisfactory. We shall therefore derive the result using a physically realistic (i.e. extended) source. In sect.3 we have shown that for an extended source, the $|\underset{\sim}{x}|^{-1}$ term is as in (4.3.1). One could continue to use the method of that section to investigate the succeeding inverse powers of $|\underset{\sim}{x}|$. However, it proves rather cumbersome, so we shall adopt a slightly different procedure.

In the following we shall drop the subscript 2, indicating that we are dealing with the first $(O(\kappa^2))$ approximation, and since the background flat metric is δ_{ab} we may bring all the indices down to the covariant position. Hence, for (4.1.17) we write

$$\psi_{ab}(\underset{\sim}{x}, it) = 4\int T_{ab}(\underset{\sim}{x}', it')dv , \quad t' = t - |\underset{\sim}{x} - \underset{\sim}{x}'| . \qquad (4.3.3)$$

where

$$T^*_{ab} = T_{ab} - \tfrac{1}{2}\,\delta_{ab}\,T_{cc} \qquad (4.3.4)$$

and

$$T_{ab} = T_{ba}, \quad T_{ab,b} = 0, \quad T_{ab} \underset{N}{=} 0 \ \text{on}\ B\ . \qquad (4.3.5)$$

The (linearized) Riemann tensor is

$$L_{abcd} = \tfrac{1}{2}\,B_{abcd:pqrs}\,g_{pq,rs} \qquad (4.3.6)$$

where

$$B_{abcd:pqrs} = (\delta_{ap}\delta_{cq}\delta_{br}\delta_{ds} + \delta_{bp}\delta_{dq}\delta_{ar}\delta_{cs} - \delta_{ap}\delta_{dq}\delta_{br}\delta_{cs} - \delta_{bp}\delta_{cq}\delta_{ar}\delta_{ds})\ , \qquad (4.3.7)$$

and so we have

$$L_{abcd} = 2\,B_{abcd:pqrs}\int T^*_{pq,rs}(x', t')\,dw\ . \qquad (4.3.8)$$

The colon in the B symbol merely serves to separate dummy indices from the others.

From (3.2.4), (3.2.8) and (3.2.15), we see that the energy tensor in first approximation is expressible as a sum of a static and time-dependent part, the (44)-component being the only non-zero component of the static part. In what follows let us consider T_{ab} to denote only the time-dependent part of the energy tensor. Since equation (4.3.5) is satisfied by the static part it must also be satisfied separately by the time-dependent part. A general solution of (4.3.5) is given by (cf. Dorn and Schild[38])

$$T_{ab} = M_{acbd,cd} , \qquad (4.3.9)$$

with

$$M_{acbd} = -M_{cabd} = M_{bdac} ,$$

$$M_{abcd} \underset{N=2}{=} 0 \quad \text{on } S . \qquad (4.3.10)$$

Although (4.3.10) implies that T_{44} changes sign within I, since

$$\int T_{44}\, d_3x = \int M_{4\beta 4\gamma,\beta\gamma}\, d_3x = 0 \qquad (4.3.11)$$

by Gauss's theorem, the existence of the static term in the (44)-component of the total energy tensor ensures that positive density is maintained. By (4.3.4)

$$T^*_{pq} = C_{pquv} T_{uv} , \quad C_{pquv} = \delta_{pu}\delta_{qv} - \tfrac{1}{2}\delta_{pq}\delta_{uv} . \qquad (4.3.12)$$

Thus (4.3.8) becomes

$$L_{abcd}(\underset{\sim}{x}, it) = N_{abcd:uvrs} \int M_{uvef,efrs}(\underset{\sim}{x}', it')\, d\omega , \qquad (4.3.13)$$

where

$$N_{abcd:uvrs} = 2\, D_{abcd:pqrs}\, C_{pquv} . \qquad (4.3.14)$$

Finally we write

$$L_{abcd}(\underset{\sim}{x}, it) = F_{abcd:uvrsefgh} \int M_{uvef,ghrs}(x', it')\, d\omega , \qquad (4.3.15)$$

where

$$F_{abcdi:uvrsefgh} = E_{abcdi:uvrs}\,\delta_{eg}\,\delta_{fh} \,. \qquad (4.3.16)$$

We now choose an origin in space-time in the following manner. We choose the spatial origin in the domain I of the source. This fixes the time-axis. In order to fix the time-origin we draw the null-cone into the past from the field event at which the Riemann tensor is being calculated and take as origin of time the point at which this null-cone cuts the time-axis. This means that at the field event in question

$$t = r = |\underset{\sim}{x}| \,. \qquad (4.3.17)$$

Let $\underset{\sim}{x}'$ be the radius vector of any point in I. Then

$$|\underset{\sim}{x} - \underset{\sim}{x}'| = |\underset{\sim}{x}| - x'_\mu \frac{\partial}{\partial x_\mu}|\underset{\sim}{x}| + \tfrac{1}{2}\, x'_\mu\, x'_\nu \frac{\partial^2}{\partial x_\mu \partial x_\nu}|\underset{\sim}{x}| + \ldots \qquad (4.3.18)$$

$$= r - l_\mu\, x'_\mu + \sigma \,, \qquad (4.3.19)$$

where

$$l_\mu = \frac{x_\mu}{r}\,, \qquad \sigma = \sum_{n=1}^{\infty} \frac{b_n}{r^n}\,, \qquad (4.3.20)$$

the coefficients b_n being functions of l_α and x'_β of the form

$$b_n = G_{\alpha_1 \ldots \alpha_n \beta_1 \ldots \beta_n}\, l_{\alpha_1} \ldots l_{\alpha_n}\, x'_{\beta_1} \ldots x'_{\beta_n} \,, \qquad (4.3.21)$$

the G-term being a combination of Kronecker deltas. The essential point of the argument is that σ is small when r is large.

Now, the integral involved in (4.3.15) is, more explicitly,

$$J = \int M_{uvwf,ghrs}(\underset{\sim}{x}', it') \frac{d_3 x'}{|\underset{\sim}{x}-\underset{\sim}{x}'|}, \quad t' = t - |\underset{\sim}{x}-\underset{\sim}{x}'| . \tag{4.3.22}$$

The second of (4.3.22) gives

$$x_4' = x_4 - i\,|\underset{\sim}{x}-\underset{\sim}{x}'| = it - ir + i\,l_\mu x_\mu' - i\sigma , \tag{4.3.23}$$

by (4.3.19), and so, by (4.3.17)

$$x_4' = i\,l_\mu x_\mu' - i\sigma . \tag{4.3.24}$$

Let us suppose that M_{uvwf} are analytic in their fourth argument so that

$$M_{uvwf,ghrs}(\underset{\sim}{x}', x_4') = \sum_{n=0}^{\infty} \frac{(-i\sigma)^n}{n!} \tilde{M}_{uvwf,ghrs(n)} , \tag{4.3.25}$$

where the last symbol means that we differentiate n times with respect to x_4' and then put $x_4' = i\,l_\mu x_\mu'$. Now, define

$$H_n = \frac{(-i\sigma)^n}{n!} \cdot \frac{1}{|\underset{\sim}{x}-\underset{\sim}{x}'|} . \tag{4.3.26}$$

Then, (4.3.22) gives

$$I = \sum_{n=0}^{\infty} I_n , \quad I_n = \int H_n \tilde{M}_{uvwf,ghrs(n)}\, d_3 x' . \tag{4.3.27}$$

In this and in the following formulae there is no summation over the suffix n unless explicitly indicated.

The integral (4.3.27) may be written in another way. In effect, let us consider any function $F(\underset{\sim}{x}', x_4')$ in I, vanishing strongly to some order on B, with $x_4' = i l_\mu x_\mu'$. We write this as

$$\tilde{F}(\underset{\sim}{x}') = F(\underset{\sim}{x}', i\, l_\mu x_\mu') . \qquad (4.3.28)$$

$D_\mu \tilde{F}$ will mean "substitute $x_4' = i l_\mu x_\mu'$ in F and then differentiate with respect to x_μ'". $\tilde{F}_{,a}$ will mean "differentiate F with respect to x_a' and then substitute $x_4' = i\, l_\mu x_\mu'$". We thus obtain

$$D_\mu \tilde{F} = \tilde{F}_{,\mu} + i\, l_\mu \tilde{F}_{,4} . \qquad (4.3.29)$$

Since $D_4 \tilde{F}$ is clearly zero, we may write (4.3.29) in the form

$$D_r \tilde{F} = \tilde{F}_{,r} + i\, k_r \tilde{F}_{,4} \qquad (4.3.30)$$

where

$$k_\mu = l_\mu , \qquad k_4 = i , \qquad (4.3.31)$$

and so, k_r is a null vector,

$$k_r k_r = 0 . \qquad (4.3.32)$$

Let us then consider the integral

$$J_r = \int_I E_n \tilde{F}_{,(n)r}\, d_3 x' . \qquad (4.3.33)$$

By virtue of (4.3.30) we obtain

$$J_r = \int H_h(D_r \tilde{F}_{,(n)} - i\, k_r \tilde{F}_{,(n+1)})\, d_3x' \qquad (4.3.34)$$

$$= \int \Big[D_r(H_h \tilde{F}_{,(n)}) - (D_r H_h)\tilde{F}_{,(n)} - i k_r \tilde{F}_{,(n+1)} \Big] d_3x' \; . \qquad (4.3.35)$$

Using Gauss's theorem and the fact that F vanishes strongly on B we find that the first three components of the first term in (4.3.35) are zero and the fourth component is clearly zero.

Hence

$$J_r = -\int (D_r + i k_r E)\, \tilde{F}_{,(n)}\, d_3x' \quad , \qquad (4.3.36)$$

where D_r acts only on H_h and E acts only on $\tilde{F}_{,(n)}$, its effect being defined by

$$E\, \tilde{F}_{,(n)} = \tilde{F}_{,(n+1)} \; . \qquad (4.3.37)$$

Applying (4.3.36) successively to (4.3.27) we obtain

$$I_n = \int (D_g + ik_g E)(D_h + ik_h E)(D_r + ik_r E)(D_s + ik_s E) H_h \tilde{M}_{uvwf,(n)}\, d_3x' \; . \qquad (4.3.38)$$

and so, finally, we have

$$R_{abcd} = F_{abcd:uvwxefgh} \sum_{n=0}^{\infty} \int (D_g + ik_g E)(D_h + ik_h E)(D_r + ik_r E)(D_s + ik_s E) H_h \tilde{M}_{uvwf,(n)}\, d_3x' \; . \qquad (4.3.39)$$

With the Riemann tensor in this form it is possible to examine the structure of the coefficients of succeeding inverse powers of r. We shall do this for the first two terms, viz. $O(r^{-1})$ and $O(r^{-2})$.

Denoting the term of $O(r^{-1})$ by $\overset{-1}{L}_{abcd}$, we have, by (4.3.39) and (4.3.26),

$$\overset{-1}{L}_{abcd}$$

$$= F_{abcd:uvrsefgh} \int (D_g + ik_g B)(D_h + ik_h B)(D_r + ik_r B)(D_s + ik_s B) \overset{-1}{H}_o \tilde{N}_{uvef} \, d_3x' , \qquad (4.3.40)$$

where $\overset{-1}{H}_o$ is the r^{-1} term of H_o. By (4.3.26),

$$\overset{-1}{H}_o = \frac{1}{r} . \qquad (4.3.41)$$

Hence,

$$\overset{-1}{L}_{abcd} = \frac{1}{r} F_{abcd:uvrsefgh} \, k_g k_h k_r k_s \int \tilde{N}_{uvef,(4)} \, d_3x' . \qquad (4.3.42)$$

Multiplying (4.3.42) by k_d and using (4.3.16), we obtain

$$\overset{-1}{L}_{abcd} \, k_d$$

$$= \frac{2}{r} k_d k_e k_f \Big[k_b k_d \delta_{au} \delta_{cv} + k_a k_c \delta_{bu} \delta_{dv} - k_b k_c \delta_{au} \delta_{dv} - k_a k_d \delta_{bu} \delta_{cv}$$

$$- \tfrac{1}{2} (k_b k_d \delta_{ac} \delta_{uv} + k_a k_c \delta_{bd} \delta_{uv} - k_b k_c \delta_{ad} \delta_{uv} - k_a k_d \delta_{bc} \delta_{uv}) \Big] \int \tilde{N}_{uvef,(4)} \, d_3x' . \qquad (4.3.43)$$

Inspection of this expression, using (4.3.32) and (4.3.10) yields

$$\overset{-1}{L}_{abcd} \, k_d = 0 \qquad (4.3.44)$$

and hence $\overset{-1}{L}_{abcd}$ is of class 3 a or Petrov type N (cf. Table 1, col. 4).

Denoting the $O(r^{-2})$ term of the Riemann tensor by $\overset{-2}{L}_{abcd}$, we have

$$\overset{-2}{L}_{abcd} = F_{abcd;uvrsefgh}\int (D_g+ik_gX)(D_h+ik_hX)(D_r+ik_rX)(D_s+ik_sX)[\overset{-2}{H}_0 \tilde{M}_{uvef} + \overset{-2}{H}_1 \tilde{M}_{uvef,(2)}]\, d_3x' , \qquad (4.3.45)$$

where

$$\overset{-2}{H}_0 = \frac{l_\mu x'_\mu}{r^2} , \quad \overset{-2}{H}_1 = -\frac{1}{2r^2}(\delta_{\mu\nu} - l_\mu l_\nu)x'_\mu x'_\nu . \qquad (4.3.46)$$

Multiplying (4.3.46) by k_a, we find, after a somewhat lengthy calculation, that $k_a\overset{-2}{L}_{abcd}$ may be expressed in the form

$$k_a \overset{-2}{L}_{abcd} = \frac{1}{r^2} k_b V_{cd} , \qquad (4.3.47)$$

where

$$V_{cd} = W_{cd} - W_{dc} , \qquad (4.3.48)$$

with

$$\begin{aligned} W_{cd} = {} & 2\Big[ik_a k_h k_d \int D_g \overset{-2}{H}_1 \tilde{M}_{agch,(4)} + ik_a k_h k_c \int D_g \overset{-2}{H}_0 \tilde{M}_{agdh,(3)} \\ & + k_a k_c \int D_g D_h \overset{-2}{H}_1 \tilde{M}_{agdh,(3)} + k_a k_h \int D_g D_c \overset{-2}{H}_1 \tilde{M}_{agdh,(3)} \\ & + \frac{1}{r^2} k_a k_h k_c \int \tilde{M}_{a4dh,(3)}\Big] + \Big[ik_g k_h k_d \int D_c \overset{-2}{H}_1 \tilde{M}_{agch,(4)} \qquad (4.3.49) \\ & + ik_g k_h k_a \int D_c \overset{-2}{H}_0 \tilde{M}_{agch,(3)} + k_h k_c \int D_g D_d \overset{-2}{H}_1 \tilde{M}_{agch,(3)} \\ & + k_g k_c \int D_d D_h \overset{-2}{H}_1 \tilde{M}_{agch,(3)} + k_g k_h k_c \delta_{4d} \frac{1}{r^2} \int \tilde{M}_{agch,(3)}\Big] . \end{aligned}$$

For brevity, we have omitted the symbol d_3x' under the integral signs.

Equation (4.1.93) implies that

$$\overset{-2}{L}_{abcd}\, k_g\, k_c - \overset{-2}{L}_{abcg}\, k_d\, k_c = 0 \ . \qquad (4.3.50)$$

Hence $\overset{-2}{L}_{abcd}$ is of class 3b or Petrov type III (cf. Table 1, col. 4) and so we have established that

$$L_{abcd} = \frac{N_{abcd}}{r} + \frac{III_{abcd}}{r^2} + O\left(\frac{1}{r^3}\right) \ . \qquad (4.3.51)$$

For the higher order terms the calculations become rather cumbersome, as they do in the case of the point source considered by Sachs, but, in principle, the same procedure as above can be carried out for these terms. In addition to (4.3.51) there will be another term arising from the static part of the energy tensor. In section 1 of the present chapter it was shown that the coefficients of r^{-1} and r^{-2} are zero in this term and so the first two terms of (4.3.51) remain undisturbed. However, for higher powers of r^{-1}, the static term must be considered and may be treated in the same manner as the time-dependent term. We have already seen that the r^{-3} static term is of type D.

CHAPTER V.

Higher approximations.

5.1 The second approximation.

In the present chapter we take up the question of the behaviour of the metric tensor as the field event goes to infinity in all directions of space-time. The present section will be concerned with the second approximation, i.e. with

$$\underset{4}{g}_{ij}(y, it) = 4\int \left(\underset{4}{T}^{ij} + \kappa^{-1}\, \underset{4}{\hat{G}}^{*ij} \right) d\omega . \qquad (5.1.1)$$

<u>We shall prove that the retarded integral (5.1.1) is bounded as one goes to infinity in all directions of space-time and that it tends to zero (in varying degrees according to the space-time direction) whenever the field event goes to infinity in a manner which entails $|y| \to \infty$.</u>

The proof of this is not necessary when we consider only the region $I_0 + E_0$ since, in this region, null-cones drawn into the past lie entirely within a static DPB domain. For the remainder of space-time there are three main cases to be considered according as the field event goes to infinity (i) in E_1, (ii) in I_2 and (iii) in E_2. We shall explain in each case what is envisaged by the term "goes to infinity".

In what follows we shall take the interior I of fig. 1 to be the history of a sphere. In so doing there is no loss of generality as, if necessary, we can make $T^{ij} = 0$ over part of the sphere. The envelopes S_1 and S_2 then become null-cones with vertices on the world-line of the centre of the sphere. Let us choose this world-line as origin of the space coordinates and the vertex of S_1 as our time origin.

(i) Going to infinity in E_1 .

Let us take an event $P(\underline{y},it)$ in E_1, and consider the value of the integral (5.1.1) at that event, the domain of integration being the null-cone N drawn into the past from P (cf. fig. 2). We shall then see how the integral behaves as P goes to infinity along the null-line which passes through this event and intersects the world-line of the centre of the sphere.

In order to do this, we project the domain of integration onto the 3-space orthogonal to the t-axis. The projection of the intersection of the null-cone N with S_1 will be given by the equation

$$|\underline{x} - \underline{y}| + |\underline{x}| = t , \qquad (5.1.2)$$

where (x_1, x_2, x_3) are the current coordinates, (y_1, y_2, y_3) and t are fixed. Equation (5.1.2) is simply the equation of an ellipsoid of revolution with foci at the origin and at $\underline{y}$.

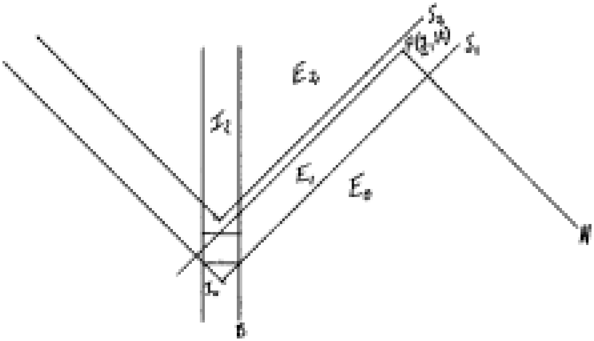

Figure. 2. The field event in R_4.

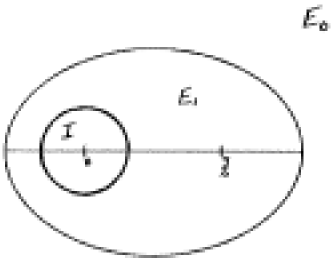

Figure 3. Space projection of figure 2.

Denoting the semi-major axis and eccentricity, in the usual way, by a and e respectively, then (5.1.2) tells us that

$$t = 2a \, , \qquad |\underset{\sim}{y}| = 2ae \, , \tag{5.1.3}$$

and hence,

$$e = |\underset{\sim}{y}| \, / \, t \, . \tag{5.1.4}$$

From now on we shall write simply y instead of $|\underset{\sim}{y}|$.

The projection of $N \cap I$ will be a sphere, where the symbol $\cap$ has the usual meaning of "intersection". Taking the case in which this sphere is wholly contained within the ellipsoid of revolution (the case in which it is not included requires only minor modifications), the 3-dimensional picture is as illustrated in Fig. 3 – the complete picture is got by rotating the figure about the major axis.

Going back, for a moment, to the original space-time picture, we derive from (5.1.1) that

$$|\underset{\sim}{g}_{ij}(\underset{\sim}{y}, it)| \leq \underset{\sim}{Y}_{ij}(\underset{\sim}{y}, it) = 4 \int_N |\underset{\sim}{m}^{ij} + \kappa^{-1} \dot{\underset{\sim}{s}}^{ij}| \, d\omega \, . \tag{5.1.5}$$

The integral on the right-hand side of (5.1.5) may be written

$$\underset{\sim}{Y}_{ij} = \overset{(1)}{\underset{\sim}{Y}_{ij}} + \overset{(2)}{\underset{\sim}{Y}_{ij}} + \overset{(3)}{\underset{\sim}{Y}_{ij}} \, , \tag{5.1.6}$$

where

$$\overset{(1)}{\underset{4}{Y}}{}_{ij} = 4\,\kappa^{-1} \int_{N \cap E_0} |\underset{4}{\dot{g}}{}^{*ij}|\, d\omega\,, \qquad (5.1.7)$$

$$\overset{(2)}{\underset{4}{Y}}{}_{ij} = 4\,\kappa^{-1} \int_{N \cap E_1} |\underset{4}{\dot{g}}{}^{*ij}|\, d\omega\,, \qquad (5.1.8)$$

$$\overset{(3)}{\underset{4}{Y}}{}_{ij} = 4 \int_{N \cap I} |\underset{4}{T}{}^{*ij} + \kappa^{-1} \underset{4}{\dot{g}}{}^{*ij}|\, d\omega\,. \qquad (5.1.9)$$

For large r, $\overset{(3)}{\underset{4}{Y}}{}_{ij}$ is clearly of order r^{-1}. We therefore confine our attention to the other two terms. Let us begin with $\overset{(2)}{\underset{4}{Y}}{}_{ij}$. The domain of integration in the 3-dimensional picture is the region E_1 of Fig. 3. Since we are dealing with retarded integrals, the contributions to the integrals at different points must be taken at different times. However, since we are interested merely in orders of magnitude, we may set upper bounds on the several integrands, which bounds are independent of time, and thus reduce the problem to one of volume integrals in 3-space. We do this in the following way.

In E_1, since $\underset{2}{g}{}_{ij}$ is time-dependent, $\underset{4}{\dot{g}}{}^{*ij}$ is of order r^{-2} for large r. This means that there exists a finite constant M such that $|\underset{4}{\dot{g}}{}^{*ij}|\,r^2 < M$ for all events in E_1, i.e. such that

$$|\underset{4}{\dot{g}}{}^{*ij}| < \frac{M}{r^2} \quad \text{in} \quad E_1\,. \qquad (5.1.10)$$

Hence by (5.1.8) and (5.1.10),

$$\overset{(2)}{\underset{4}{\gamma}}_{ij}(y,t) \;<\; 4\kappa^{-1}\int_{E_1}\frac{M}{x^2}\,\frac{d_3x}{|y-x|} \quad \text{in } E_1 \,. \tag{5.1.11}$$

To integrate the right-hand side of (5.1.11), we take spherical polar coordinates (x, θ, ϕ). Let us call this integral J, then we have

$$J = J_1 + J_2 + J_3 \,, \tag{5.1.12}$$

where

$$J_1 = 4\kappa^{-1} M \int_0^{a(1-e)}\int_0^{\pi}\int_0^{2\pi} \frac{\sin\theta}{|y-x|}\, dx\, d\theta\, d\phi \,, \tag{5.1.13}$$

$$J_2 = 4\kappa^{-1} M \int_{a(1-e)}^{y}\int_0^{\alpha}\int_0^{2\pi} \frac{\sin\theta}{|y-x|}\, dx\, d\theta\, d\phi \,, \tag{5.1.14}$$

$$J_3 = 4\kappa^{-1} M \int_{y}^{a(1+e)}\int_0^{\alpha}\int_0^{2\pi} \frac{\sin\theta}{|y-x|}\, dx\, d\theta\, d\phi \,, \tag{5.1.15}$$

where α is the value of θ at the points where the sphere $r = x$ cuts the ellipsoid. We already know the values of a and e in terms of y and t by (5.1.3) and (5.1.4). From these we obtain

$$\begin{aligned} a(1-e) &= \tfrac{1}{2}(t-y) = \tfrac{1}{2}\eta \ \ \text{(say)}, \\ a(1+e) &= \tfrac{1}{2}(t+y) \,. \end{aligned} \tag{5.1.16}$$

At any event in E_1, $t - y = \eta$ is of the order of magnitude of $\varepsilon + \tau$, where τ is the period over which $\underset{2}{T}^{ij}$ is time-dependent. Hence, if we let y go to infinity keeping η

constant (i.e. along a null line in Σ_1 drawn from the central world-line), it is evident that J_1 is of order y^{-1} for large y.

To integrate J_2, we first of all consider

$$\int_0^{\pi} \frac{\sin\theta\, d\theta}{R} \tag{5.1.17}$$

where

$$R^2 = |\underset{\sim}{y} - \underset{\sim}{x}|^2 = y^2 + x^2 - 2xy\cos\theta\,. \tag{5.1.18}$$

From (5.1.18) we obtain

$$R\, dR = xy \sin\theta\, d\theta \tag{5.1.19}$$

and so, (5.1.17) becomes

$$\int_0^{\pi} \frac{dR}{xy} = \frac{1}{xy}(R)\,\Big|_0^{\pi} \tag{5.1.20}$$

$$= \frac{1}{xy}\,[(t-x) - (y-x)]\,. \tag{5.1.21}$$

Hence,

$$J_2 = \frac{M}{y}\int_{\frac{1}{2}\eta}^{y} \frac{\eta}{x}\,dx = \frac{M\eta}{y}\,(\log y - \log \tfrac{1}{2}\eta) \tag{5.1.22}$$

$$\rightarrow\ M\eta\,\frac{\log y}{y} \quad \text{as } y \rightarrow \infty\,. \tag{5.1.23}$$

The procedure for J_3 is the same except that we have

$$\int_0^{\pi} \frac{dR}{xy} = \frac{1}{xy}[(t - x) - (x - y)] \tag{5.1.24}$$

and therefore,

$$J_3 = \frac{M}{y}\int_y^{\frac{1}{2}(t+y)} \left(\frac{t+y}{x} - 2\right) dx \qquad (5.1.25)$$

$$= \frac{M}{y}\left[(t+y)\log\frac{t+y}{2y} - (t-y)\right] \qquad (5.1.26)$$

$$= \frac{M}{y}\left[(2y+\eta)\log\frac{2y+\eta}{2y} - \eta\right] . \qquad (5.1.27)$$

Letting y tend to infinity and keeping η constant as before, we obtain

$$J_3 \rightarrow \frac{\eta^2 M}{4y^2} \quad \text{as} \quad y \rightarrow \infty . \qquad (5.1.28)$$

Combining these results, we may state that $\underset{4}{Y}^{(2)}_{ij}$ tends to zero like $y^{-1}\log y$ as y tends to infinity along a null-line in R_1.

We must now establish the asymptotic behaviour of $\underset{4}{Y}^{(1)}_{ij}$ given by (5.1.7). Since $\underset{2}{g}_{ij}$ is independent of time in R_0, it follows that $\underset{4}{\dot{D}}^{ij}$ is of order x^{-4} for large x in R_0. By an argument similar to the preceding, we find that $\underset{4}{Y}^{(1)}_{ij}$ is of order y^{-1} for large y (going to infinity in the manner prescribed above).

Collecting all the foregoing results, we have, by (5.1.5), that as y tends to infinity along a null-line in R_1, $|\underset{4}{g}_{ij}(y,t)|$ tends to zero at least as fast as $y^{-1}\log y$.

(ii) Going to infinity in I_2.

The method followed here is similar to that employed above, so it will be sufficient to give merely a brief outline of the argument and state the eventual results.

We consider $|g_{ij}|$ at an event $(\underline{y}, it)$ contained in I_2 and, keeping y fixed, see how it behaves as t tends to infinity (cf. Fig. 4). The 3-space projection is illustrated by Fig. 5. The intersections of N with S_1 and S_2 form two confocal ellipsoids of revolution. The projection of $N \cap B$ is a sphere as before. The foci of the confocal ellipsoids are the centre of the sphere and the point $\underline{y}$, which in the present case is an interior point of the sphere. We note that for large t, the two ellipsoids are very nearly spheres, but, for safety, we shall not use this approximation at this stage.

We denote the semi-major axis and eccentricity of the outer and inner ellipsoids by a_1, e_1 and a_2, e_2, respectively. Let us take the vertex of S_1 to be the origin of time and let $t = h$ be the time coordinate of the vertex of S_2 (the space coordinates of the vertex of S_2 are, of course, the same as those for S_1, viz. $(0,0,0)$). From Fig. 4 we see that h is of the same order of magnitude as $\sigma + \tau$. We then have

$$2\, a_1\, e_1 = y, \qquad 2\, a_1 = t,$$
$$2\, a_2\, e_2 = y, \qquad 2\, a_2 = t - h, \qquad (5.1.29)$$

and therefore,

$$e_1 = \frac{y}{t}, \quad e_2 = \frac{y}{t-h}, \quad a_1 = \frac{t}{2}, \quad a_2 = \frac{t-h}{2}. \qquad (5.1.30)$$

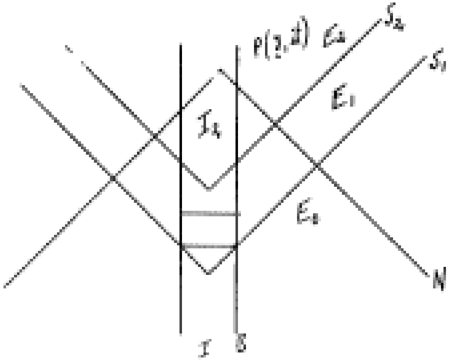

Figure 4. The field event in I_2.

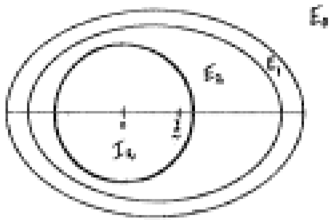

Figure 5. Space projection of figure 4.

Now, in I_2,

$$\left| \underset{4}{g}_{ij} \right| \leq \underset{4}{\gamma}_{ij} = 4 \int_N \left| \underset{4}{T}^{*ij} + \kappa^{-1} \underset{4}{\hat{G}}^{*ij} \right| d\omega \qquad (5.1.31)$$

$$= \underset{4}{\overset{(1)}{\gamma}}_{ij} + \underset{4}{\overset{(2)}{\gamma}}_{ij} + \underset{4}{\overset{(3)}{\gamma}}_{ij} + \underset{4}{\overset{(4)}{\gamma}}_{ij} ,$$

where

$$\underset{4}{\overset{(1)}{\gamma}}_{ij} = 4 \int_{N \cap E_2} \left| \underset{4}{T}^{*ij} + \kappa^{-1} \underset{4}{\hat{G}}^{*ij} \right| d\omega$$

$$\underset{4}{\overset{(2)}{\gamma}}_{ij} = 4\, \kappa^{-1} \int_{N \cap E_3} \left| \underset{4}{\hat{G}}^{*ij} \right| d\omega$$

$$\underset{4}{\overset{(3)}{\gamma}}_{ij} = 4\, \kappa^{-1} \int_{N \cap E_1} \left| \underset{4}{\hat{G}}^{*ij} \right| d\omega \qquad (5.1.32)$$

$$\underset{4}{\overset{(4)}{\gamma}}_{ij} = 4\, \kappa^{-1} \int_{N \cap E_4} \left| \underset{4}{\hat{G}}^{*ij} \right| d\omega .$$

The integrand on the right-hand side of the first of (5.1.32) is time-independent (cf. (3.2.17)) and therefore bounded as t tends to infinity. We may majorize the other three integrals of (5.1.32) by ordinary volume integrals in 3-space in a manner similar to that already used. When we calculate the volume integrals, we obtain expressions involving h, y and t. Keeping h and y constant, we find that for large t,

$${}_4Y^{(2)}_{ij} < \text{finite term independent of } t + \text{term of } O\left(\frac{1}{t^2}\right),$$

$${}_4Y^{(3)}_{ij} < \text{term of } O\left(\frac{1}{t}\right), \qquad (5.4.33)$$

$${}_4Y^{(4)}_{ij} < \text{term of } O\left(\frac{1}{t^2}\right).$$

Collecting these results, we may therefore state that $|{}_4g_{ij}|$ remains finite as we go to infinity in the manner described above.

(iii) Going to infinity in E_2.

The space-time and projected 3-space pictures in this case are as illustrated in Figs. 6 and 7. The latter is similar to the previous case except for the fact that the second focus is outside the sphere I. If the field event is close to the radiative region, the sphere will intersect the inner ellipsoid, but this will entail only minor modifications which do not affect the essential argument.

The process of "going to infinity" in the previous cases could be defined in a relatively straightforward manner, however, in the present case, we must distinguish between the various directions in which one may go to infinity. We begin by deriving an expression in terms of y and t which majorizes $|{}_4g_{ij}|$ at an event (y, it) in E_2. This expression will be valid for <u>any</u> event in E_2, in the sense that it does not assume the large-

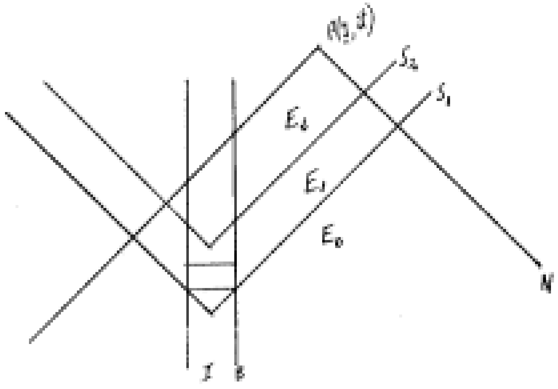

Figure 6. The field event in [illegible].

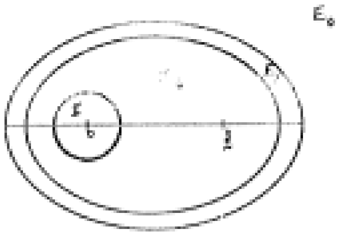

Figure 7. Space projection of figure 6.

ness or smallness of y or t or any quantities involved in it. We may then see how this upper bound behaves as we go to infinity in the following three ways: (a) fix y and allow t to tend to infinity; (b) fix $\eta = t - y$ and allow t (and consequently, y) to tend to infinity; (c) let $y = \alpha(t-h)$, $0 < \beta_1 \leq \alpha \leq \beta_2 < 1$, and allow t (and consequently y) to tend to infinity.

We have, as before,

$$|\underset{4}{g}_{ij}(\underset{\sim}{y},t)| \leq \underset{4}{Y}_{ij}(\underset{\sim}{y},t) = 4\int_{N} |\underset{4}{T}^{*ij} + \varkappa^{-1}\underset{4}{S}^{*ij}|\, d\omega\,, \quad (5.1.34)$$

and

$$\underset{4}{Y}_{ij} = \overset{(1)}{\underset{4}{Y}}_{ij} + \overset{(2)}{\underset{4}{Y}}_{ij} + \overset{(3)}{\underset{4}{Y}}_{ij} + \overset{(4)}{\underset{4}{Y}}_{ij}\,, \quad (5.1.35)$$

where

$$\begin{aligned}
\overset{(1)}{\underset{4}{Y}}_{ij} &= 4\int_{N\cap E_2} |\underset{4}{T}^{*ij} + \varkappa^{-1}\underset{4}{S}^{*ij}|\, d\omega\,, \\
\overset{(2)}{\underset{4}{Y}}_{ij} &= 4\varkappa^{-1}\int_{N\cap E_2} |\underset{4}{S}^{*ij}|\, d\omega\,, \\
\overset{(3)}{\underset{4}{Y}}_{ij} &= 4\varkappa^{-1}\int_{N\cap E_1} |\underset{4}{S}^{*ij}|\, d\omega\,, \\
\overset{(4)}{\underset{4}{Y}}_{ij} &= 4\varkappa^{-1}\int_{N\cap E_0} |\underset{4}{S}^{*ij}|\, d\omega\,.
\end{aligned} \quad (5.1.36)$$

By (3.2.17) and (3.2.36), the integrand in the first of (5.1.36) is independent of time and hence $\overset{(1)}{\underset{4}{Y}}_{ij}$ is time-independent and is of order y^{-1} for large y. Some rather obvious considerations likewise show that $\left(\overset{(2)}{\underset{4}{Y}}_{ij} + \overset{(4)}{\underset{4}{Y}}_{ij}\right)$ may be majorised by an expression which is independent of time and is of order y^{-1} as y tends to infinity. This leaves us with $\overset{(3)}{\underset{4}{Y}}_{ij}$, where the domain of integration contains the radiative region.

As in the previous case, we take the vertex of S_1 to be the origin of space-time and the time coordinate of S_2 to be h. If a_1, e_1 and a_2, e_2 are the semi-major axes and eccentricities of the outer and inner ellipsoids respectively, we have

$$2a_1 = t , \quad 2a_1 e_1 = y , \quad 2a_2 = t - h , \quad 2a_2 e_2 = y . \tag{5.1.37}$$

By the same argument as in case (i) we can show that

$$\overset{(3)}{\underset{4}{Y}}_{ij} \le J = 4\pi^{-1} M \int \frac{d_3 x}{x^2 |y - x|} , \tag{5.1.38}$$

where M is a constant. Using spherical polar coordinates, we denote by α_1, α_2 the values of θ on the intersections of the sphere $r = x$ with the outer (S_1) and inner (S_2) ellipsoids respectively. We then have

$$J = J_1 + J_2 + J_3 \tag{5.1.39}$$

where

$$J_1 = M \int_{\alpha_2}^{\pi} \int_{a_2(1-e_2)}^{a_1(1-e_1)} \frac{\sin\theta \, d\theta \, dx}{|\underline{y} - \underline{x}|} ,$$

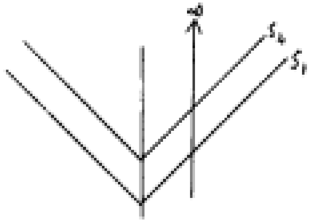

Figure 8. Going to infinity in E_2, case (a).

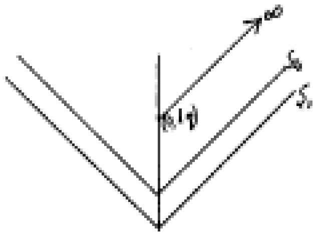

Figure 9. Going to infinity in E_2, case (b).

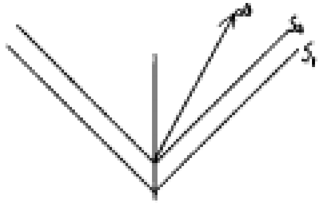

Figure 10. Going to infinity in E_2, case (c)

$$J_2 = M\int_0^{\theta_1}\int_{a_1(1-e_1)}^{a_2(1+e_2)} \frac{\sin\theta \, d\theta \, dx}{|\underline{y}-\underline{x}|} , \qquad (5.1.40)$$

$$J_3 = M\int_0^{\theta_1}\int_{a_2(1+e_2)}^{a_1(1+e_1)} \frac{\sin\theta \, d\theta \, dx}{|\underline{y}-\underline{x}|} .$$

A straightforward integration yields

$$J_1 = \frac{M}{y}\left[2x + (y-t+h)\log x\right]\Big|_{\frac{1}{2}(t-y-h)}^{\frac{1}{2}(t-y)} ,$$

$$J_2 = \frac{Mh}{y}\left[\log x\right]\Big|_{\frac{1}{2}(t-y)}^{\frac{1}{2}(t-h+y)} , \qquad (5.1.41)$$

$$J_3 = \frac{M}{y}\left[(t+y)\log x - 2x\right]\Big|_{\frac{1}{2}(t+y-h)}^{\frac{1}{2}(t+y)} .$$

Note that (5.1.41) are the <u>exact</u> integrals of (5.1.40). By (5.1.39) and (5.1.41) we obtain finally

$$J = \frac{M}{y}\left[(y-t+h)\log\frac{t-y}{t-y-h} + h\log\frac{t-h+y}{t-y} + (t+y)\log\frac{t+y}{t+y-h}\right] , \qquad (5.1.42)$$

and this is the exact integral of (5.1.38).

To see how J behaves as one "goes to infinity", we consider the following three cases:

(a) Fixing y and allowing t to tend to infinity (cf. Fig. 8), we obtain

$$J = \frac{2Mh}{t} . \qquad (5.1.43)$$

(b) Let $t-y = \eta$ and take η to be constant. Then let y (and t) tend to infinity – in other words, we allow the field point to go to infinity along a null-line through the event $(0,i\eta)$ (cf. Fig. 9). We obtain

$$J = Mh \frac{\log y}{y} . \qquad (5.1.44)$$

(c) Letting $y = \alpha(t-h)$, $0 < \beta_1 \leq \alpha \leq \beta_2 < 1$ and allowing t (and y) to tend to infinity (cf. Fig. 10), we obtain

$$J = \frac{Mh}{\alpha t} \log \frac{1+\alpha}{1-\alpha} . \qquad (5.1.45)$$

To sum up all the foregoing, we have found that $\underset{4}{g}_{ij}$ as given by (5.1.1) has a finite value everywhere in space-time and, furthermore, that it tends to zero whenever the field point goes to infinity in a way that involves $y \to \infty$. In the latter case, it tends to zero most slowly, like $y^{-1} \log y$, along null-lines drawn from the world-line of the centre of a sphere constructed in the manner described above.

5.2 Further approximations.

So far we have examined in some detail the model universe defined by an energy tensor

$$T^{ij} = \underset{2}{T}^{ij} + \underset{4}{T}^{ij} \quad \text{in } I \qquad (5.2.1)$$

and a metric tensor

$$g_{ij} = \delta_{ij} + \underset{2}{g}_{ij} + \underset{4}{g}_{ij} \quad \text{in } I + E , \qquad (5.2.2)$$

such that

$$\underset{2}{G}^{ij} = -\kappa \underset{2}{T}^{ij} , \quad \underset{4}{G}^{ij} = -\kappa \underset{4}{T}^{ij} \quad \text{in } I + E ,$$
$$\underset{2}{T}^{ij} = \underset{4}{T}^{ij} = 0 \quad \text{in } E . \qquad (5.2.3)$$

In E there is a residual energy tensor of order k^6 which tends to zero like r^{-4} as r tends to infinity in E_0, where $r = |x|$. In E_1 and E_2, we have seen that we must be careful in defining how one goes to infinity. However, we found that the metric tensor is bounded everywhere in space-time and that when the field point goes to infinity in such a way that r tends to infinity, the metric tensor tends to zero (in the several ways we have described in the previous section). In the latter case, the residual energy tensor also tends to zero, its order at infinity being, at most, that of the square of the metric tensor. We now examine two further stages of the approximation.

A. The third approximation.

By (3.2.53), we must first of all find $\underset{6}{T}^{ij}$ to satisfy

$$\underset{6}{T}^{ij}{}_{,j} = -\underset{6}{E}^{i} . \qquad (5.2.4)$$

By virtue of the reflectional symmetry which we have imposed on the model, we shall again have

$$\int_E \underset{6}{E}^{\mu} \xi_\mu \, d_3x = 0 , \quad \text{for all } t , \qquad (5.2.5)$$

where ξ_μ is a Euclidean Killing 3-vector. From (3.2.54) we obtain

$$\underset{6}{K}^l = \underset{2}{\Gamma}_{jk,l}\,\underset{4}{T}^{jk} + \underset{2}{\Gamma}_{lk,j}\,\underset{4}{T}^{lk} - (\underset{2}{g}_{ln}\,\underset{2}{\Gamma}_{jk,n} - \underset{4}{\Gamma}_{jk,l})\,\underset{2}{T}^{jk} - (\underset{2}{g}_{jn}\,\underset{2}{\Gamma}_{lk,n} + \underset{4}{\Gamma}_{lk,j})\,\underset{2}{T}^{lk} \,. \qquad (5.2.6)$$

Taking the fourth component and separating the indices, we obtain

$$\begin{aligned}
\underset{6}{K}^4 = {} & \underset{2}{\Gamma}_{\beta\gamma,4}\,\underset{4}{T}^{\beta\gamma} + (3\underset{2}{\Gamma}_{4\gamma,4} + \underset{2}{\Gamma}_{\beta\gamma,\beta})\,\underset{4}{T}^{4\gamma} + (2\underset{2}{\Gamma}_{44,4} + \underset{2}{\Gamma}_{\beta4,\beta})\,\underset{4}{T}^{44} \\
& + (\underset{4}{\Gamma}_{\beta\gamma,4} - \underset{2}{g}_{4\delta}\underset{2}{\Gamma}_{\beta\gamma,\delta} - \underset{2}{g}_{44}\underset{2}{\Gamma}_{\beta\gamma,4})\,\underset{2}{T}^{\beta\gamma} \\
& + (3\underset{4}{\Gamma}_{4\gamma,4} + \underset{4}{\Gamma}_{\beta\gamma,\beta} - 3\underset{2}{g}_{4\delta}\underset{2}{\Gamma}_{4\gamma,\delta} - 3\underset{2}{g}_{44}\underset{2}{\Gamma}_{4\gamma,4} - \underset{2}{g}_{\beta\delta}\underset{2}{\Gamma}_{\beta\gamma,\delta} - \underset{2}{g}_{\beta4}\underset{2}{\Gamma}_{\beta\gamma,4})\,\underset{2}{T}^{4\gamma} \\
& + (2\underset{4}{\Gamma}_{44,4} + \underset{4}{\Gamma}_{\beta4,\beta} - 2\underset{2}{g}_{4\delta}\underset{2}{\Gamma}_{44,\delta} - 2\underset{2}{g}_{44}\underset{2}{\Gamma}_{44,4} - \underset{2}{g}_{\beta\delta}\underset{2}{\Gamma}_{\beta4,\delta} - \underset{2}{g}_{\beta4}\underset{2}{\Gamma}_{\beta4,4})\,\underset{2}{T}^{44} .
\end{aligned} \qquad (5.2.7)$$

By (3.2.17) and (3.2.39), we find that

$$\underset{6}{K}^4 = 0 \quad \text{in } I_0 \,. \qquad (5.2.8)$$

In I_1, there is no simplification in the expression for $\underset{6}{K}^4$ but that poses no particular problem. In I_2, we shall have, by (3.2.17) (3.2.15), (3.2.36) and (5.2.7),

$$\underset{6}{K}^4 = (\underset{4}{g}_{44,4} + \tfrac{1}{2}\,\underset{4}{g}_{\beta\beta,4})\,\underset{2}{T}^{44} \,, \qquad (5.2.9)$$

or, since $\underset{2}{T}^{44}$ is independent of t in I_2,

$$\underset{6}{K}^{4} = [(\underset{4}{g}_{44} + \tfrac{1}{2}\,\underset{4}{g}_{\beta\beta})\,\underset{2}{T}^{44}]_{,4} \quad . \tag{5.2.10}$$

We may now return to (5.2.4). We must find $\underset{6}{T}^{ij}$ to satisfy

$$\underset{6}{T}^{\mu\nu}{}_{,\nu} + \underset{6}{T}^{\mu 4}{}_{,4} = -\underset{6}{K}^{\mu} \quad \text{in } I \, , \tag{5.2.11}$$

$$\underset{6}{T}^{4\nu}{}_{,\nu} + \underset{6}{T}^{44}{}_{,4} = -\underset{6}{K}^{4} \quad \text{in } I \, , \tag{5.2.12}$$

with the requisite boundary conditions. In accordance with the general method, we take

$$\underset{6}{T}^{\mu 4} = 0 \quad \text{everywhere,} \tag{5.2.13}$$

which leaves us with the task of finding $\underset{6}{T}^{\mu\nu}$ and $\underset{6}{T}^{44}$ to satisfy

$$\underset{6}{T}^{\mu\nu}{}_{,\nu} = -\underset{6}{K}^{\mu} \quad \text{in } I \, , \tag{5.2.14}$$

$$\underset{6}{T}^{44}{}_{,4} = -\underset{6}{K}^{4} \quad \text{in } I, \tag{5.2.15}$$

and vanishing smoothly to order N on the boundary. From the preceding steps, we already know that $\underset{6}{K}^{i}$ vanishes smoothly to order N on B .

By (5.2.6) and the results of the preceding section, $\underset{6}{K}^{\mu}$ is bounded everywhere in I , that is, for all t . Hence, by our general method, it is possible to find $\underset{6}{T}^{\mu\nu}$ to satisfy (5.2.14) for all t and which vanishes smoothly to order N on the boundary. We satisfy (5.2.15) by taking

$$\underset{6}{T}^{44} = -\underset{6}{\rho}(\underline{x}) - i\int_0^t \underset{6}{E}^4 \, dt \; , \qquad (5.2.16)$$

where $-\underset{6}{\rho}(\underline{x})$ is the arbitrary time-independent value of $\underset{6}{T}^{44}$ in I_0. In I_2 we shall have, by (5.2.10),

$$\underset{6}{T}^{44} = -\underset{6}{\rho}(\underline{x}) - i\int_0^{S_2} \underset{6}{E}^4 \, dt - (\underset{4}{g}_{44} + \tfrac{1}{2}\underset{4}{g}_{\rho\rho})\Big|_{S_2}^{t} \underset{2}{T}^{44} \; . \qquad (5.2.17)$$

Hence, in virtue of the results of the preceding section, it is clear that $\underset{6}{T}^{44}$ is bounded as t tends to infinity in I.

We have thus found $\underset{6}{T}^{ij}$ to satisfy the required conditions and, as in (5.2.6?), we then define

$$\underset{8}{g}_{ij} = 4\int_{E+I} (\underset{6}{T}^{*ij} + \kappa^{-1} \underset{6}{\dot{G}}^{*ij}) \, dv \; . \qquad (5.2.18)$$

In $I_0 + E_0$, this integral is independent of time. Hence, by DFS, it converges and is of order r^{-1} for large r. At any event in $I_1 + E_1 + I_2 + E_2$, for which t is finite, the integral (5.2.18) also converges since, outside a finite domain of the null-cone, the values of the integrand are those of the static DFS case.

From (5.2.18), we can then show, in the usual way, that

$$\underset{6}{G}^{ij} = -\kappa \underset{6}{T}^{ij} \quad \text{in } I + E \; , \qquad (5.2.19)$$

where

$$\underset{6}{T}^{ij} = 0 \quad \text{in } E \; . \qquad (5.2.20)$$

From what has been done up to this, we see that we can go a certain number of steps in our method without making any restriction as to the domain of space-time in which we are certain that all the integrals, which occur, converge. These steps may be represented schematically as follows:

$$\underset{2}{T} \to \underset{2}{g} \to \underset{4}{T} \to \underset{4}{g} \to \underset{6}{T} \quad . \qquad (5.2.21)$$

Up to $\underset{6}{T}^{ij}$ all the quantities in (5.2.21) are bounded everywhere in space-time. It is with $\underset{6}{g}_{ij}$ that difficulties begin to arise. We have seen that this is bounded in $I_0 + E_0$ and at all events of $I_1 + E_1 + I_2 + E_2$ for which t is finite, but we can go further. We know that along a null-line through the point $(0, i\eta)$, where η is finite, $\underset{4}{g}_{ij}$ is of order $x^{-1} \log x$ and $\underset{2}{g}_{ij}$ of order x^{-1}, for large x. Hence, $\underset{6}{\dot{G}}^{ij}$ is of order $x^{-2} \log x$, at the most, for large x along such a line. By going through the work of the preceding section, we find that $\underset{6}{g}_{ij}$ is of order $x^{-1}(\log x)^2$ at large distances along a null-line through the point $(0, i\eta)$, η being fixed and finite. Hence it is bounded and tends to zero as x tends to infinity along the null-line. We can repeat the same argument for the further steps in the approximation and find that, in general, as x tends to infinity along a null-line through $(0, i\eta)$, $\underset{2N}{g}_{ij}$ tends to zero as $x^{-1}(\log x)^{N-1}$. In particular therefore, for an observer in E_1 at any distance from the source, $g_{\underset{2N}{}ij}$ is bounded for finite N.

If one were satisfied with the first three orders of approximation one would have a model with a residual energy tensor of order $(m/a)^4$ outside the body. In the case of the sun $(m/a)^4$ is of order 10^{-24} and so the residual energy density outside would be 10^{-18} that of the sun's interior mean density.

B. The fourth approximation.

Again, taking $\underset{8}{T}^{44} = 0$ everywhere, we must find $\underset{8}{T}^{ij}$, $\underset{8}{T}^{i4}$ satisfying the requisite conditions on the boundary and

$$\underset{8}{T}^{ij}{}_{,j} = -\underset{8}{K}^{i}, \qquad \underset{8}{T}^{i4}{}_{,i} = -\underset{8}{K}^{4} \quad \text{in } I. \tag{5.2.22}$$

For $\underset{8}{K}^{i}$ we obtain

$$\begin{aligned} \underset{8}{K}^{i} = {} & \underset{2}{\Gamma}_{jk,i}\,\underset{6}{T}^{jk} + \underset{2}{\Gamma}_{jk,j}\,\underset{6}{T}^{ik} + \left(\underset{4}{\Gamma}_{jk,i} - \underset{2}{g}_{im}\,\underset{2}{\Gamma}_{jk,m}\right)\underset{4}{T}^{jk} \\ & + \left(\underset{4}{\Gamma}_{jk,j} - \underset{2}{g}_{jm}\,\underset{2}{\Gamma}_{jk,m}\right)\underset{4}{T}^{ik} \\ & + \left(\underset{6}{\Gamma}_{jk,i} - \underset{2}{g}_{im}\,\underset{4}{\Gamma}_{jk,m} - \underset{4}{g}_{im}\,\underset{2}{\Gamma}_{jk,m} + \underset{2}{g}_{ir}\,\underset{2}{g}_{rm}\,\underset{2}{\Gamma}_{jk,m}\right)\underset{2}{T}^{jk} \\ & + \left(\underset{6}{\Gamma}_{jk,j} - \underset{2}{g}_{jm}\,\underset{4}{\Gamma}_{jk,m} - \underset{4}{g}_{jm}\,\underset{2}{\Gamma}_{jk,m} + \underset{2}{g}_{jr}\,\underset{2}{g}_{rm}\,\underset{2}{\Gamma}_{jk,m}\right)\underset{2}{T}^{ik}. \end{aligned} \tag{5.2.23}$$

For the same reasons as before, the first of (5.2.22) admits a solution, for all t in I, vanishing smoothly to order N on the boundary. Straightforward calculation yields

$$\underset{8}{K}^{4} = 0 \quad \text{in } I_0 \tag{5.2.24}$$

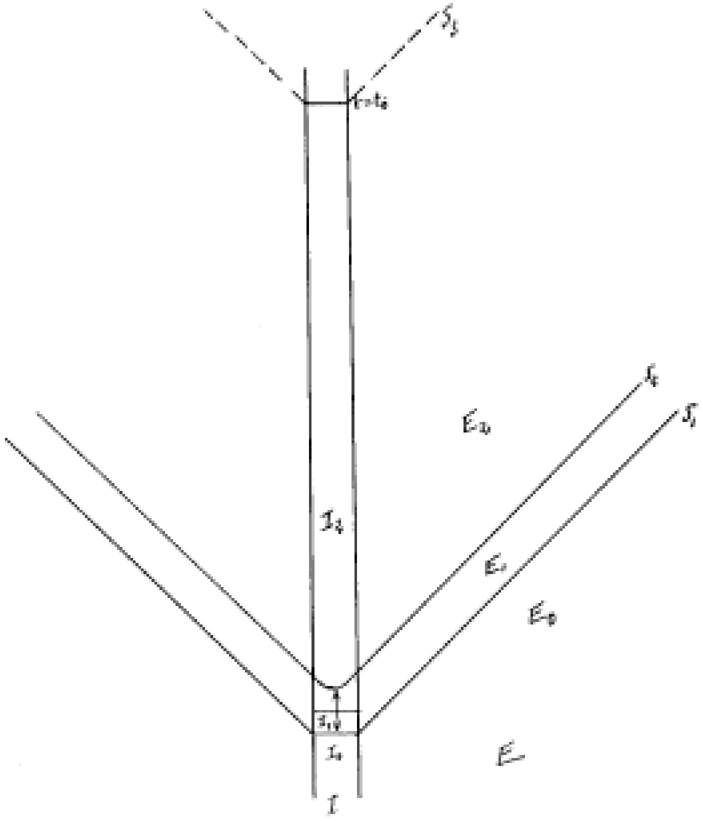

Figure 11. Incomplete non-stationary model universe.

and, in I_2,

$$\underset{8}{K}^4 = [(\underset{4}{g}_{44} + \tfrac{1}{2}\underset{4}{g}_{\nu\nu})\underset{4}{T}^{44}$$

$$- (\underset{2}{g}_{44}\underset{4}{g}_{44} + \tfrac{1}{2}\underset{2}{g}_{\mu\lambda}\underset{4}{g}_{\mu\lambda} + \underset{6}{g}_{44} + \tfrac{1}{2}\underset{6}{g}_{\nu\nu})\underset{2}{T}^{44} - \tfrac{1}{2}\underset{4}{g}_{\mu\lambda}T^{\mu\lambda}]_{,4}$$

$$+ \underset{4}{g}_{\nu 4,\lambda}\underset{4}{T}^{\nu\lambda} + \tfrac{1}{2}\underset{4}{g}_{4\lambda}\underset{2}{g}_{44,\lambda}\underset{2}{T}^{44} \,. \qquad (5.2.25)$$

We satisfy the second of (5.2.22) by taking

$$\underset{8}{T}^{44} = -\underset{8}{\rho}(x) - i\int_0^t \underset{8}{K}^4\,dt \,, \qquad (5.2.26)$$

where $-\underset{8}{\rho}(x)$ is the arbitrary time-independent value of $\underset{8}{T}^{44}$ in I_0. In I_2 we shall have, by (5.2.25),

$$\underset{8}{T}^{44} = -\underset{8}{\rho}(x) - i\int_0^{s}\underset{8}{K}^4\,dt - i\int_{s_2}^{t}(\underset{4}{g}_{\nu 4,\lambda}\underset{4}{T}^{\nu\lambda} + \tfrac{1}{2}\underset{4}{g}_{4\lambda}\underset{2}{g}_{44,\lambda}\underset{2}{T}^{44})dt$$

$$- [(\underset{4}{g}_{44} + \tfrac{1}{2}\underset{4}{g}_{\nu\nu})\underset{4}{T}^{44} - (\underset{2}{g}_{44}\underset{4}{g}_{44} + \tfrac{1}{2}\underset{2}{g}_{\mu\lambda}\underset{4}{g}_{\mu\lambda} + \underset{6}{g}_{44} + \tfrac{1}{2}\underset{6}{g}_{\nu\nu})\underset{2}{T}^{44} - \tfrac{1}{2}\underset{4}{g}_{\mu\lambda}\underset{4}{T}^{\mu\lambda}]\Big|_{s_2}^{t} .$$

$$(5.2.27)$$

This is finite provided t is finite. Otherwise we see that in the third term we have an integral of terms involving $\underset{4}{g}_{\nu 4}$ which for large t is of order t^{-1}, and hence on integration yields logarithmic terms. We therefore take a certain value t_0 of t, large compared with the dimensions of the body and with τ, but finite, and assert that what we propose to present is an incomplete model universe consisting of all of space-time except that part which is in the future relative to the intersection of $t = t_0$ with I (cf. fig.11). In other words, the

the lower boundary of the excluded region is formed by the intersection of $t = t_0$ with I and the lower bound S_3 of null-cones drawn into the future from all events in this intersection. In adjusting the value of t we have the arbitrary $\underset{8}{\rho}(\underline{x})$ at our disposal to ensure positive density, and corresponding quantities for the higher approximations, as mentioned in Chapter III. Note that it is only at this (4th) stage of the approximation that t_0 must be introduced.

Finally we define

$$\underset{8}{g}_{ij} = 4\int_{I+E} (\underset{8}{T}^{*ij} + \kappa^{-1} \underset{8}{t}^{*ij}) \, dx \, . \qquad (5.2.28)$$

We have already shown that as one goes to infinity along a null-line from $(0, \underline{x}_0)$, τ being finite, $\underset{8}{g}_{ij}$ is of order $x^{-1}(\log x)^3$ for large x and is therefore bounded, tending to zero as x tends to infinity.

CHAPTER VI.

Loss of Mass.

As in the case of energy, there is no generally accepted invariant definition, in general relativity, of the total mass of an extended body. Several definitions exist for what is termed the "total energy" or "gravitational mass" of an isolated system (cf. Trautman[3]). All these definitions agree, when applied to the Schwarzschild metric, yielding the constant m which occurs in the line element. However, it is not obvious that the different expressions give the same answer when applied to fields more complicated than that of the Schwarzschild metric, or that they have any meaning when applied to a non-stationary system. Furthermore, these expressions are not invariant under general coordinate transformations.

In order to find a definition of "mass" applicable to our model universe, we shall try to avoid the confusion of pseudo-tensors and take the Schwarzschild field as a basis of comparison. First of all, let us consider the body and its gravitational field in their initial static state. Although the body and its energy tensor are not spherically symmetric, one would expect that at large distances from the source the field approaches that of a spherically symmetric source i.e. that it is asymptotically a Schwarzschild field. If we take the exterior Schwarzschild metric in its usual form

$$\Phi = \left(1-\frac{2m}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) - \left(1-\frac{2m}{r}\right)dt^2 , \tag{6.1}$$

we may define the "mass" of the central body to be the constant m. Introducing Cartesian coordinates x_s by

$$x_1 = r\sin\theta\cos\phi , \quad x_2 = r\sin\theta\sin\phi , \quad x_3 = r\cos\theta \tag{6.2}$$

we have

$$r^2 = x_s\, x_s , \qquad r\, dr = x_s\, dx_s \tag{6.3}$$

and putting $x_4 = it$, we may transform the line-element (6.1) into

$$\Phi = g_{ij}\, dx_i\, dx_j ,$$

where

$$g_{\rho\sigma} = \delta_{\rho\sigma} + \left[\left(1-\frac{2m}{r}\right)^{-1} - 1\right]\frac{dx_\rho\, dx_\sigma}{r^2} ,$$

$$g_{\rho 4} = 0 , \tag{6.4}$$

$$g_{44} = 1 - \frac{2m}{r} .$$

Florides and Synge[37] have applied the DFB method to calculate the metric tensor due to a spherically symmetric body up to the second order of approximation (inclusive). For $r >> a$ (a being the radius of the body), their metric tensor may be written in the form

$$
\begin{aligned}
g_{\alpha\beta} &= \left(1 + \frac{2(m_1+m_2)}{r}\right)\delta_{\alpha\beta} + O\left(\frac{1}{r^2}\right) , \\
g_{\alpha 4} &= 0 , \qquad\qquad\qquad\qquad\qquad\qquad (6.5) \\
g_{44} &= 1 - \frac{2(m_1+m_2)}{r} + O\left(\frac{1}{r^2}\right) ,
\end{aligned}
$$

where, in terms of the present work, m_1/a is of order k^2 and m_2/a is of order k^4. In particular,

$$m_1 = \int_{\Sigma} \rho_1(\underline{x})\, d_3x \qquad (6.6)$$

and m_2 is proportional to $m_1{}^2/a$. The essential point for our purposes is that (as they have shown) the metric (6.5) is in fact a disguised Schwarzschild metric (to order k^4) and that the mass of the central body, to this order, is, in accordance with our definition,

$$m = m_1 + m_2 . \qquad (6.7)$$

We therefore adopt the following procedure in the case of our model. First of all, we shall calculate the metric tensor in Σ_0 up to order k^4. Then for $r >> a$, where a is a typical radius of the body and $r = |\underline{x}|$, we shall compare the coefficients of r^{-1} in the components of the metric tensor to those of (6.5). In this way we hope to obtain an expression for the mass of the body in its initial static state, up to order k^4. We shall see that although this procedure gives an unequivocal result at the first order of approximation, serious difficulties of interpretation arise at the second order. Although the $O(k^4)$ part of the metric tensor is not independent of time in E_2 it becomes so as $t \to \infty$, in a sense which will be defined presently. We may therefore apply the same procedure as outlined above to the metric tensor $(t = \infty)$ in E_2 and compare the result so obtained to that of the initial static state. This will enable us to make some statements with regard to the change in the mass of the body due to radiation.

Before proceeding with the explicit calculation of the metric tensor in E_0 and E_2, we must explain in what sense the field in E_2 becomes static as t tends to infinity. We have seen that our method for determining the mass involves the calculation of the metric tensor at large distances from the source. Let us take r fixed with $r >> a$ and consider how g_{ij} behaves as we allow t to tend to infinity along a line $\underline{x}$ = const., $|\underline{x}| = r >> a$, where

$$g_{ij} = \delta_{ij} + \underset{2}{g}_{ij} + \underset{4}{g}_{ij} \quad . \tag{6.8}$$

In E_2, g_{ij} is independent of time and $\underset{4}{g}_{ij}$ is given by

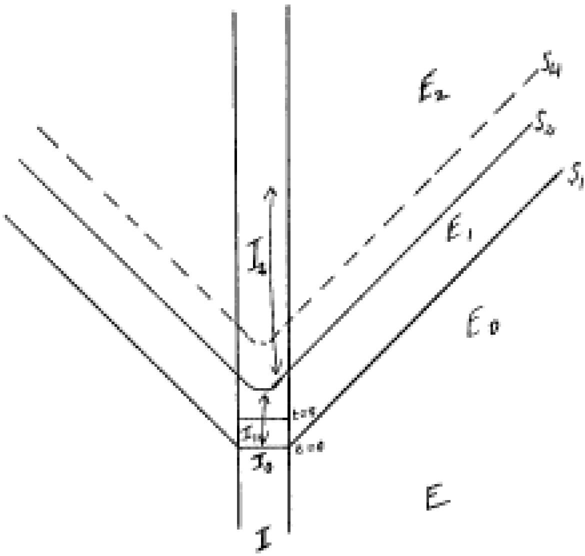

Figure 12. Non-stationary model universe in second approximation.

$$\underset{4}{g}_{ij} = 4\int(\underset{4}{T}^{*ij} + \kappa^{-1}\,\underset{4}{\dot{t}}^{*ij})d\omega\,. \qquad (6.9)$$

Let S_4 (Fig.12) be the upper boundary of the region traversed by null-cones drawn from all events in I_1. Then, since $\underset{4}{T}^{ij}$ is independent of time in I_2 so also is $\int\underset{4}{T}^{*ij}d\omega$ in that part of E_2 which lies above S_4. Hence we need only consider $\int\underset{4}{\dot{t}}^{*ij}d\omega$. By arguments analogous to those of Chapter V, one may show that the contribution to this integral from the intersection of the null-cone with E_1 goes to zero like t^{-1} and the contribution from E_0 goes to zero like t^{-2}, as $t \to \infty$ along the specified line. Hence as $t \to \infty$, the integral becomes, in the limit,

$$\int\underset{4}{\dot{t}}^{*ij}\,d\omega\,, \qquad (6.10)$$

where the functions $\underset{4}{\dot{t}}^{*ij}$ are given by their time-independent values in E_2 and the integral is taken over the whole infinite range of the three variables x_1, x_2, x_3. Hence, proceeding to the limit in the manner prescribed above, $\underset{4}{g}_{ij}$ tends towards a static value given by (6.9), where $\underset{4}{T}^{ij}$ is given by its time-independent values in I_2 and $\int\underset{4}{\dot{t}}^{*ij}d\omega$ is defined as for (6.10).

We are therefore in a position to compare the coefficients of r^{-1}, for $r >> a$, in the metric tensors of two static fields and consequently the mass of the body for the two cases, $t < 0$ and $t = \infty$.

A. Comparison of mass at $O(k^2)$.

(i) $t < 0$.

By (3.2.4 - 6), (3.2.10) and (3.2.16) we have, in Σ_0,

$$\underset{2}{g}_{\alpha\beta} = 2\underset{2}{V}\delta_{\alpha\beta}\ , \quad \underset{2}{g}_{\alpha 4} = 0\ , \quad \underset{2}{g}_{44} = -2\underset{2}{V}\ , \qquad (6.11)$$

where

$$\underset{2}{V} = \int_{\Sigma_0} \frac{\underset{2}{\rho}(\underline{x}')}{|\underline{x}-\underline{x}'|}\, d_3x'\ . \qquad (6.12)$$

For $r >> a$ (a being a typical radius of the body), we may write

$$\underset{2}{V} = \frac{\underset{2}{m}'}{r} + O(\frac{1}{r^2}) \qquad (6.13)$$

where

$$\underset{2}{m}' = \int_{\Sigma} \underset{2}{\rho}(\underline{x}')\, d_3x'\ . \qquad (6.14)$$

Hence, to order k^2,

$$g_{\alpha\beta} = (1 + \frac{2\underset{2}{m}'}{r})\,\delta_{\alpha\beta} + O(\frac{1}{r^2})\ ,$$

$$g_{\alpha 4} = 0\ , \qquad (6.15)$$

$$g_{44} = 1 - \frac{2\underset{2}{m}'}{r} + O(\frac{1}{r^2})\ .$$

Comparing this with (6.5), we find that the mass of the body for $t < 0$ is $\underset{2}{m}'$, to this order of approximation.

(ii) $t = \infty$.

Performing the same calculations for (3.2.4), (3.2.5), (3.2.9), (3.2.14) and (3.2.16) we obtain in E , for $r \gg a$,

$$g_{\alpha\beta} = \left(1 + \frac{2\underset{2}{m}^*}{r}\right)\delta_{\alpha\beta} + O\left(\frac{1}{r^2}\right)$$

$$g_{\alpha 4} = 0 \qquad (6.16)$$

$$g_{44} = 1 - \frac{2\underset{2}{m}^*}{r} + O\left(\frac{1}{r^2}\right)$$

where

$$\underset{2}{m}^* = \int_{I_2} \underset{2}{\sigma}(x')\, d_3x' \qquad (6.17)$$

and, by (3.2.14),

$$\underset{2}{\sigma} = \underset{2}{\rho} + q\, \underset{2}{\phi}^{\mu\nu}{}_{,\mu\nu} , \quad q = \int_0^\infty d\eta \int_0^\eta F(\xi)\, d\xi \quad \text{(a constant)} . \qquad (6.18)$$

Applying Gauss's theorem to (6.17) and using the fact that $\underset{2}{\phi}^{\mu\nu}{}_{,\mu} = 0$ on the boundary of I_2 , we obtain

$$\underset{2}{m}^* = \underset{2}{m}' . \qquad (6.19)$$

Hence, as already mentioned at the end of section (4.1), we conclude that there is no loss of mass at this order of approximation.

B. Comparison of mass at $O(k^4)$.

By (3.2.38), we have

$$\underset{4}{g}_{ij}(\mathbf{x}) = 4\int_{I_0+E_0}\left(\underset{4}{T}^{*ij}(\mathbf{x}') + k^{-4}\,\underset{4}{\hat{G}}^{*ij}(\mathbf{x}')\right)\frac{d_3x'}{|\mathbf{x}-\mathbf{x}'|} \quad \text{in } I_0+E_0 \,. \tag{6.20}$$

From the definition of $\underset{4}{\hat{G}}^{ij}$ as the $O(k^4)$ part of the Einstein tensor for the metric, $g_{ij} = \delta_{ij} + \underset{2}{g}_{ij}$, we obtain

$$\begin{aligned}\underset{4}{\hat{G}}^{ij} = & -\underset{2}{g}_{ia}\underset{2}{R}^{*}{}_{aj} - \underset{2}{g}_{ja}\underset{2}{R}^{*}{}_{ai} \\ & - \tfrac{1}{2}\underset{2}{g}_{ab}\left(\underset{2}{g}_{ab,ij}+\underset{2}{g}_{ij,ab}-\underset{2}{g}_{aj,bi}-\underset{2}{g}_{bi,aj}\right) + \tfrac{1}{2}\underset{2}{g}_{ij}\underset{2}{R}^{*}{}_{aa} \\ & + \delta_{ij}\underset{2}{g}_{ab}\underset{2}{R}^{*}{}_{ab} - \underset{2}{\Gamma}_{ai,b}\underset{2}{\Gamma}_{aj,b} + \tfrac{1}{2}\delta_{ij}\underset{2}{\Gamma}_{ab,c}\underset{2}{\Gamma}_{ab,c} \,.\end{aligned} \tag{6.21}$$

Furthermore, we know that, in $I_0 + E_0$,

$$\underset{2}{g}_{\mu\nu} = 2V\delta_{\mu\nu} \,, \quad \underset{2}{g}_{\mu 4} = 0 \,, \quad \underset{2}{g}_{44} = -2V \,, \tag{6.22}$$

where

$$V = \int \frac{\underset{2}{\rho}(\mathbf{x}')\,d_3x'}{|\mathbf{x}-\mathbf{x}'|} \,. \tag{6.23}$$

Substituting (6.22) into (6.21) and taking star conjugates, we obtain, in $I_0 + E_0$,

$$\begin{aligned}\underset{4}{\hat{G}}^{*\mu\nu} &= -(V^2)_{,\gamma\gamma}\,\delta_{\mu\nu} - 4VV_{,\mu\nu} - 2V_{,\mu}V_{,\nu} \,, \\ \underset{4}{\hat{G}}^{*\mu 4} &= 0 \,, \\ \underset{4}{\hat{G}}^{*44} &= -(V^2)_{,\gamma\gamma} \,.\end{aligned} \tag{6.24}$$

Finally, noting that

$$4\kappa^{-1}\int_{I_o+E_o}(V^2)_{,\nu\nu}\frac{d_3x'}{|\underline{x}-\underline{x}'|} = -2V^2(\underline{x}) = O(\tfrac{1}{r^2}), \text{ for } r \gg a, \tag{6.25}$$

we obtain, from (6.20),

$$\begin{aligned} \underset{4}{g}_{\mu\nu} &= \frac{4}{r}\int_{I_o}\underset{4}{T}^{\mu\nu}(\underline{x}')d_3x' - 16\kappa^{-1}\int_{I_o+E_o}(VV_{,\mu\nu}+2V_{,\mu}V_{,\nu})\frac{d_3x'}{|\underline{x}-\underline{x}'|} + O(\tfrac{1}{r^2}) \\ \underset{4}{g}_{\mu 4} &= 0 \\ \underset{4}{g}_{44} &= \frac{4}{r}\int\underset{4}{T}^{44}(\underline{x}')d_3x' + O(\tfrac{1}{r^2}), \end{aligned} \tag{6.26}$$

for $r \gg a$ in E_o. Note that the second term in the first of (6.26) has not been expanded in inverse powers of r. In principle, it would be possible to carry out this expansion, but the resulting coefficient of r^{-1} would be rather complicated.

Comparison of the first of (6.26) with the first of (6.5) does not allow us to draw any conclusions about the mass of the body at $O(\lambda^4)$. This is due to the fact that the r^{-1} term of (6.26) is <u>not</u> spherically symmetric and hence, despite what one might expect, the field of the body, when calculated beyond the first approximation, is not asymptotically a Schwarzschild field.

While keeping this difficulty in mind, we think it is worthwhile to point out that the second and third of (6.5) and (6.26) admit a comparison which, if accepted, would yield

$$\underset{4}{m'} = -2\int \underset{4}{T}^{**44}(\underset{\sim}{x}')d_3x' , \qquad (6.27)$$

as the $O(k^4)$ part of the mass. Combining this with (6.14) we obtain as the mass of the body to $O(k^4)$, for $t < 0$,

$$m' = \int_{\Sigma_0} (\underset{2}{\rho}(\underset{\sim}{x}') - 2\, T^{**44}(\underset{\sim}{x}'))d_3x' . \qquad (6.28)$$

It is interesting to note that (6.28) is what one would obtain from Tolman's expression[40] for the total energy of a static system, viz.,

$$U = \int \sqrt{-g}\,(T_4^{\ 4} - T_\mu^{\ \mu})\, d_3x . \qquad (6.29)$$

He obtains this by calculating, in the case of a static system, the usual expression,

$$U = \int \sqrt{-g}\,(T_4^{\ 4} + t_4^{\ 4})\, d_3x \qquad (6.30)$$

which (allegedly) gives the total energy, where $t_j^{\ i}$ are the components of Einstein's pseudotensor in an appropriate coordinate system.

Continuing a little further on the present line of discussion, we may calculate the corresponding quantity for $t = \infty$ in the same manner and obtain

$$m'' = \int_{\Sigma_2} [\underset{2}{\rho}(\underset{\sim}{x}') - (\underset{4}{T}^{**44}(\underset{\sim}{x}') - \underset{4}{T}^{\dagger\dagger 44}(\underset{\sim}{x}'))]\, d_3x' \qquad (6.31)$$

where the $\underset{4}{T}^{ij}$ are those of I_2. Denoting the values of $\underset{4}{T}^{ij}$ in I_o and I_2 by superscripts (o) and (2) respectively, we have, by (3.2.29) and a simple calculation, that $\overset{(o)}{\underset{4}{T}}$ and $\overset{(2)}{\underset{4}{T}}$ satisfy

$$\overset{(o)}{\underset{4}{T}}{}^{\mu\nu}{}_{,\nu} = -\overset{(o)}{\underset{4}{K}}{}^{\mu} = \underset{2}{\rho}\,\underset{2}{V}{}_{,\mu} \quad \text{in } I_o\,,$$
$$\overset{(2)}{\underset{4}{T}}{}^{\mu\nu}{}_{,\nu} = -\overset{(2)}{\underset{4}{K}}{}^{\mu} = \underset{2}{\sigma}\,\underset{2}{W}{}_{,\mu} \quad \text{in } I_2\,, \tag{6.32}$$

respectively, where

$$\underset{2}{W} = \int_{I_2} \frac{\sigma(x')}{|x-x'|}\, d_3x' \,. \tag{6.33}$$

Also, by (3.2.33)

$$\overset{(o)}{\underset{4}{T}}{}^{44} = -\underset{4}{\rho}\,, \quad \overset{(2)}{\underset{4}{T}}{}^{44} = -\underset{4}{\sigma} - i\int_0^{S_2} \underset{4}{K}{}^4 dt\,. \tag{6.34}$$

On comparing m' (6.28) and m'' (6.34), we see that although the integral of the first term is the same in both cases this will not be so for the remaining $(O(k^4))$ terms. Hence, we would conclude that, at this order, there is a difference between the mass of the body for $t < 0$ and $t \to \infty$. The explicit calculation for the difference in mass cannot be carried out because of the indeterminacy of the equations (6.32).

The main conclusion of the present chapter is therefore a negative one, viz., that the field of a single static body calculated by the EIH method is not asymptotically a Schwarz-schild field when one goes beyond the first approximation. The agreement of the rather uncertain definition of mass obtained by considering the g_{44} term with the mass obtained by the pseudotensor approach would only seem to cast further doubt on the validity of the latter. It is the author's belief that the difficulty encountered here, and indeed universally, of finding a satisfactory definition for "mass" and "energy" is essentially due to the fact that these Newtonian concepts cannot be transferred directly into the scheme of general relativity. Instead, one must look for invariant quantities, capable of physical interpretation, which may perhaps play a role in general relativity analogous to those of mass and energy in Newtonian physics.

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