Sgríbhinní Institiuid Árd-Léinn Bhaile Átha Cliath
Sraith A, Uimh. 21

Communications of thé Dublin Institute for
Advanced Studies. Series A, No. 21

QUATERNIONS, LORENTZ TRANSFORMATIONS, AND THE CONWAY-DIRAC-EDDINGTON MATRICES

by

J. L Synge

Dedicated to the memory of A.W. Conway (1875 - 1950)

DUBLIN
INSTITIÚID ÁRD-LÉINN BHAILE ÁTHA CLIATH
DUBLIN INSTITUTE FOR ADVANCED STUDIES
1972
Price £1·50

C O N T E N T S

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1. Introduction.

It is now sixty years since Conway (1911) introduced quaternions as a notation in the special theory of relativity. He was followed soon after independently by Silberstein (1912), and a number of papers on the subject were subsequently written by Conway and others. Yet, on the whole, quaternions have been used very little in relativity, and one may ask whether this is due to some inadequacy in the notation or to other reasons.

An ordinary quaternion is composed of four real numbers, and ordinary quaternions form an algebra under addition and multiplication. They are therefore easy to use. However, their particular field of applicability is Euclidean 4-space, not Euclidean 3-space nor space-time. Reflections and rotations in Euclidean 4-space lend themselves admirably to quaternionic treatment, as has been shown by Coxeter (1946) in a very clearly written paper. For the application of quaternions to reflections in Euclidean 3-space, see Tuckerman (1947) and Wagner (1951).

Since Minkowskian space-time can be converted formally into Euclidean 4-space by the simple expedient of imaginary time $(x_4 = it)$, and since a Lorentz transformation may be regarded as a rotation (with perhaps a reflection) in Minkowskian space-time, it might appear that Lorentz transformations could be treated by quaternions with nothing more than minor adjustments arising out of the imaginary time.

But that is not the case. The intrusion of the imaginary element is not trivial. To explain this, let us use the expression Minkowskian quaternion (or minquat for short) to denote a quaternion which is the sum of a real vector and an imaginary number. Minquats belong to the wider class of complex quaternions (biquaternions Hamilton called them), and complex quaternions form an algebra under addition and multiplication. But, and this is the difficulty, minquats do not form an algebra; the sum of two minquats is a minquat, but the product of two minquats is not in general a minquat.

To deal adequately with Lorentz transformations, we must (as will be seen later) use complex quaternions, and indeed there is no trouble here because they form an algebra. What is troublesome is the fact that we are constantly compelled to come down out of this wider domain into the domain of (non-algebraic) minquats. If quaternions are to receive due attention in relativity, this point must not be slurred over. If it is slurred over - if, in fact, attention is not constantly directed as to what is real, what imaginary, and what complex - then the intending user is likely to reject quaternions as a confusing notation.

The primary aim of the present paper is to give a brief but complete account of the application of quaternions to Lorentz transformations, general and singular. There is little novelty in the results, except perhaps in regard to the singular transformations, which do not appear to be well known. I take this opportunity to acknowledge that on a previous occasion (Synge, 1956, p. 93) I overlooked the singular transformations, and to thank I. Robinson and A. H. Taub for drawing my attention to the omission; in a later edition (Synge, 1965), this is corrected.

This paper being in fact a tribute to the work of the late Professor Conway, I have included a section dealing with his quaternionic approach to the matrices associated with the names of Dirac and Eddington. The paper ends with a list of references, in which the titles of papers are included to make it more useful. But since my aim is expository rather than historical, I have not attempted to assign specific formulae to their originators.

I thank Professor C. Lanczos for discussions which have greatly illuminated the subject for me.

2. The three standpoints.

To gain a complete understanding of the Lorentz transformation, one must view it from three different standpoints: (i) physics, (ii) geometry, (iii) algebra.

(i) Physics. In modern quantum physics the Lorentz transformation is so closely woven into the basic equations that it is impossible to separate the physics from the algebra. I shall consider here only the traditional physical approach, in which two observers in uniform relative motion compare observations made on moving particles or photons, describing what they observe in quasi-Newtonian terms, velocity being the 3-vector $(dx/dt, dy/dt, dz/dt)$ and so on. Since the Newtonian notation is badly suited to the discussion of relativity, calculations become very complicated unless the two observers cooperate in their choices of spatial axes in such a way that the Lorentz transformation takes the simple familiar form

$$x' = \gamma(x - vt),\ y' = y,\ z' = z,\ t' = \gamma(t - vx),\ \gamma^{-2} = 1 - v^2. \qquad (2.1)$$

Hence emerge the formulae for rod-contraction and clock-retardation. But the Newtonian notation obscures essential features of the Lorentz transformation, and in that notational jungle the famous 'clock paradox' continues to be reborn, no matter how often it is brought out into the light and killed. Nevertheless, in so far as the Lorentz transformation belongs to physics and not to pure mathematics, it is from the physical standpoint that any geometrical or algebraic formula is ultimately to be assessed for meaning.

(ii) Geometry. Space-time in special relativity is a flat 4-space for which, if the coordinates are suitably chosen, the metric form is

$$dx^2 + dy^2 + dz^2 - dt^2. \qquad (2.2)$$

We have then an orthonormal tetrad of axes of reference, and a Lorentz transformation presents itself in two different but equivalent ways. We may hold the axes fixed and give to space-time a displacement which is rigid in terms of the metric (2.2), or we may hold space-time fixed and change the axes rigidly. However, it is more in the true spirit of geometry to forget about coordinates, and to think of a Lorentz transformation as a rigid displacement of space-time into itself, in much the same way as we are accustomed to think of an ordinary rotation in Euclidean 3-space.

There are four types of Lorentz transformation: it may be proper or improper, future-preserving or future-reversing. The following symbolism may be used:

	future-preserving	future-reversing	
proper	(+ +)	(+ -)	
improper	(- +)	(- -)	(2.3)

These types are best defined by considering what happens to an orthonormal tetrad as a result of the transformation. Let T and T' denote the tetrad before and after. The transformation is <u>proper</u> if T can be changed into T' either (a) by continuous transformation preserving orthonormality, or (b) by this followed by reflection in the origin of space-time; otherwise it is <u>improper</u>. It is <u>future-preserving</u> if the timelike members of T and T' point into the same half of the null cone, and <u>future-reversing</u> if they point into different halves. Future-reversal, important in modern physics, is without meaning in the traditional physical view described above.

(iii) <u>Algebra</u>. A Lorentz transformation is a linear transformation $(x,y,z,t) \longrightarrow (x',y',z',t')$ with the conservation

$$x^2 + y^2 + z^2 - t^2 = x'^2 + y'^2 + z'^2 - t'^2. \tag{2.4}$$

If X stands for the column matrix with elements (x,y,z,t), a Lorentz transformation may be written

$$X' = LX, \tag{2.5}$$

where L is a 4×4 matrix satisfying

$$\tilde{L}\eta L = \eta , \tag{2.6}$$

where the tilde denotes transpose and $\eta = \text{diag}(1,1,1,-1)$, the Minkowskian matrix. The symbols in the scheme (2.3) may be obtained by writing down ($\det L$, L_{44}) and replacing these quantities by their signs. It is obvious from (2.6) that $\det L = +1$ or -1; the former belongs to a proper transformation, the latter to an improper one. L_{44} is positive and negative for future-preserving and future-reversing, respectively.

A Lorentz transformation conserves scalar products; if it carries (x,y,z,t) into (x',y',z',t') and (X,Y,Z,T) into (X',Y',Z',T'), then

$$xX + yY + zZ - tT = x'X' + y'Y' + z'Z' - t'T'. \tag{2.7}$$

Indeed this is the best way to look at a Lorentz transformation, for, if (2.7) holds for every pair (x,y,z,t), (X,Y,Z,T), then the transformation is linear with the conservation (2.4), and so it is a Lorentz transformation. Thus (2.7) is both necessary and sufficient for Lorentz character.

Any proper future-preserving Lorentz transformation may be written in the form

$$u' = D\, u\, \tilde{D}^*, \tag{2.8}$$

where

$$u = \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix}, \quad D = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \det D = 1, \tag{2.9}$$

the star indicating complex conjugate.

That this gives the conservation (2.4) is verified immediately by taking the determinants of the matrices in (2.8).

As will be shown below, we can also express a Lorentz transformation in terms of quaternions. The formula (2.8), which belongs to spinor theory, is a link between matrix methods and quaternion methods.

This completes a brief review of the background against which a quaternionic treatment of the Lorentz transformation may fittingly be displayed.

3. Quaternions and minquats.

We have to deal with <u>numbers</u> (in general complex - complex conjugates will be indicated by a star) and <u>quaternions</u>. A quaternion is an expression of the form

$$q = q_1 e_1 + q_2 e_2 + q_3 e_3 + q_4 , \qquad (3.1)$$

where q_1, q_2, q_3, q_4 are numbers and e_1, e_2, e_3 are <u>quaternionic units</u>, which satisfy the non-commutative multiplication rules

$$e_1^2 = e_2^2 = e_3^2 = -1,$$

$$e_2 e_3 = e_1 = -e_3 e_2, \; e_3 e_1 = e_2 = -e_1 e_3, \; e_1 e_2 = e_3 = -e_2 e_1 . \qquad (3.2)$$

A quaternion is essentially an ordered tetrad of numbers, and we may write (3.1) as

$$q = (q_1, q_2, q_3, q_4), \qquad (3.3)$$

with the understanding that it is a matter of indifference whether we write $q_1 e_1$ or $e_1 q_1$, and so on. We write $q = 0$ if, and only if, the four numbers in the tetrad vanish. We write $q = q'$ (and say that the two quaternions are equal) if, and only if, their two ordered tetrads are the same. To add or subtract two quaternions, we add or subtract the numbers in their respective tetrads. To multiply a quaternion by a number, we multiply the numbers in its tetrad by that number.

To multiply two quaternions, we proceed to multiply out two expressions of the type (3.1) as we would in ordinary algebra, but with one important reservation: we do not treat the multiplication of quaternionic units as commutative. Instead, we substitute for ordered products as in (3.2). Thus the product of two quaternions is itself a quaternion. In fact, quaternions form an algebra under addition and multiplication.

We have been considering complex quaternions (the biquaternions of Hamilton). If the tetrad of numbers as in (3.3) are real, then we have real quaternions. It is easy to see that they too form an algebra under addition and multiplication, for at no stage can a complex number leak in.

A quaternion as in (3.1) (in general complex) possesses two types of conjugate:

$$\begin{aligned} &\text{Complex conjugate:} && \overset{*}{q} = \overset{*}{q}_1 e_1 + \overset{*}{q}_2 e_2 + \overset{*}{q}_3 e_3 + \overset{*}{q}_4 \\ &\text{Hamiltonian conjugate:} && \bar{q} = - q_1 e_1 - q_2 e_2 - q_3 e_3 + q_4 \,. \end{aligned} \tag{3.4}$$

For two or more quaternions we have

$$(q'q'')^* = q'^* q''^*, \quad (q'q''q''')^* = q'^* q''^* q'''^*, \ \dots \tag{3.5}$$

$$\overline{(q'q'')} = \overline{q''}\ \overline{q'}, \qquad \overline{(q'q''q''')} = \overline{q'''}\ \overline{q''}\ \overline{q'}\ , \ \ldots \tag{3.6}$$

Here (3.5) is obvious; the first of (3.6) is easily verified, and the others then proved by induction.

The norm of a quaternion q as in (3.1) is

$$q\,\overline{q} = q_1^{\,2} + q_2^{\,2} + q_3^{\,2} + q_4^{\,2} = \overline{q}\,q, \tag{3.7}$$

and the scalar product of two quaternions p, q is

$$\tfrac{1}{2}(p\,\overline{q} + q\,\overline{p}) = p_1q_1 + p_2q_2 + p_3q_3 + p_4q_4 = \tfrac{1}{2}(\overline{p}\,q + \overline{q}\,p). \tag{3.8}$$

The norm and scalar product are numbers, in general complex, the norm being of course the self-scalar-product. As an alternative notation, we may write

$$(q,q) = q\,\overline{q}, \qquad (p,q) = (q,p) = \tfrac{1}{2}(p\,\overline{q} + q\,\overline{p}). \tag{3.9}$$

Then from (3.2) we have

$$(e_1,\ e_1) = (e_2,\ e_2) = (e_3,\ e_3) = 1,$$
$$(e_2,\ e_3) = (e_3,\ e_2) = (e_3,\ e_1) = (e_1,\ e_3) = (e_1,\ e_2) = (e_2,\ e_1) = 0, \tag{3.10}$$

which suggests that we might regard e_1, e_2, e_3 as an orthogonal triad of unit vectors in Euclidean 3-space.

A quaternion q is a unit quaternion if

$$(q,q) = q\,\overline{q} = \pm 1, \tag{3.11}$$

and we shall call it a positive or negative unit quaternion according as the sign is plus or minus.

For the application of quaternions to Lorentz transformations it is essential to introduce Minkowskian quaternions (or, briefly, minquats): a minquat is a quaternion

$$q = q_1 e_1 + q_2 e_2 + q_3 e_3 + q_4, \tag{3.12}$$

where (q_1, q_2, q_3) are real numbers and q_4 a pure imaginary.

Silberstein (1912) called this a physical quaternion; Fischer (1957, p. 4) called it a technical physical quaternion.

A minquat is of course a complex quaternion. If we add two minquats, we get a minquat. But if we multiply two minquats the result, in general, is a complex quaternion which is not a minquat - the imaginary element leaks into the first part. It is clear then that minquats must be handled with circumspection.

The reason for introducing minquats is this. Let x_1, x_2, x_3, x_4 be an event in space-time, with $x_4 = it$ (imaginary time). Then the minquat

$$x = x_1 e_1 + x_2 e_2 + x_3 e_3 + x_4 \tag{3.13}$$

may be regarded as a 4-vector in space-time. Although this is a complex quaternion, the norm and scalar product are real:

$$\begin{aligned} (x,x) &= x\bar{x} = x_1^2 + x_2^2 + x_3^2 - t^2, \\ (x,x') &= \tfrac{1}{2}(x\bar{x}' + x'\bar{x}) = x_1 x_1' + x_2 x_2' + x_3 x_3' - tt'. \end{aligned} \tag{3.14}$$

These are invariant under Lorentz transformations. The 4-vector is spacelike, null, or timelike according as its norm is positive, zero, or negative.

It is important to note that $\underline{q}$ is a minquat if, and only if,

$$\boxed{\bar{q} + q^{*} = 0;} \tag{3.15}$$

This is obvious by (3.4). If we use S and B to denote the operators "star" and "bar" as in (3.4), with I for the identity operator, we have

$$S^2 = B^2 = I, \qquad SB = BS. \tag{3.16}$$

The minquat condition (3.15) may be written in the following forms:

$$(B + S)q = 0, \quad (SB + I)q = 0, \quad \bar{q}^{*} + q = 0. \tag{3.17}$$

4. Reflections and Lorentz transformations.

Let $\underline{p}$ be a unit quaternion, so that

$$(p,p) = p\bar{p} = \varepsilon(p) = \pm 1. \tag{4.1}$$

Let $\underline{q}$ be any quaternion. Then the formula

$$q' = -\varepsilon(p)\, p\, \bar{q}\, p \tag{4.2}$$

transforms $\underline{q}$ into another quaternion $\underline{q}'$. This formula may be inverted by multiplying by $\bar{p}$ on the right and on the left, giving

$$\bar{q} = -\varepsilon(p)\, \bar{p}\, q'\, \bar{p}, \qquad q = -\varepsilon(p)\, p\, \bar{q}'\, p, \tag{4.3}$$

which last is of the same form as (4.2) with $\underline{q}$ and $\underline{q}'$ interchanged. Another useful form is obtained by multiplying (4.2) on the right by $\bar{p}$:

$$q'\bar{p} + p\bar{q} = 0. \tag{4.4}$$

There is still another form:

$$\bar{q}' = -\varepsilon(p)\,\bar{p}\,q\,\bar{p}. \tag{4.5}$$

All these several forms represent the same transformation $q \longrightarrow q'$. We shall now see that this transformation has an important property: *it conserves scalar products*. To prove this, let q and Q transform into q' and Q', so that we have

$$q' = -\varepsilon(p)\,p\,\bar{q}\,p, \qquad Q' = -\varepsilon(p)\,p\,\bar{Q}\,p. \tag{4.6}$$

Then

$$\begin{aligned} q'\,\bar{Q}' &= p\,\bar{q}\,p\,\bar{p}\,Q\,\bar{p} = \varepsilon(p)\,p\,\bar{q}\,Q\,\bar{p}, \\ Q'\,\bar{q}' &= p\,\bar{Q}\,p\,\bar{p}\,q\,\bar{p} = \varepsilon(p)\,p\,\bar{Q}\,q\,\bar{p}. \end{aligned} \tag{4.7}$$

Adding and using the notation (3.8) for scalar products, we get

$$(q', Q') = \varepsilon(p)\,p\,(q,Q)\,\bar{p}. \tag{4.8}$$

But the scalar product is a number and so can be carried out in front, giving

$$(q', Q') = (q, Q). \tag{4.9}$$

The conservation of scalar products is established, and this includes the conservation of norms.

The conservation of scalar products is the essential property of Lorentz transformations, but it would be wrong to call (4.2) a Lorentz transformation, since a Lorentz transformation acts on 4-vectors (minquats), whereas so far q is any complex quaternion.

Let us now take both p and q to be minquats, so that

$$\bar{p} + p^* = 0, \; p\,\bar{p} = \varepsilon(p) = \pm 1, \; \bar{q} + q^* = 0. \tag{4.10}$$

Apart from being a minquat, q is free. To show that (4.2) is indeed a Lorentz transformation, we have to show two things:

(a) Minquats transform into minquats.

(b) Scalar products are conserved.

To establish (a), we carry out application of star and bar as follows, using (4.10):

$$\begin{aligned} q' &= -\varepsilon(p)\, p\, \bar{q}\, p = \varepsilon(p)\, p\, q^*\, p, \\ q'^* &= \varepsilon(p)\, p^*\, q\, p^* = \varepsilon(p)\, \bar{p}\, q\, \bar{p} = -\bar{q}', \\ & q'^* + \bar{q}' = 0, \end{aligned} \tag{4.11}$$

so that q' is a minquat. The conservation of scalar products has already been established.

Let us restate this result which is of basic importance. Let p be any unit minquat, so that $\bar{p} + p^* = 0$, $p\,\bar{p} = \varepsilon(p) = \pm 1$. Then when applied to any minquat q, the transformation

$$q' = -\varepsilon(p)\, p\, \bar{q}\, p \tag{4.12}$$

is a Lorentz transformation (i.e. it conserves minquats and their scalar products).

When applied to p itself, (4.12) gives

$$p' = -\varepsilon(p)\, p\, \bar{p}\, p = -p. \tag{4.13}$$

The minquat (or 4-vector) p is reversed by the transformation, which is in fact a reflection along p, i.e. a reflection of space-time in the 3-flat orthogonal to p.

We have now to ask this question: Given two minquats, q and q', can we find a unit minquat p such that the transformation (4.12) carries q into q' ?

Since the transformation conserves the norm, it is clear that the answer is negative unless q and q' satisfy

$$q\,\bar{q} = q'\,\bar{q}'. \tag{4.14}$$

We shall accordingly assume this to be the case. Then, using the form (4.4) for the transformation, we seek to find p to satisfy

$$q'\,\bar{p} + p\,\bar{q} = 0, \quad \bar{p} + p^* = 0, \; p\,\bar{p} = \varepsilon(p) = \pm 1. \tag{4.15}$$

The solution is far from unique. Let us seek one of the form

$$p = \theta(q' - q), \tag{4.16}$$

suggested by the reflectional property, θ being real number. Substitution in the first of (4.15) gives

$$q'\,\bar{p} + p\,\bar{q} = \theta[q'(\bar{q}' - \bar{q}) + (q' - q)\bar{q}] = 0, \tag{4.17}$$

by (4.14). The second of (4.15) is also satisfied, since q and q' are minquats. It remains only to satisfy the last of (4.15), that is,

$$\theta^2(q' - q)(\bar{q}' - \bar{q}) = \varepsilon(p) = \pm 1. \tag{4.18}$$

If (q' - q) is spacelike, we are to take the upper sign; if timelike, the lower sign. In either case θ exists, with an ambiguity of sign, which is trivial since (4.12) is even in p.

Thus $\underline{p}$ exists, satisfying the required equations (4.15) unless $(q' - q)$ is a null minquat. For our purposes this does not impose a serious restriction. It means that, given $\underline{q}$, we are to avoid minquats $\underline{q}'$ of the form

$$q' = q + n \tag{4.19}$$

where $\underline{n}$ is null, i.e. $n\bar{n} = 0$. Since the norms of $\underline{q}'$ and $\underline{q}$ are equal, (4.19) gives

$$q'\,\bar{q}' = (q + n)(\bar{q} + \bar{n}) = q\,\bar{q} + q\,\bar{n} + n\,\bar{q}$$

or

$$(q,n) = 0. \tag{4.20}$$

Thus the reflective process $\underline{q} \longrightarrow \underline{q}'$ can fail only when $\underline{q}$ is orthogonal to a null vector. From our knowledge of Minkowskian geometry we know that no timelike vector can be orthogonal to a null vector, and so there exists a reflection (or Lorentz transformation) which carries a timelike vector into any other timelike vector of the same norm. If $\underline{q}$ is spacelike, we cannot reflect it into a timelike $\underline{q}'$ because their norms differ in sign. But if $\underline{q}$ and $\underline{q}'$ are both spacelike (with the same norm) and they are both orthogonal to some timelike vector, then $(q' - q)$ is spacelike and we can effect the reflection taking $\underline{q}$ into $\underline{q}'$. It would be possible, but tedious, to validate these conclusions algebraically.

We are now to see that any Lorentz transformation can be accomplished by a succession of three or four reflections. We start with the idea that a Lorentz transformation is a displacement of all the events of space-time such that some specified orthonormal tetrad of vectors is carried into another specified orthonormal tetrad. Each of these tetrads includes a timelike minquat, say $\underline{T}$ for the initial tetrad and $\underline{T}'$ for the final tetrad. Our first task is to carry $\underline{T}$ into $\underline{T}'$ by a reflection along some unit minquat, say $\underline{v}$ with $v\,\bar{v} = \varepsilon(v) = \pm 1$. We know that this can be done: then the transformation of all minquats $\underline{q}$ is of the form

$$q' = -\,\varepsilon(v)\; v\,\bar{q}\,v. \qquad (4.21)$$

This is only the first step in the complete Lorentz transformation, but we must pause to consider the meaning of the sign of $\varepsilon(v)$. In Minkowskian space-time the complete null cone has two distinct parts. If the 4-vectors (or minquats) $\underline{T}$ and $\underline{T}'$ point into the same part of the null cone, we have a future-preserving transformation; then the difference $(T' - T)$ is spacelike, and [cf. (4.18)] $\varepsilon(v) = 1$. On the other hand if $\underline{T}$ and $\underline{T}'$ point into different parts of the null cone, we have a future-reversing transformation; the difference $(T' - T)$ is timelike and $\varepsilon(v) = -1$.

Having made the transformation (4.21), our two tetrads have a timelike member in common. The remaining members of each tetrad form an orthonormal triad, orthogonal to the common timelike member. A reflection, say $\underline{u}$ with $\varepsilon(u) = 1$ since $\underline{u}$ is necessarily spacelike, will bring the first member of a triad into coincidence with the first member of the other triad. This gives a transformation

$$q'' = -u\bar{q}'u, \tag{4.22}$$

and so, by (4.21), the whole transformation up to this point is

$$q'' = \varepsilon(v)\, u\bar{v}\, q\, \bar{v} u. \tag{4.23}$$

Another coincidence of members of the tetrads will be achieved by some reflection _s_ with $\varepsilon(s) = 1$, and this gives the transformation

$$q''' = -s\bar{q}''s, \tag{4.24}$$

so that the complete transformation to this point is

$$q''' = -\varepsilon(v) s\bar{u} v \bar{q} v \bar{u} s. \tag{4.25}$$

We have now brought three members of the old tetrad into coincidence with three members of the new tetrad. Each of the three reflections reverses the orientation of the tetrad, and so we are left with a tetrad with an orientation opposite to that of the original tetrad. Now a _proper_ Lorentz transformation is one which conserves orientation; an _improper_ one reverses orientation. Thus if the transformation is improper, it has been completed as in (4.25) by means of three reflections, since, if three members of an orthonormal tetrad are given, the fourth member is defined within a reversal of direction. But if the transformation is proper, we shall have to add one more reflection, say _r_ with $\varepsilon(r) = 1$:

$$q'''' = -r\bar{q}'''r. \tag{4.26}$$

With (4.25) this gives, for the complete transformation,

$$q'' = \varepsilon(v)\ r\ \bar{s}\ u\ \bar{v}\ q\ \bar{v}\ u\ \bar{s}\ r. \tag{4.27}$$

Let us now simplify the formulae by writing q' for the left hand side of (4.25) and (4.26), and summarise as follows: All Lorentz transformations (proper and improper, future-preserving and future-reversing) may be exhibited as follows:

$$\text{proper:}\quad q \longrightarrow q' = \varepsilon(v)\ r\ \bar{s}\ u\ \bar{v}\ q\ \bar{v}\ u\ \bar{s}\ r, \tag{4.28}$$

$$\text{improper:}\quad q \longrightarrow q' = -\ \varepsilon(v)\ s\ \bar{u}\ v\ \bar{q}\ v\ \bar{u}\ s,$$

$$\text{future-preserving:}\quad \varepsilon(v) = 1,$$

$$\text{future-reversing:}\quad \varepsilon(v) = -1.$$

If we write

$$r\ \bar{s}\ u\ \bar{v} = a, \tag{4.29}$$

then, since these factors are minquats, it follows that

$$\bar{v}\ u\ \bar{s}\ r = \bar{a}^{*}. \tag{4.30}$$

Further,

$$a\ \bar{a} = \varepsilon(v). \tag{4.31}$$

Treating the second of (4.28) in a similar way, we may suppress the individual reflection vectors, and state that all Lorentz transformations are included in the scheme

$$\text{proper:}\quad q \longrightarrow q' = \varepsilon\ a\ q\ \bar{a}^{*},$$

$$\text{improper:}\quad q \longrightarrow q' = -\ \varepsilon\ a\ \bar{q}\ \bar{a}^{*}, \tag{4.32}$$

$$\text{future-preserving:}\quad \varepsilon = 1,$$

$$\text{future-reversing:}\quad \varepsilon = -1,$$

$$a\ \bar{a} = \varepsilon\ ,$$

or, in tabular form,

	proper	improper
future-preserving	$q' = a q \bar{a}^*$ $a \bar{a} = 1$	$q' = - a \bar{q} \bar{a}^*$ $a \bar{a} = 1$
future-reversing	$q' = - a q \bar{a}^*$ $a \bar{a} = -1$	$q' = a \bar{q} \bar{a}^*$ $a \bar{a} = -1$

(4.33)

We are to note that, although a is built up out of minquats, it is itself a complex unit quaternion, not a minquat in general.

In the formulae (4.33) we have succeeded in expressing all Lorentz transformations in terms of quaternions. The argument has been somewhat long because it was desirable to make sure that all Lorentz transformations were included and classified into proper and improper, future-preserving and future-reversing.

Henceforth we shall concentrate on the proper future-preserving transformation

$$q \longrightarrow q' = a q \bar{a}^* , \quad a \bar{a} = 1. \tag{4.34}$$

As derived above, the unit quaternion a is a product of four minquats, and the question arises whether (4.34) is a Lorentz transformation only for an a constructed in that way. Let us test the matter, using (4.34) with no restriction on a except

$$a \bar{a} = 1 \tag{4.35}$$

Let q and Q be any two minquats and q', Q' their transforms. We have

$$q' = a\,q\,\bar{a}^*\,, \quad Q' = a\,Q\,\bar{a}^*\,, \quad \bar{Q}' = a^*\,\bar{Q}\,\bar{a},$$

$$q'\,\bar{Q}' = a\,q\,\bar{a}^*\,a^*\,\bar{Q}\,\bar{a} = a\,q\,\bar{Q}\,\bar{a}, \qquad (4.36)$$

using the complex conjugate of (4.35). Likewise

$$Q'\,\bar{q}' = a\,Q\,\bar{q}\,\bar{a}.$$

Adding this to the preceding equation, we get

$$(q', Q') = a\,(q, Q)\,\bar{a} = (q, Q)\,a\,\bar{a} = (q, Q). \qquad (4.37)$$

Thus the transformation (4.34), for any complex a satisfying (4.35), conserves scalar products. But does it conserve minquat character ? Let us see:

$$q' = a\,q\,\bar{a}^*, \quad q'^* = a^*\,q^*\,\bar{a} = -\,a^*\,\bar{q}\,\bar{a}\,, \qquad (4.38)$$

since q is a minquat. But

$$\bar{q}' = a^*\,\bar{q}\,\bar{a} = -\,q'^*\,, \qquad (4.39)$$

which establishes the minquat character of q'. Thus the transformation (4.34) conserves minquat character and also scalar products. But what is the number of degrees of freedom ? We know that a Lorentz transformation has in general six degrees of freedom. Now a, as complex quaternion, contains four complex numbers, i.e. eight real numbers. The condition (4.35), being complex, is equivalent to two real conditions. Thus the transformation (4.34) has 8 - 2 = 6 degrees of freedom, the same as a Lorentz transformation. Since (4.34) conserves scalar products, we are entitled to conclude that it adequately represents all Lorentz transformations of one of the four classes. But which class ? This is answered by choosing, as we may, $\underline{a} = 1$. Then the transformation is the identity, $\underline{q}' = \underline{q}$, and this is surely a proper future-preserving transformation. Since change of orientation and future-reversal are discontinuous operations, they are inconsistent with (4.34), since any positive unit quaternion $\underline{a}$ can be reached continuously from $\underline{a} = 1$.

Thus although the construction of Lorentz transformations in quaternionic form by a succession of reflections is of interest, it is possible to start with (4.34) and verify directly that it represents the most general proper future-preserving Lorentz transformation. A similar verification can be carried out for the other three transformations shown in (4.33).

But is the positive unit quaternion $\underline{a}$ in (4.34) unique ? Might we not express the same transformation in the form

$$q' = b\, q\, \bar{b}^{*}, \quad b\, \bar{b} = 1, \tag{4.40}$$

with $b \neq a$? If we can, then

$$a\, q\, \bar{a}^{*} = b\, q\, \bar{b}^{*} \tag{4.41}$$

for all minquats $\underline{q}$. Let us multiply in front by $\underline{\bar{b}}$ and behind by $\underline{a}^{*}$. This gives

$$\bar{b}\, a\, q = q\, \bar{b}^* a^* ,$$

or

$$c\, q = q\, c^*, \quad c = \bar{b}\, a. \tag{4.42}$$

Put $\underline{q} = i$ (a minquat): this gives $c = c^*$ and so $\underline{c}$ is a real quaternion, say

$$c = c_1 e_1 + c_2 e_2 + c_3 e_3 + c_4 , \tag{4.43}$$

Put $q = e_1$ in (4.42): we get

$$- c_1 - c_2 e_3 + c_3 e_2 + c_4 e_1 = - c_1 + c_2 e_3 - c_3 e_2 + e_1 c_4 ,$$

and so $c_2 = c_3 = 0$. Put $q = e_2$: this gives $c_1 = 0$, and so $\underline{c}$ is merely a real number. But

$$c\, \bar{c} = \bar{b}\, a\, \bar{a}\, b = 1,$$

and so $c = \pm 1$. It follows from (4.42) that

$$b = \pm a. \tag{4.44}$$

Thus, if we are to preserve the transformation (4.34), the only change we can make in the unit quaternion a is to reverse its sign.

To sum up: Any unit quaternion a defines a Lorentz transformation as in (4.33). A Lorentz transformation defines a unit quaternion a up to an ambiguity in sign. Although the details have been filled in only for a proper future-preserving transformation, the statements are true for all four types in (4.33). It is of course obvious that those formulae are invariant under sign-reversal of the quaternion $\underline{a}$.

This duality is of importance since it corresponds to the duality of spin-transformations yielding a given Lorentz transformation. But for present purposes it is unimportant, and, in dealing with a Lorentz transformation, we shall assume that the unit quaternion a is given one of its two possible values.

5. General and singular Lorentz transformations.

Henceforth let us confine our attention to the simplest and most interesting type of Lorentz transformations, the proper future-preserving type. Under such a transformation, any minquat (or Minkowskian 4-vector) q goes into q' where

$$q' = a\, q\, \bar{a}^{*}, \qquad (5.1)$$

where a is any positive unit quaternion:

$$a\, \bar{a} = 1. \qquad (5.2)$$

A study of Lorentz transformations thus reduces to a study of positive unit quaternions, in general complex.

It is convenient to refer to a quaternion of the form

$$c_1 e_1 + c_2 e_2 + c_3 e_3$$

as a **vector**, c_1, c_2, c_3 being numbers, in general complex. Then $(a - \bar{a})$ is a vector, while $(a + \bar{a})$ is a number, and we may resolve the quaternion a into a vector and a number by writing the identity

$$a = \tfrac{1}{2}(a - \bar{a}) + \tfrac{1}{2}(a + \bar{a}). \qquad (5.3)$$

Then, from the unitary condition (5.2), we have

$$(a + \bar{a})^2 - (a - \bar{a})^2 = 4\, a\, \bar{a} = 4. \qquad (5.4)$$

We have now to consider two separate cases:

(i) General case: $\frac{1}{2}(a + \bar{a}) \neq \pm 1.$ (5.5)

(ii) Singular case: $\frac{1}{2}(a + \bar{a}) = \pm 1.$ (5.6)

We shall examine each in turn.

(i) General case. Noting that the square of a vector is a number, we can, in view of (5.4), define a complex number χ (uniquely except for sign) by the equations

$$\cos \chi = \tfrac{1}{2}(a + \bar{a}), \quad \sin^2 \chi = -\frac{1}{4}(a - \bar{a})^2 . \qquad (5.7)$$

Because of (5.5), χ is not zero or a multiple of π, and so (having chosen one sign for χ) we can define a vector I by

$$I \sin \chi = \tfrac{1}{2}(a - \bar{a}); \qquad (5.8)$$

We have then

$$I^2 = -1, \quad I + \bar{I} = 0, \qquad (5.9)$$

the last of these holding for any vector. Returning to (5.3), we note that in the general case any positive unit quaternion a may be expressed in the form

$$a = \cos \chi + I \sin \chi , \qquad (5.10)$$

where χ is a non-zero complex number. On account of (5.9), this may also be written

$$a = \exp(\chi I). \qquad (5.11)$$

Either of these expressions may be called the <u>first standard form</u> of $\underline{a}$ in the general case.

(ii) <u>Singular case</u>. In the singular case, (5.4) gives

$$(a - \bar{a})^2 = 0. \tag{5.12}$$

Define the vector $\underline{J}$ by

$$J = \tfrac{1}{2}(a - \bar{a}). \tag{5.13}$$

Then

$$J^2 = 0, \qquad J + \bar{J} = 0\,. \tag{5.14}$$

Since a reversal in the sign of $\underline{a}$ does not alter the transformation (5.1), we may arrange, without essential loss of generality, that the upper sign holds in (5.6). Then (5.3) gives

$$a = 1 + J. \tag{5.15}$$

This may be called the <u>first standard form</u> of a positive unit quaternion $\underline{a}$ in the singular case.

To sum up, the Lorentz transformations read as follows:

(i) General case:

$$q' = \exp(\chi I)\, q\, \exp(-\chi^* I^*), \qquad I^2 = -1, \quad I + \bar{I} = 0. \tag{5.16}$$

(ii) Singular case:

$$q' = (1 + J)\, q\, (1 - J^*), \qquad J^2 = 0, \quad J + \bar{J} = 0. \tag{5.17}$$

Note that the vectors $\underline{I}$ and $\underline{J}$ are in general complex vectors.

Let us check up on the numbers of parameters. A Lorentz transformation in general has six parameters. The complex quaternion $\underline{a}$ contains eight real parameters, but this is reduced to six by (5.2), which contains two real equations. In (5.16) the complex vector $\underline{\mathbf{I}}$ contains six real parameters, but these are reduced to four by $\mathbf{I}^2 = -1$; the two additional parameters, making six in all, are contained in χ. As for the singular case, the complex vector $\mathbf{J}$ contains six parameters, but these are reduced to two by $\mathbf{J}^2 = 0$. Thus <u>the singular Lorentz transformation contains only four parameters.</u>

It is of course to be understood that a 'singular Lorentz transformation' is one satisfying the condition (5.6); it is not singular in the usual sense of possessing no inverse. The inverse of (5.17) is

$$q = (1 - J)\, q'\, (1 + J^*). \tag{5.18}$$

The most familiar form of Lorentz transformation reads

$$x' = \gamma(x - vt),\ y' = y,\ z' = z,\ t' = \gamma(t - vx),\ \gamma^{-2} = 1 - v^2. \tag{5.19}$$

To show how this is included in (5.16), we first put $\mathbf{I} = e_1$, so that the transformation becomes

$$q' = (\cos\chi + e_1 \sin\chi)\, q\, (\cos\chi^* - e_1 \sin\chi^*)\ . \tag{5.20}$$

Substituting

$$\begin{aligned} q &= xe_1 + ye_2 + ze_3 + it, \\ q' &= x'e_1 + y'e_2 + z'e_3 + it', \end{aligned} \tag{5.21}$$

working out the right hand side of (5.20), and comparing it with the left hand side, we obtain the transformation

$$\begin{aligned} x' &= x\cos(\chi - \chi^*) + it\sin(\chi - \chi^*), \\ y' &= y\cos(\chi + \chi^*) - z\sin(\chi + \chi^*), \\ z' &= y\sin(\chi + \chi^*) + z\cos(\chi + \chi^*), \\ t' &= ix\sin(\chi - \chi^*) + t\cos(\chi - \chi^*). \end{aligned} \tag{5.22}$$

In the second and third of these equations we recognize a rotation in the (y,z)-plane. If we now choose $\chi = \frac{1}{2}i\phi$, with ϕ real, this rotation disappears, and we are left with

$$\begin{aligned} x' &= x\cosh\phi - t\sinh\phi, \\ y' &= y, \quad z' = z, \\ t' &= t\cosh\phi - x\sinh\phi. \end{aligned} \tag{5.23}$$

On putting $\tanh\phi = v$, this reduces to the familiar form (5.19).

It may be noted that when we take, as above, $\underline{I}$ real and χ a pure imaginary, then we have $\bar{a}^* = a$. Silberstein (1924, p. 154) did not make it clear that a transformation $q' = aqa$, although adequate to describe the transformation (5.19), does not in general preserve minquat character.

6. Invariant null rays.

(i) General case. With $\underline{I}$ as in (5.16), define quaternions $\underline{m}$ and $\underline{n}$ by

$$m = i(I + i)(I^* + i), \quad n = i(I - i)(I^* - i). \tag{6.1}$$

We have

$$\bar{m} = i(-I^* + i)(-I + i), \quad m^* = -i(I^* - i)(I - i), \tag{6.2}$$

and so

$$\bar{m} + m^* = 0. \tag{6.3}$$

Thus by (3.15) $\underline{m}$ (and likewise $\underline{n}$) is a minquat. Further, since

$$(I^* + i)(-I^* + i) = 0, \tag{6.4}$$

and the same equation holds with $\underline{i}$ changed to $-\underline{i}$, we have

$$m\,\bar{m} = 0, \quad n\,\bar{n} = 0. \tag{6.5}$$

Thus m and n are null minquats, i.e. 4-vectors lying on the null cone.

To see what happens to $\underline{m}$ under the Lorentz transformation (5.16), we write (remembering that χ is a number)

$$\begin{aligned} m \longrightarrow m' &= \exp(\chi\ I)\ m\ \exp(-\chi^*\ I^*) \\ &= \exp\ i(\chi - \chi^*)\ \exp\ \chi(I - i)\ i(I+i)(I^*+i)\ \exp[-\chi^*(I^* - i)]. \end{aligned} \tag{6.6}$$

When we expand the last exponential and multiply the series by $(I^* + i)$, every term except the first is destroyed by virtue of (6.4). A similar destruction of terms occurs with the middle exponential, and we get, with a similar result for $\underline{n}$,

$$m \longrightarrow m' = km, \quad n \longrightarrow n' = k^{-1}n, \tag{6.7}$$

where <u>k</u> is the real number

$$k = \exp i(\chi - \chi^*). \tag{6.8}$$

<u>Thus the null rays on which m and n lie are invariant under the Lorentz transformation, the null vectors being extended and contracted in the ratios k and k^{-1}</u>. The 2-flat containing <u>m</u> and <u>n</u> is unchanged as a whole by the transformation.

It follows that the 2-flat orthogonal to <u>m</u> and <u>n</u> is unchanged as a whole; in fact, it undergoes a rigid rotation. To examine this, we introduce the complex quaternions

$$\begin{aligned} P &= i(I + i)(I^* - i) = iII^* + i + I - I^*, \\ Q &= i(I - i)(I^* + i) = i I I^* + i - I + I^*. \end{aligned} \tag{6.9}$$

These are not minquats, but by addition and subtraction we obtain the minquats

$$r = \tfrac{1}{2}(P + Q) = i(I I^* + 1), \quad s = \tfrac{1}{2} i(P - Q) = i(I - I^*). \tag{6.10}$$

Under the Lorentz transformation (5.16), we get

$$\begin{aligned} P &\longrightarrow P' = P \exp i(\chi + \chi^*), \\ Q &\longrightarrow Q' = Q \exp [-i(\chi + \chi^*)], \end{aligned} \tag{6.11}$$

and hence

$$\begin{aligned} r &\longrightarrow r' = r \cos \theta + s \sin \theta \\ s &\longrightarrow s' = - r \sin \theta + s \cos \theta, \\ &\theta = \chi + \chi^* . \end{aligned} \tag{6.12}$$

It is easy to see that **r** and **s** are orthogonal to **m** and **n**. Thus the Lorentz transformation (5.16) rotates the 2-flat orthogonal to the invariant null rays through an angle $(\chi + \chi^*)$.

(ii) Singular case. With $\underline{J}$ as in (5.17), define $\underline{n}$ by

$$n = -iJJ^* . \tag{6.13}$$

Then

$$\begin{aligned} \bar{n} &= -iJ^* J, \quad n^* = iJ^* J, \quad \bar{n} + n^* = 0, \\ n\bar{n} &= -JJ^* J^* J = 0. \end{aligned} \tag{6.14}$$

Thus $\underline{n}$ is a null minquat. Under the singular Lorentz transformation (5.17) we get

$$\begin{aligned} n \longrightarrow n' &= (1+J)\, n\, (1-J^*) \\ &= -i(1+J)\, JJ^*\, (1-J^*) \\ &= -i\, JJ^*(1-J^*) = -iJJ^* = n. \end{aligned} \tag{6.15}$$

Thus the null minquat n is unchanged by the transformation, and indeed the whole null ray on which it lies is unchanged pointwise.

Define now two real vectors $\underline{r}$ and $\underline{s}$ as follows:

$$r = J + J^*, \qquad s = i(J - J^*). \tag{6.16}$$

It is easy to show that

$$r\bar{n} + n\bar{r} = 0, \quad s\bar{n} + n\bar{s} = 0, \quad r\bar{s} + s\bar{r} = 0, \tag{6.17}$$

so that $\underline{n}$, $\underline{r}$, $\underline{s}$ form an orthogonal triad; in fact, $\underline{r}$ and $\underline{s}$ lie in the 3-flat tangent to the null cone along $\underline{n}$. Under the transformation (5.17), we get

$$r \longrightarrow r' = (1 + J)(J + J^*)(1 - J^*)$$
$$= (1 + J)(J + J^* - J J^*)$$
$$= J + J^* = r, \tag{6.18a}$$

and

$$s \longrightarrow s' = (1 + J)i(J - J^*)(1 - J^*)$$
$$= i(1 + J)(J - J^* - J J^*)$$
$$= i(J - J^*) - 2i J J^*$$
$$= s + 2n. \tag{6.18b}$$

Thus $\underline{r}$, as well as $\underline{n}$, is unchanged, and $\underline{s}$ is pushed in the direction of $\underline{n}$. The 2-flat of $\underline{n}$ and $\underline{r}$ is unchanged pointwise, and the 3-flat of $\underline{n}$, $\underline{r}$ and $\underline{s}$ is unchanged as a whole. From the condition of Minkowskian rigidity, the transformation of this 3-space determines the transformation of all space-time.

Let us now obtain expressions for Lorentz transformations in terms of null minquats.

(i) General case.

Let us recall the null minquats $\underline{m}$ and $\underline{n}$ of (6.1):

$$\begin{aligned} m &= i(I + i)(I^* + i) = iII^* - I - I^* - i, \\ n &= i(I - i)(I^* - i) = iII^* + I + I^* - i. \end{aligned} \tag{6.19}$$

We are to solve these equations for $\underline{I}$, which also satisfies

$$I^2 = -1, \quad I + \overline{I} = 0.$$

We obtain directly

$$
\begin{aligned}
m\,\bar{n} &= -(I + i)(I^* + i)(-I^* - i)(-I - i) \\
&= -2i(I + i)(I^* + i)(I + i) \\
&= -2m(I + i), \\
n\,\bar{m} &= 2n(I - i).
\end{aligned}
$$

Multiplying these on the left by $\bar{n}$ and $\bar{m}$ respectively, and subtracting, we get

$$\bar{n}\,m\,\bar{n} - \bar{m}\,n\,\bar{m} = -4b\,I + 2i(\bar{m}\,n - \bar{n}\,m),$$

where $\underline{b}$ is the (real) scalar product

$$b = (m, n) = \tfrac{1}{2}(m\,\bar{n} + n\,\bar{m}) = \tfrac{1}{2}(\bar{m}\,n + \bar{n}\,m).$$

We have essentially solved for $\underline{I}$ above, but the expression can be simplified. Since $\underline{m}$ and $\underline{n}$ are null, we have, with the help of (6.19),

$$
\begin{aligned}
\bar{n}\,m\,\bar{n} - \bar{m}\,n\,\bar{m} &= \bar{n}(2b - n\,\bar{m}) - \bar{m}(2b - m\,\bar{n}) \\
&= 2b(\bar{n} - \bar{m}) \\
&= -4b(I + I^*),
\end{aligned}
$$

and so our equation for $\underline{I}$ becomes

$$
\begin{aligned}
-4b\,I^* &= 2i(\bar{m}\,n - \bar{n}\,m) \\
&= 4ib - 4i\,\bar{n}\,m,
\end{aligned}
$$

or

$$I^* + i = i\,\bar{n}\,mb^{-1}.$$

Taking the complex conjugate and remembering that $\underline{m}$ and $\underline{n}$ are minquats, we get

$$I - i = - i n \bar{m} b^{-1} .$$

Combining this with its Hamiltonian conjugate, we have what we need:

$$I + i = i m \bar{n} b^{-1} , \quad I - i = -i n \bar{m} b^{-1} . \tag{6.20}$$

Defining the complex scalar γ by

$$\gamma = \exp i \chi , \tag{6.21}$$

we may write (5.10) in the form

$$\begin{aligned} a &= \cos \chi + I \sin \chi \\ &= - \tfrac{1}{2} i \, [\gamma(I + i) - \gamma^{-1}(I - i)], \end{aligned} \tag{6.22}$$

and hence by (6.20)

$$a = (m \bar{n} + n \bar{m})^{-1} (\gamma m \bar{n} + \gamma^{-1} n \bar{m}). \tag{6.23}$$

This may be called the <u>second standard form</u> of a positive unit quaternion <u>a</u> in the general case. It displays <u>a</u> in terms of the two invariant null rays (<u>rays</u>, not <u>4-vectors</u>, since only the ratios of components are involved) and the complex scalar γ, related to χ by (6.21). The first factor in (6.23) is of course a real scalar. Since each null ray involves two parameters and γ involves two also, there are six parameters in (6.23), the correct number.

It takes only a moment to verify that <u>a</u> as in (6.23) satisfies $a\bar{a} = 1$, so that $\underline{a}$ does in fact yield a Lorentz transformation. It is typical of quaternion formulae that, though they may be difficult to find, once found they are immediately verifiable. Having found a formula such as (6.23), and having verified that it does yield a Lorentz transformation, one might be tempted to assert that the quaternion $\underline{a}$ of any proper future-preserving Lorentz transformation could be so written. But that would be false. For (6.23) is available only for a general Lorentz transformation, not for a singular one.

(ii) Singular case. Collected from (5.14), (5.15), (6.13), (6.16), the relevant formulae for the singular case are

$$a = 1 + J, \quad n = -iJJ^*, \quad r = J + J^*, \quad J^2 = 0, \quad J + \bar{J} = 0. \qquad (6.24)$$

We recall that $\underline{n}$ is the invariant null minquat and $\underline{r}$ an invariant real vector. Define a real number β by

$$\beta = -\tfrac{1}{2}(JJ^* + J^*J) = J_1J^*_1 + J_2J^*_2 + J_3J^*_3 \; ; \qquad (6.25)$$

this vanishes only in the trivial case where $J = 0$. Note that

$$r\bar{r} = -(J + J^*)(J + J^*) = 2\beta \; . \qquad (6.26)$$

Now

$$n\bar{r} = iJJ^*(J + J^*) = iJ(JJ^* + J^*J) = -2i\beta J, \qquad (6.27)$$

and so we have

$$a = 1 + \tfrac{1}{2}\, i\beta^{-1} n\,\bar{r} \; . \qquad (6.28)$$

This expresses $\underline{a}$ in terms of the invariant null minquat $\underline{n}$ and the invariant (real) vector $\underline{r}$. It may be called the <u>second standard form</u> of a positive unit quaternion in the singular case.

We have already seen that in the singular case $\underline{a}$ contains four parameters. It might appear that there were six parameters in (6.28) viz. three in $\underline{n}$ and three in $\underline{r}$. But one parameter is lost through the known orthogonality of $\underline{n}$ and $\underline{r}$, and a second parameter is lost since the expression in (6.28) is unchanged if $\underline{n}$ and $\underline{r}$ are both magnified in the same ratio.

We may also write (6.28) in the form

$$a = 1 + i\,\frac{n\,\bar{r}}{r\,\bar{r}}\,, \tag{6.29}$$

and the corresponding singular Lorentz transformation

$$q \longrightarrow q' = \left(1 + i\,\frac{n\,\bar{r}}{r\,\bar{r}}\right) q \left(1 + i\,\frac{\bar{n}\,r}{r\,\bar{r}}\right). \tag{6.30}$$

The singular Lorentz transformation may also be presented in a rather simpler form. Let $\underline{n}$ be a null minquat and $\underline{u}$ a unit minquat orthogonal to $\underline{n}$ (and so necessarily spacelike). We have then

$$n\,\bar{n} = 0, \quad u\,\bar{u} = 1, \quad n\,\bar{u} + u\,\bar{n} = 0. \tag{6.31}$$

Consider the quaternion

$$a = 1 + n\,\bar{u}\,. \tag{6.32}$$

Then

$$a\,\bar{a} = (1 + n\,\bar{u})(1 + u\,\bar{n})$$
$$= 1 + n\,\bar{u} + u\,\bar{n} + n\,\bar{u}\,u\,\bar{n} = 1, \qquad (6.33)$$

by (6.31). It is clear that $a + \bar{a} = 2$, and so

$$q \longrightarrow q' = a\,q\,\bar{a}^{*} = (1 + n\,\bar{u})q(1 + \bar{u}\,n) \qquad (6.34)$$

is a singular Lorentz transformation. Here we count three parameters in n and three in the unit minquat u, but we deduct one on account of their mutual orthogonality, and another one because a is unchanged by adding to u a minquat proportional to n; that leaves four parameters, the correct number.

We note that, to put (6.32) into the form 1 + J, as in (5.15), we are to put

$$J = n\,\bar{u}\,; \qquad (6.35)$$

We verify immediately with the aid of (6.31) that $J^2 = 0$, $J + \bar{J} = 0$, as in (5.14). We can check very easily that the transformation (6.34) leaves n unchanged. As for the invariant spacelike vector r, by (6.24) and (6.35) it is

$$r = J + J^{*} = n\,\bar{u} + \bar{n}\,u, \qquad (6.36)$$

where, as throughout, we make use of the relation (3.15) for minquats. (The right hand side of (6.36) would not be a vector but for the orthogonality of the minquats n and u.)

7. Resolution into minquats.

Any complex quaternion can be resolved uniquely into the complex sum of two minquats:

$$a = R + iS, \quad \bar{R} + R^{*} = 0, \quad \bar{S} + S^{*} = 0, \qquad (7.1)$$

where **R** and **S** are minquats. Since the minquat or 4-vector is a geometrical object in space-time, whereas a 3-vector (as **I** or **J** in the preceding work) is not, it might be expected that this resolution into minquats would be very useful in dealing with Lorentz transformations. Actually, the results are somewhat disappointing.

The condition that **a** is a positive unit quaternion ($a\bar{a} = 1$) leads to

$$R\bar{R} - S\bar{S} = 1, \quad R\bar{S} + S\bar{R} = 0, \qquad (7.2)$$

the second being the condition of orthogonality of **R** and **S**. There are the following possibilities (apart from the trivial $R = S = 0$):

(a) $R = 0$; S unit timelike.

(b) $S = 0$; R unit spacelike.

(c) $S\bar{S} > 0$; R and S spacelike orthogonal.

(d) $S\bar{S} = 0$, $S \neq 0$; R unit spacelike orthogonal to null S.

(e) $0 > S\bar{S} > -1$; R spacelike orthogonal to timelike S.

The Lorentz transformation reads

$$q \longrightarrow q' = (R + iS)q(-R + iS). \tag{7.3}$$

The condition for the singular case [cf. (5.6)] is that the scalar part of $\underline{a}$ should be ± 1. This means that $\underline{R}$ shall be a (real) vector,

$$R + \bar{R} = 0, \tag{7.4}$$

and that $\underline{S}$ shall be of the form

$$S = V \mp i, \tag{7.5}$$

where $\underline{V}$ is a (real) vector. The general case obtains when one or both of these conditions is violated.

For both general and singular cases, we have the following result for the transformation of $\bar{R}$ under (7.3):

$$\begin{aligned} \bar{R} \longrightarrow\ & (R + iS)\bar{R}(-R + iS) \\ &= (R\bar{R} + iS\bar{R})(-R + iS) \\ &= R\bar{R}(-R + iS) - iR\bar{S}(-R + iS) \\ &= R(-R\bar{R} + S\bar{S}) = -R. \end{aligned} \tag{7.6}$$

Thus, combining a similar result, we get

$$\bar{R} \longrightarrow -R, \quad \bar{S} \longrightarrow S. \tag{7.7}$$

Thus we know what happens to the 2-flat $(\bar{R}, \bar{S})$ as a whole: it goes into the 2-flat (R, S).

(i) General case. Identifying the unit quaternions in (5.10) and (7.1), we have

$$R + iS = \cos\chi + I\sin\chi \tag{7.8}$$

The quantities on the right hand side have already been given geometrical meanings (rotations, null minquats), and by identifying the real and imaginary parts and the vector and scalar parts of the two sides, we can transfer those geometrical meanings to R and S. But the resulting formulae are complicated and therefore uninteresting.

(ii) Singular case. This is somewhat more rewarding. Identifying (5.15) and (7.1), we have

$$R + iS = 1 + J. \tag{7.9}$$

Let us split the complex vector J into real and imaginary parts:

$$J = L + iM. \tag{7.10}$$

Here L and M are real vectors. The condition $J^2 = 0$ gives

$$L^2 - M^2 = 0, \quad LM + ML = 0, \tag{7.11}$$

so that L and M are equal orthogonal vectors. Remembering that R and S are minquats, (7.9) gives

$$R = L, \quad S = M - i. \tag{7.12}$$

The invariant null minquat $\underline{n}$ of (6.13) is

$$\begin{aligned} n &= -i\, JJ^* = -i(L + iM)(L - iM) \\ &= -LM + ML - i(L^2 + M^2) \\ &= -R(S + i) + (S + i)R - iR^2 - i(S + i)^2 \\ &= -RS + SR - iR^2 - i(S + i)^2. \end{aligned} \tag{7.13}$$

The invariant vector $\underline{r}$ of (6.16) is simply

$$r = J + J^* = 2L = 2R. \tag{7.14}$$

In fact, $\underline{R}$ itself is an invariant vector under the singular Lorentz transformation.

8. Physical interpretations of general and singular Lorentz transformations.

When we think in physical terms about a Lorentz transformation, the primary concept is that of two frames of reference in uniform relative motion. The choice of spatial axes by each of the two observers is a somewhat secondary matter from this standpoint. Unfortunately this separation of primary and secondary is not maintained in the mathematical theory. A transformation of spatial axes, quite apart from a change in the frame of reference, changes the quaternion which generates the transformation. Indeed, the distinction between general and singular Lorentz transformations, although interesting and important in the mathematical theory, is really quite trivial when regarded from a physical standpoint. It amounts to no more than a special cooperation between the two observers in their choices of spatial axes. This is explained below.

(i) General case. In a general Lorentz transformation (we consider only those which are proper and future-preserving), there are two (and only two) null rays which are conserved. There are in fact just two null 4-vectors which maintain their directions in space-time, one being magnified and the other contracted in the same ratio as in (6.7). Now the momentum and energy of a photon are the four components of a null 4-vector or minquat, and so we can give the argument a physical turn by speaking of photons. Taking the speed of light to be unity, the photonic minquat is

$$h\nu(u + i) \tag{8.1}$$

where h is Planck's constant, ν the frequency, and u a unit vector drawn in the direction of the photon's motion.

To magnify a photonic minquat by a factor amounts to multiplying its frequency by that factor, leaving its direction unchanged. The existence of the two invariant null rays may then be interpreted as follows. Imagine photons travelling in all directions, but only one photon in each direction. Let them be observed by two observers, S and S', in uniform relative motion. Any particular photon will, in general, appear to be travelling in different directions (i.e. with different direction cosines) and with different frequencies when viewed by S and S'. But in the totality of photons, there exist two, and only two, for which the directions appear the same to S and S'. These correspond to the invariant null rays. We may call these two photons communal photons.

If either observer changes his spatial axes, or if both observers do so, after the change there will again be two communal photons, but they will not be the same photons as before. If the observers play with their spatial axes, they can arrange that the two communal photons have direction cosines (1,0,0) and (-1,0,0) for both observers. If that is done, and if there is a further cooperation with regard to the directions of the spatial axes perpendicular to the direction of the photons' motion, the Lorentz transformation connecting the observations of S and S' takes the simple standard form (2.1). The magnification and contraction of the null minquats imply Doppler effects (violet-shift and red-shift) connecting the frequency observations made by S and S' on these two communal photons.

(ii) Singular case. In the singular case, there is only one communal photon, not two, and since there is, as in (6.15), an invariant null minquat (not merely an invariant null ray), there is no Doppler shift. Again, given two observers, S and S', in uniform relative motion, it is a question of playing with the spatial axes in order to achieve the desired result when moving photons are observed as described above. What the observers have to do is to change their spatial axes in such a way that, of all the photons, there is just one which appears to both S and S' to be travelling in the same direction (i.e. with the same direction cosines) and to have the same frequency. If that is done, then the Lorentz transformation connecting S and S' is singular.

To illustrate this, and to show the quaternionic method at work, consider the complex vector

$$J = -k(e_3 - ie_1). \tag{8.2}$$

Here k is a real number; a complex number would complicate the calculations a little, without any real gain in generality. We have $J^2 = 0$, and so, as in (5.17),

$$q' = (1 + J)q(1 - J^*) \tag{8.3}$$

is a singular Lorentz transformation. Since

$$J^* = -k(e_3 + ie_1), \tag{8.4}$$

the invariant null minquat is, by (6.13),

$$n = -i\,J\,J^* = 2\,k^2(e_2 + i). \tag{8.5}$$

Explicitly, (8.3) gives

$$x'e_1 + y'e_2 + z'e_3 + it'$$
$$= (1 - ke_3 + ike_1)(xe_1 + ye_2 + ze_3 + it)(1 + ke_3 + ike_1). \tag{8.6}$$

To obtain the transformation $(x,y,z,t) \longrightarrow (x',y',z',t')$, all we have to do is to work out the right hand side, reducing it to quaternionic form, and then equate coefficients of quaternionic elements. The result is

$$\begin{aligned} x' &= x + 2k(y - t), \\ y' &= y - 2kx - 2k^2(y - t), \\ z' &= z, \\ t' &= t - 2kx - 2k^2(y - t). \end{aligned} \tag{8.7}$$

It may be immediately checked that

$$x'^2 + y'^2 + z'^2 - t'^2 = x^2 + y^2 + z^2 - t^2. \tag{8.8}$$

The transformation may also be written

$$x' = x + 2k(y - t),\ y' - t' = y - t,\ z' = z,\ t' + kx' = t - kx. \tag{8.9}$$

To see what 4-vectors are invariant under this transformation, we put $x' = x$, $y' = y$, $z' = z$, $t' = t$; we find that there is a double infinity of invariant 4-vectors, viz. those satisfying the two conditions $x = 0$, $y = t$. The invariant null 4-vectors must satisfy the further condition $z = 0$, and so we have a single infinity of invariant null 4-vectors all lying on a single null ray, i.e. the particular null minquat $\underline{n}$ of (8.5), multiplied by an arbitrary real constant.

In order to study, in physical terms, the relationship between two observers S and S' connected by the Lorentz transformation (8.7) or (8.9), we must not attempt to use a single diagram to depict the two spaces. We need two, as in Fig. 1. Since the coordinates $\underline{z}'$, $\underline{z}$ are connected by the identical transformation, we can suppress the z-coordinates, and show merely the axes Oxy and O'x'y'.

We note that if S sees a photon travelling along Oy with frequency ν, then S' sees that photon travelling along O'y' with that same frequency ν; this follows from (8.5).

Let $(\underline{u},\underline{v})$ be the components on Oxy of the velocity of S' as observed by S. To find these components, we write down the inverse of (8.7):

$$\begin{aligned} x &= x' - 2k(y' - t'), \\ y &= y' + 2kx' - 2k^2(y' - t'), \\ z &= z', \\ t &= t' + 2kx' - 2k^2(y' - t'). \end{aligned} \tag{8.10}$$

The transformation matrix is obtained from that of (8.7) simply by reversing the sign of $\underline{k}$. We now put $dx' = dy' = dz' = 0$, in order to follow a point fixed in S', and obtain

$$u = dx/dt = 2k(1 + 2k^2)^{-1}, \quad v = dy/dt = 2k^2(1 + 2k^2)^{-1}. \tag{8.11}$$

The components (u',v') on O'x'y' of the velocity of S as observed by S' are similarly found by putting $dx = dy = dz = 0$ in (8.7):

$$u' = dx'/dt' = -2k(1 + 2k^2)^{-1}, \quad v' = dy'/dt' = 2k^2(1 + 2k^2)^{-1} \tag{8.12}$$

These velocities are shown in Fig. 1.

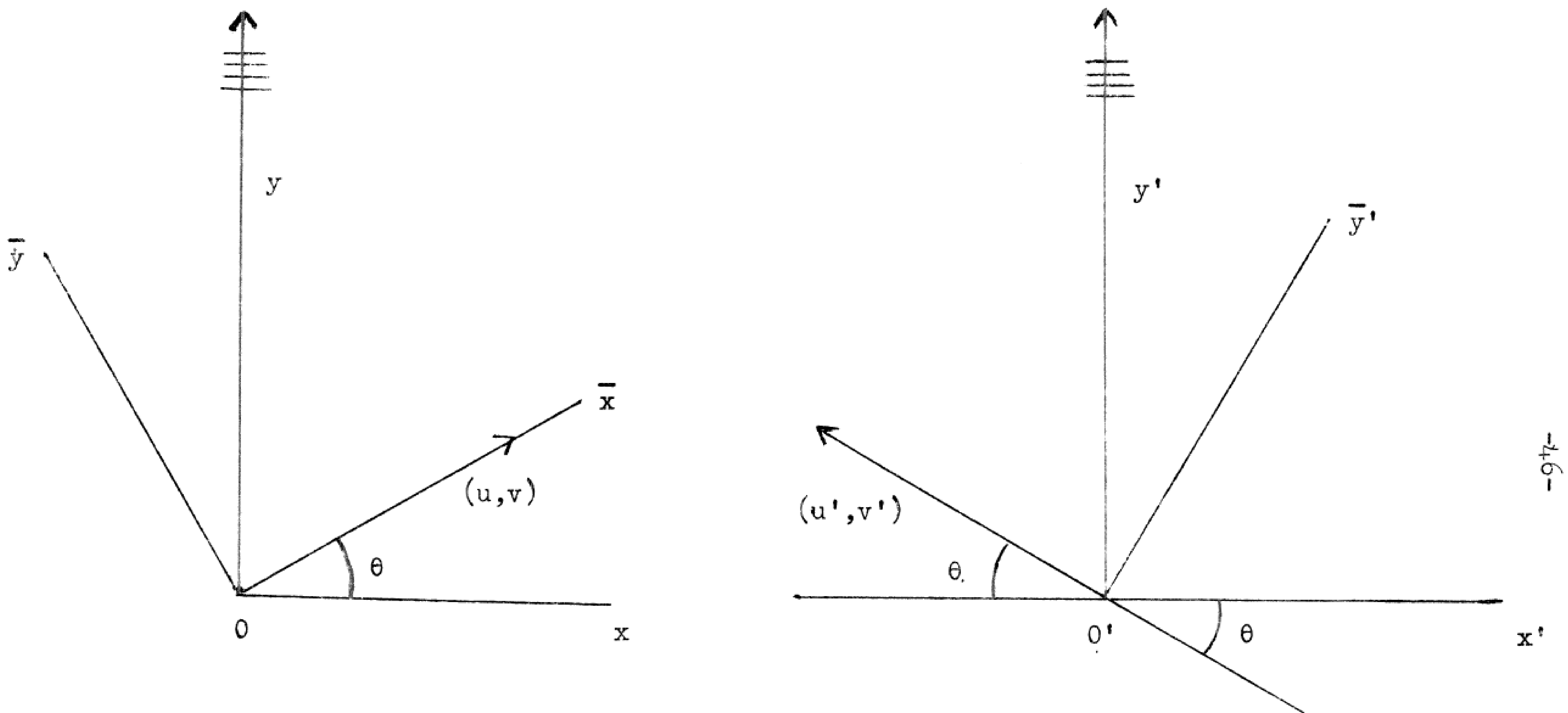

Fig.1. The spaces of two observers, S and S', with axes Oxy and O'x'y' such that the Lorentz transformation is of the singular type. A photon observed by S to have frequency ν and to be travelling along $\underline{Oy}$ appears to S' to have the same frequency ν and to be travelling along $\underline{O'y'}$. The vector $(\underline{u},\underline{v})$ is the velocity of S' relative to S, and $(\underline{u}',\underline{v}')$ is the velocity of S relative to S'. If the observers rotate their axes to $\underline{O}\ \bar{\underline{x}}\ \bar{\underline{y}}$ and $\underline{O}'\bar{\underline{x}}'\bar{\underline{y}}'$, the Lorentz transformation assumes the usual standard form.

We shall now show that the Lorentz transformation (8.7) of singular type may be changed to the usual standard form by a very simple operation. Let S rotate his axes to $\underline{O}\,\underline{\bar{x}}\,\underline{\bar{y}}$ with $\underline{O}\,\underline{\bar{x}}$ along $(\underline{u},\underline{v})$, and let S' rotate his axes to $\underline{O}'\underline{\bar{x}}'\underline{\bar{y}}'$ with $\underline{O}'\underline{\bar{x}}'$ in the direction of $(\underline{u}',\underline{v}')$ reversed. With θ as shown in the figure, so that

$$\tan\theta = k, \tag{8.13}$$

we have then the formulae

$$\begin{aligned} x &= \bar{x}\cos\theta - \bar{y}\sin\theta, & x' &= \bar{x}'\cos\theta + \bar{y}'\sin\theta, \\ y &= \bar{x}\sin\theta + \bar{y}\cos\theta, & y' &= -\bar{x}'\sin\theta + \bar{y}'\cos\theta, \\ z &= \bar{z}, \quad t = \bar{t}, & z' &= \bar{z}', \quad t' = \bar{t}'. \end{aligned} \tag{8.14}$$

Substituting these expressions in (8.7) and solving for $(\underline{\bar{x}}', \underline{\bar{y}}', \underline{\bar{z}}', \underline{\bar{t}}')$, we get

$$\bar{x}' = \Gamma(\bar{x} - V\,\bar{t}), \quad \bar{y}' = \bar{y}, \quad \bar{z}' = \bar{z}, \quad \bar{t}' = \Gamma(\bar{t} - V\,\bar{x}), \tag{8.15}$$

where

$$V = \frac{2\sin\theta}{1+\sin^2\theta}, \quad \Gamma = 1 + 2\tan^2\theta = 1 + 2k^2, \tag{8.16}$$

so that $\Gamma^{-2} = 1 - V^2$. We recognize in (8.15) the usual form (2.1) of Lorentz transformation, the relative speed of the observers being V.

When the axes have been rotated in this way, the photon indicated in Fig. 1 is no longer a communal photon. Its frequency is still the same when observed by S and by S', but it moves in different directions; for S its direction cosines are $(\sin\theta, \cos\theta)$, but for S' its direction cosines are $(-\sin\theta, \cos\theta)$.

9. The Conway-Dirac-Eddington matrices.

In two papers Conway (1937, 1945) showed briefly the connection of quaternions with the Dirac-Eddington matrices. For the sake of physicists who wish to use this notation, it is desirable to set out the argument with logical completeness, with reference in particular to the five anti-commuting Dirac matrices, which, in the notation of Pauli (1958, p. 143), are as follows:

$$\gamma_1 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \qquad \gamma_2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma_3 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \qquad \gamma_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{9.1}$$

$$\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

Let $\underline{a}$ and $\underline{b}$ be any two quaternions. Then

$$q \longrightarrow q' = a\,q\,b \tag{9.2}$$

is a linear transformation for the elements (q_1, q_2, q_3, q_4), and may be written in matrix form

$$q' = M(a,b)\, q, \tag{9.3}$$

where M is a 4×4 matrix and q, q' are 4×1 column matrices. (There is no question of minquats here; all elements are complex.) The notation indicates that M(a,b) is determined by <u>a</u> and <u>b</u>.

Consider a pair of linear transformations and their resultant:

$$q' = aqb, \quad q'' = cq'd, \quad q'' = caqbd. \tag{9.4}$$

In matrix form we have

$$q' = M(a,b)q, \quad q'' = M(c,d)q', \quad q'' = M(c,d)M(a,b)q. \tag{9.5}$$

Therefore

$$M(c,d)\, M(a,b) = M(ca, bd). \tag{9.6}$$

In particular,

$$[M(a,b)]^2 = M(a^2, b^2). \tag{9.7}$$

If we put $a = b = -1$ in (9.2), we get the identical transformation, and therefore $M(-1, -1) = I$, the unit matrix. Hence, by (9.7), we have the following basic theorem: <u>If a and b are any two quaternions satisfying</u>

$$a^2 = b^2 = -1, \tag{9.8}$$

then

$$[M(a,b)]^2 = I; \tag{9.9}$$

in fact, the matrix generated by a and b is a square root of unity.

Conway proposed the suggestive notation

$$M(a,b) = a(\ \ )b. \tag{9.10}$$

Now any two of the four quaternions e_1, e_2, e_3, i satisfy (9.8), and so we have at once the following sixteen matrices, each a square root of unity:

$$\begin{array}{cccc} e_1(\ \ )e_1 & e_1(\ \ )e_2 & e_1(\ \ )e_3 & e_1(\ \ )i \\ e_2(\ \ )e_1 & e_2(\ \ )e_2 & e_2(\ \ )e_3 & e_2(\ \ )i \\ e_3(\ \ )e_1 & e_3(\ \ )e_2 & e_3(\ \ )e_3 & e_3(\ \ )i \\ i(\ \ )e_1 & i(\ \ )e_2 & i(\ \ )e_3 & i(\ \ )i \end{array} \tag{9.11}$$

It is easy to calculate these matrices explicitly. For example, to calculate the first, we write

$$\begin{aligned} q_1'e_1 + q_2'e_2 + q_3'e_3 + q_4' &= e_1(q_1e_1 + q_2e_2 + q_3e_3 + q_4)e_1 \\ &= e_1(-q_1 - q_2e_3 + q_3e_2 + q_4e_1) \\ &= -q_1e_1 + q_2e_2 + q_3e_3 - q_4, \end{aligned} \tag{9.12}$$

so that the transformation is

$$q_1' = -q_1,\ q_2' = q_2,\ q_3' = q_3,\ q_4' = -q_4. \tag{9.13}$$

Thus $e_1(\ \ )e_1 = \mathrm{diag}(-1, 1, 1, -1)$.

The whole set of sixteen Conway matrices is as follows:

$$
\begin{array}{cccc}
e_1(\ )e_1 & e_1(\ )e_2 & e_1(\ )e_3 & e_1(\ )i \\
\begin{matrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{matrix} &
\begin{matrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{matrix} &
\begin{matrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{matrix} &
\begin{matrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{matrix} \\
e_2(\ )e_1 & e_2(\ )e_2 & e_2(\ )e_3 & e_2(\ )i \\
\begin{matrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{matrix} &
\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{matrix} &
\begin{matrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{matrix} &
\begin{matrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{matrix} \\
e_3(\ )e_1 & e_3(\ )e_2 & e_3(\ )e_3 & e_3(\ )i \\
\begin{matrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{matrix} &
\begin{matrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{matrix} &
\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{matrix} &
\begin{matrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{matrix} \\
i(\ )e_1 & i(\ )e_2 & i(\ )e_3 & i(\ )i \\
\begin{matrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{matrix} &
\begin{matrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{matrix} &
\begin{matrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{matrix} &
\begin{matrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{matrix}
\end{array}
\tag{9.14}
$$

The square of each of these matrices is unity. The set does not contain the unit matrix but its negative.

Comparing the above with (9.1), we see that they are related to the Dirac matrices as follows:

$$-\gamma_1 = i(\ \ )e_1,\quad -\gamma_2 = e_2(\ \ )e_3,\ \gamma_3 = i(\ \ )e_2,\ \gamma_4 = e_3(\ \ )e_3,$$
$$\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4 = e_1(\ \ )e_3\ . \tag{9.15}$$

In multiplying these, we are to remember the rule contained in (9.6): the quaternions belonging to the <u>first</u> factor are to be placed <u>outside</u>. Thus

$$-\gamma_2\gamma_3 = e_2 i(\ \ )e_2 e_3 = ie_2(\ \ )e_1,$$
$$-\gamma_3\gamma_2 = ie_2(\ \ )e_3 e_2 = -ie_2(\ \ )e_1\ . \tag{9.16}$$

Eddington (1946, p. 142) preferred to use matrices with square − I. The connection between Eddington's matrices and those of Conway is as follows:

$$e_1(\ \ )e_1 = iE_{14},\ e_1(\ \ )e_2 = -iE_{24},\ e_1(\ \ )e_3 = -iE_{34},\ e_1(\ \ )i = -iE_{05},$$
$$e_2(\ \ )e_1 = iE_{01},\ e_2(\ \ )e_2 = -iE_{02},\ e_2(\ \ )e_3 = -iE_{03},\ e_2(\ \ )i = -iE_{45},$$
$$e_3(\ \ )e_1 = iE_{15},\ e_3(\ \ )e_2 = -iE_{25},\ e_3(\ \ )e_3 = -iE_{35},\ e_3(\ \ )i = -iE_{12},$$
$$i(\ \ )e_1 = -iE_{23},\ i(\ \ )e_2 = iE_{31},\ i(\ \ )e_3 = -iE_{04},\ i(\ \ )i = iE_{16}. \tag{9.17}$$

The minus signs occurring in (9.15) and (9.17) are not perhaps of much importance, but it does seem of importance that there should be a standard notation for the set of sixteen matrices. If we write $e_4 = i$, then the whole set of Conway matrices is contained in the symbol $e_m(\ \ )e_n$ with <u>m</u> and <u>n</u> taking the values

1,2,3,4. This is so simple and natural that there is much to be said for accepting the Conway matrices as standard.

Multiplication of Conway matrices is very simple, as indicated in (9.16). It is evident that any two of the matrices must either commute or anti-commute. By displaying the matrices in the table

$$\begin{array}{lllll} e_1(\ \ )e_1 & e_2(\ \ )e_1 & e_3(\ \ )e_1 & i(\ \ )e_2 & i(\ \ )e_3 \\ e_1(\ \ )e_2 & e_2(\ \ )e_2 & e_3(\ \ )e_2 & i(\ \ )e_3 & i(\ \ )e_1 \\ e_1(\ \ )e_3 & e_2(\ \ )e_3 & e_3(\ \ )e_3 & i(\ \ )e_1 & i(\ \ )e_2 \\ e_2(\ \ )i & e_3(\ \ )i & e_1(\ \ )i & & \\ e_3(\ \ )i & e_1(\ \ )i & e_2(\ \ )i & & \end{array} \qquad (9.18)$$

Conway was able to state the commutation law very simply: <u>If two matrices appear in the same row or column they anti-commute; otherwise they commute</u>. This may be verified by inspection if we recall that, to find a product, we take the second factor and embrace it with the quaternions occurring in the first factor. Note that $i(\ \ )i$ is absent from (9.18), while each matrix with one i-factor appears <u>twice</u>.

The Conway matrices lend themselves to a geometrical representation in which the anti-commuting pentads and triads can be shown. We start with a tetrahedron with vertices labelled e_1, e_2, e_3, i (Fig. 2). The sixteen matrices are then put into correspondence with the four vertices and twelve directed edges

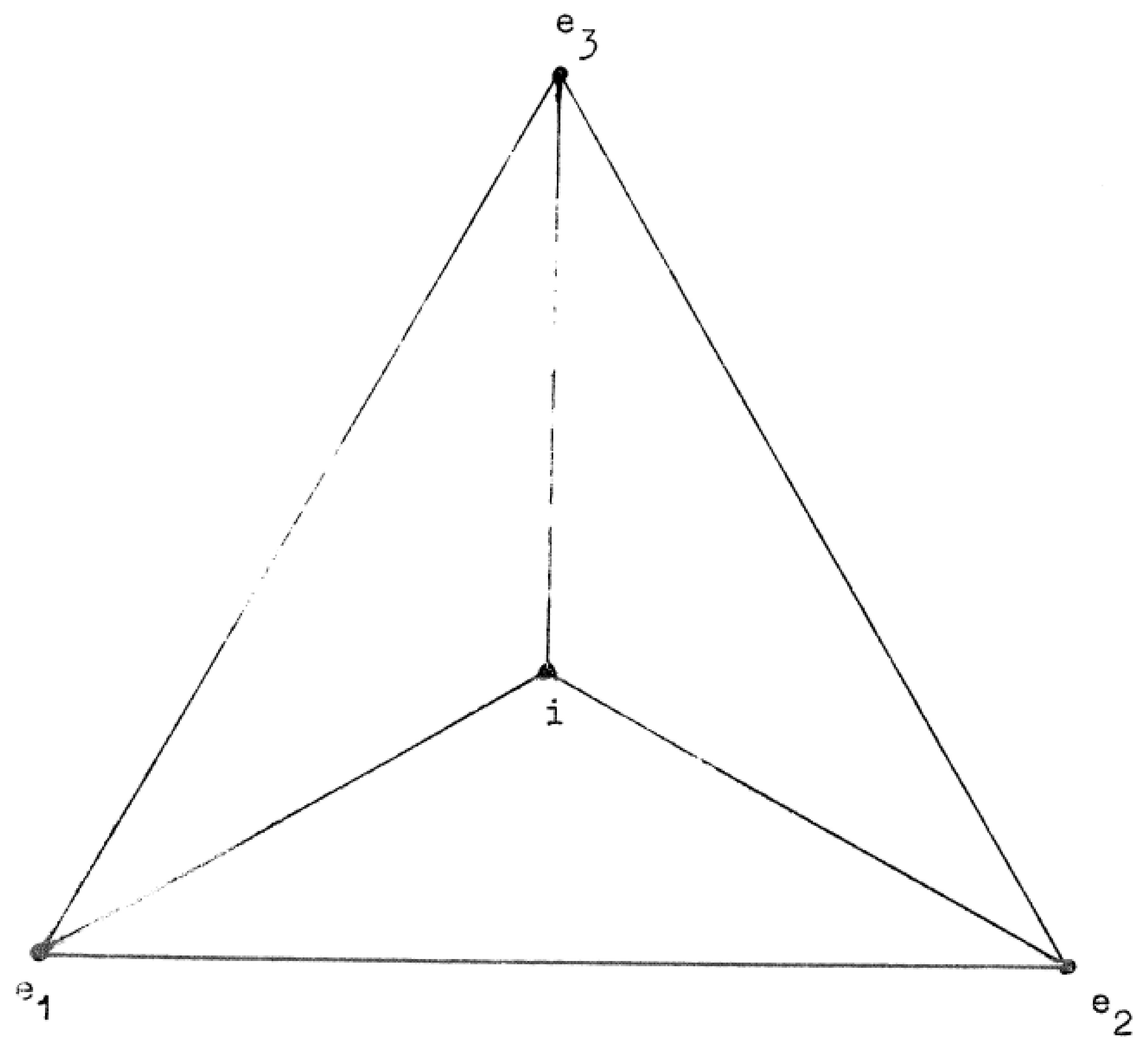

Fig. 2. The elements e_1, e_2, e_3, i (= e_4) associated with the vertices of a tetrahedron.

according to the following plan. With $e_4 = i$, the matrix $e_m(\ \ )e_m$ is associated with the vertex e_m and the matrix $e_m(\ \ )e_n$ with the directed edge running from e_m to e_n (Fig. 3).

Figs. 4 and 5 show two typical anti-commuting pentads corresponding to the first row and the first column in (9.18); Figs. 6 and 7 show the two anti-commuting triads corresponding to the fourth or fifth row and the fourth or fifth column in (9.18).

It is well known that the choice of basic γ's as in (9.15) is to some extent arbitrary. The essential requirement is that they should be four members of an anti-commuting pentad. Since there are six such pentads, there are 24 possible choices, if we disregard the order in which the subscripts 1,2,3,4 are distributed. Looking at the table (9.18), we may ask whether there is any reason to prefer one choice to another.

In making a choice, it is necessary to decide which of the three coordinates $\underline{x}$, $\underline{y}$, $\underline{z}$ should be associated with $\underline{t}$. In elementary treatments of the Lorentz transformation, the association is ($\underline{x}$, $\underline{t}$) as in (2.1). But in dealing with spinors, the usual association is ($\underline{z}$, $\underline{t}$) and this corresponds to the association (e_3, i). Now in each of the pentads in (9.18), $\underline{i}$ occurs twice, and in order to treat $\underline{i}$ symmetrically with respect to e_1 and e_2 we pick out the third row or the third column. Should we wish to reduce the number of basic matrices to four, we must throw away one member of a pentad, and symmetry tells us to throw away $e_3(\ \ )e_3$. Thus Conway's table suggests as the proper choice of basic matrices one of the following sets, related to the standard γ's as indicated:

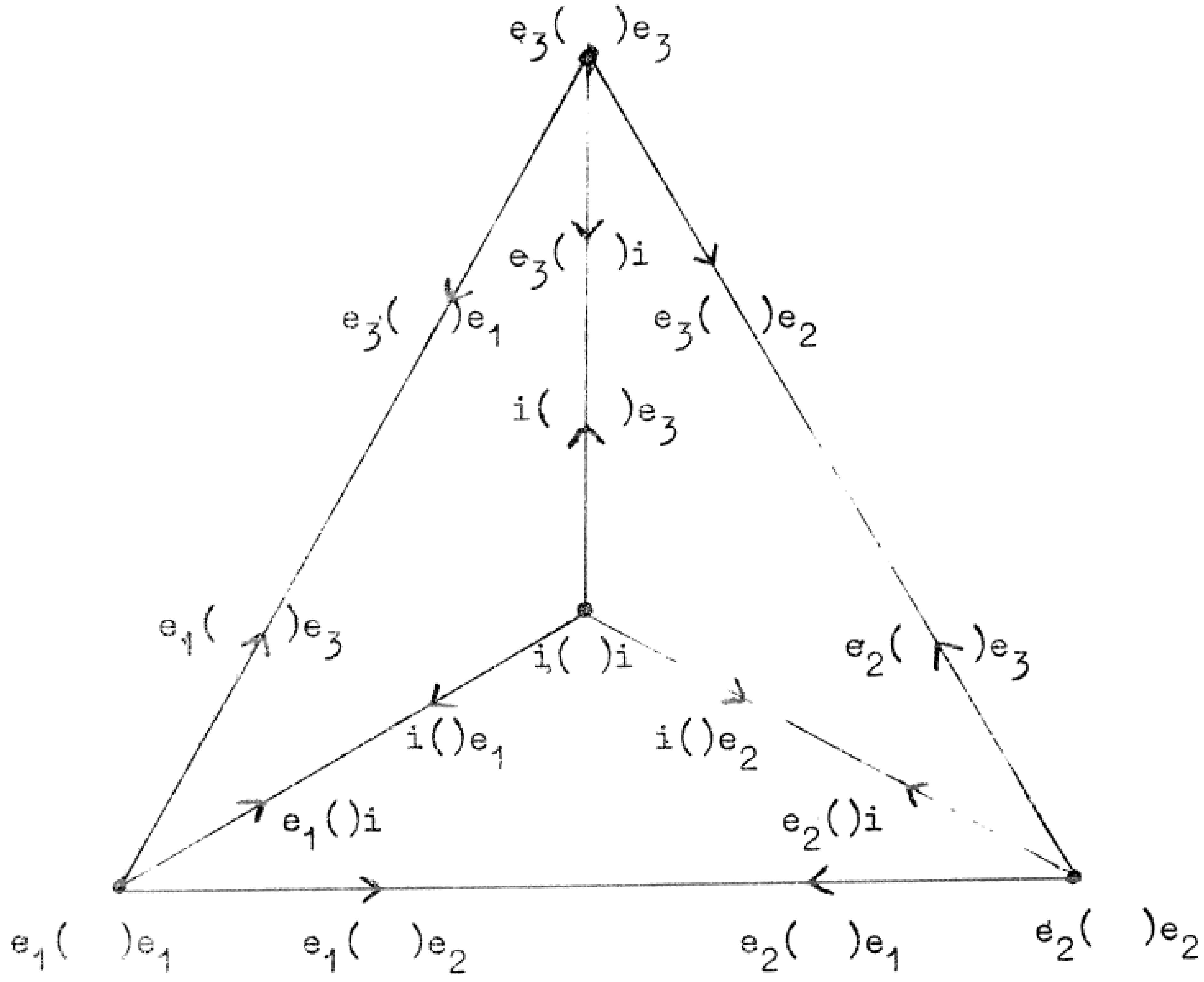

Fig. 3. The Conway matrices $e_m(\ \ )e_n$ associated with the vertices and directed edges of the tetrahedron.

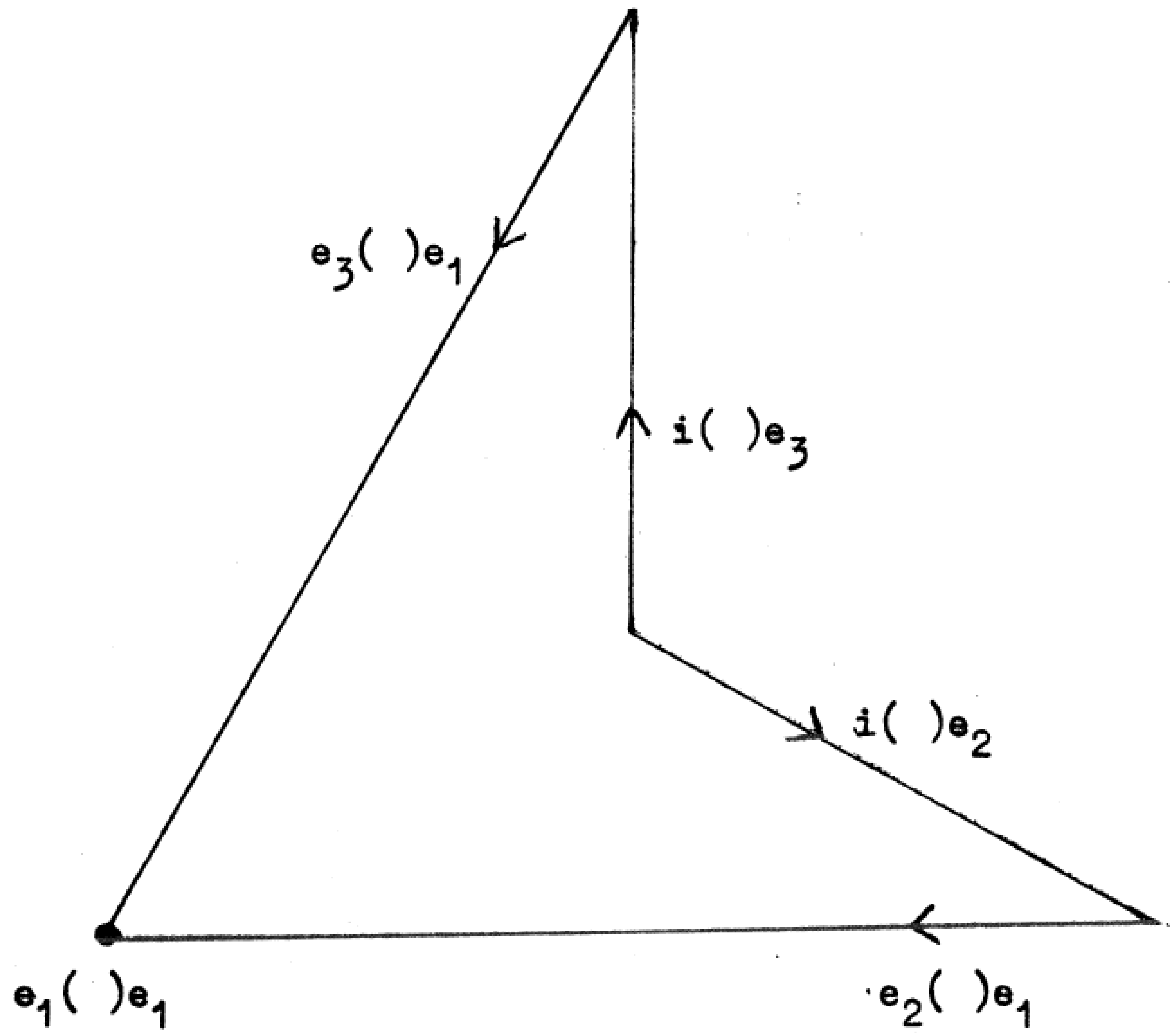

Fig. 4. An anti-commuting pentad (there are three of this type).

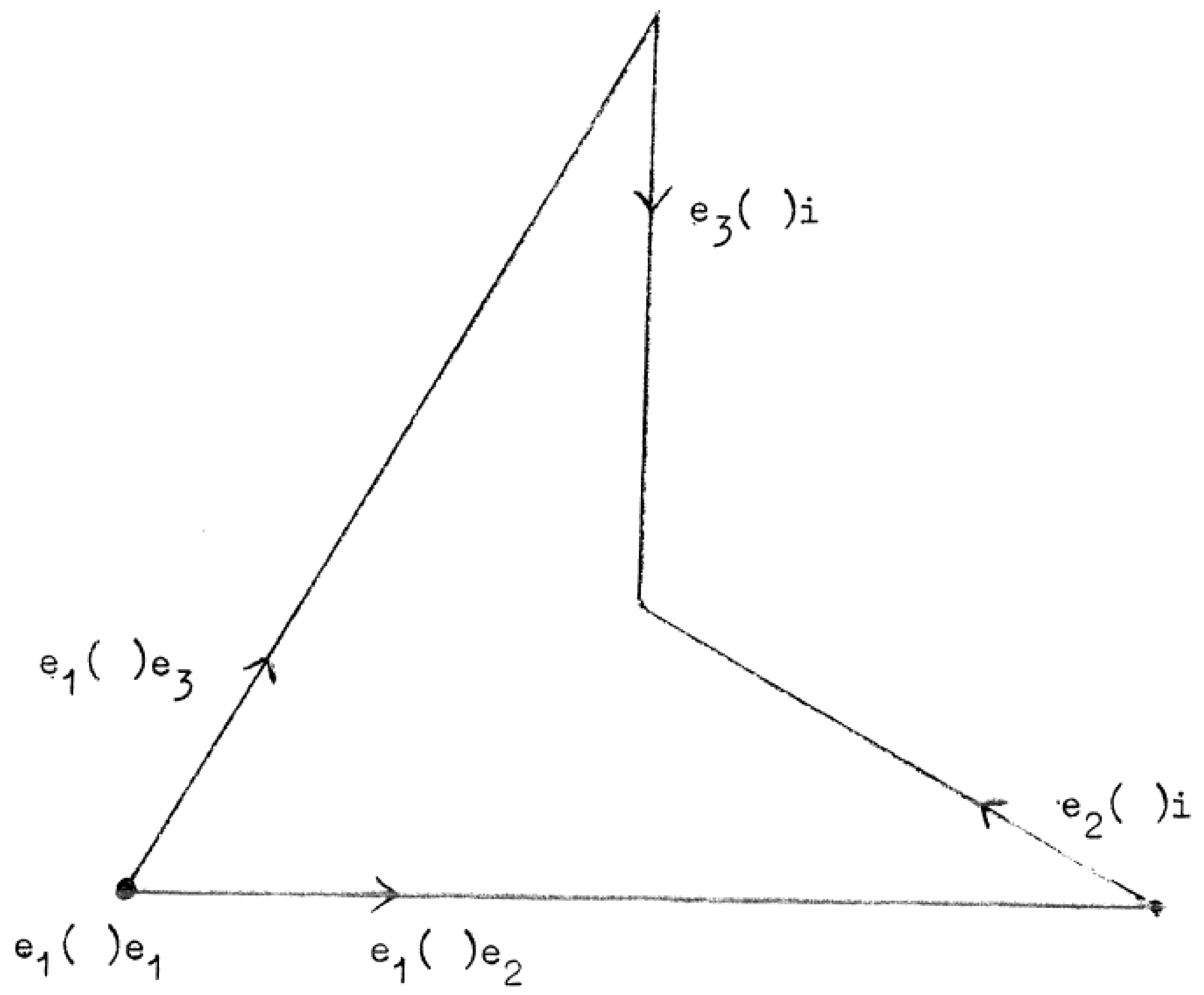

Fig. 5. An anti-commuting pentad (there are three of this type).

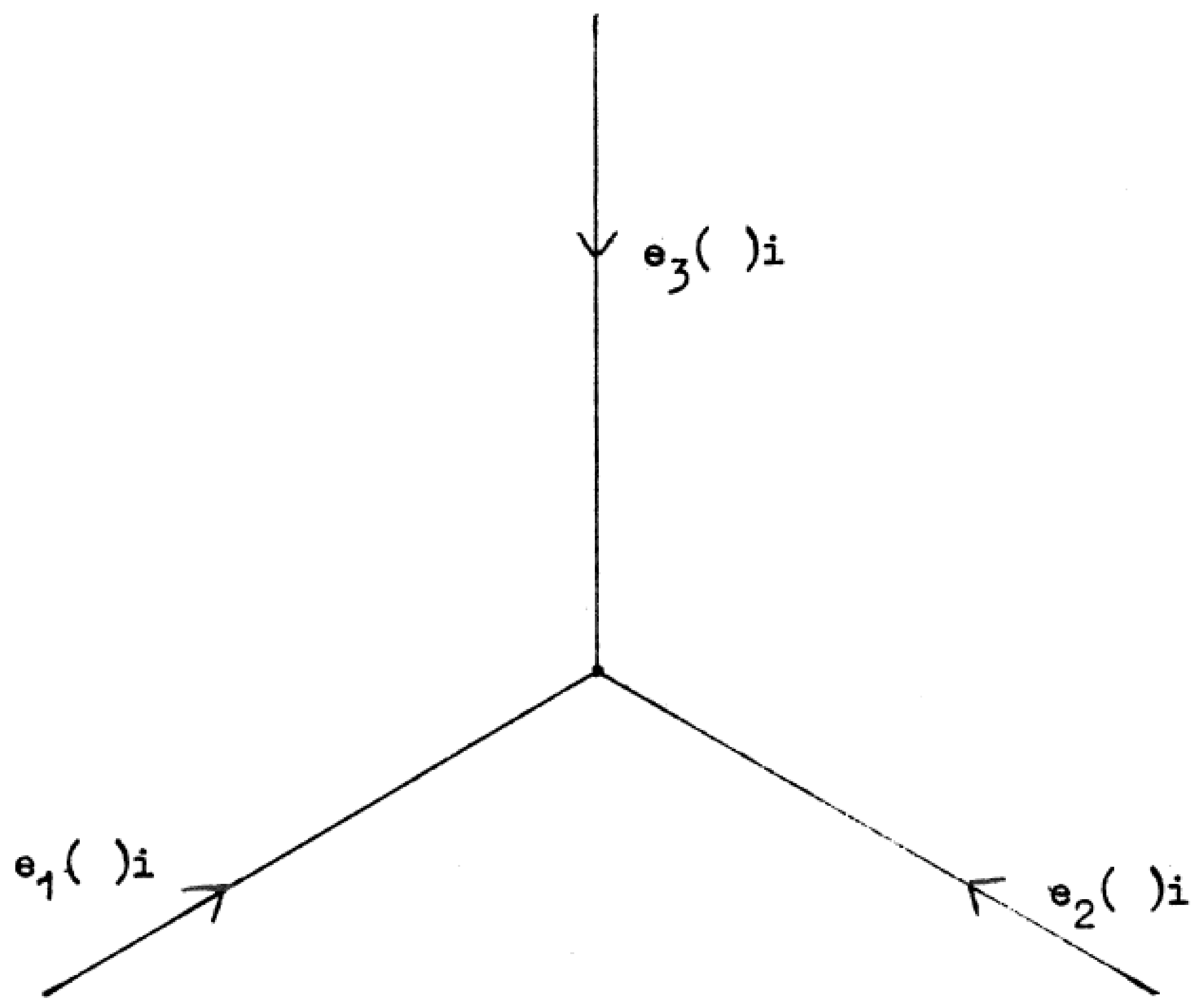

Fig. 6. First anti-commuting triad.

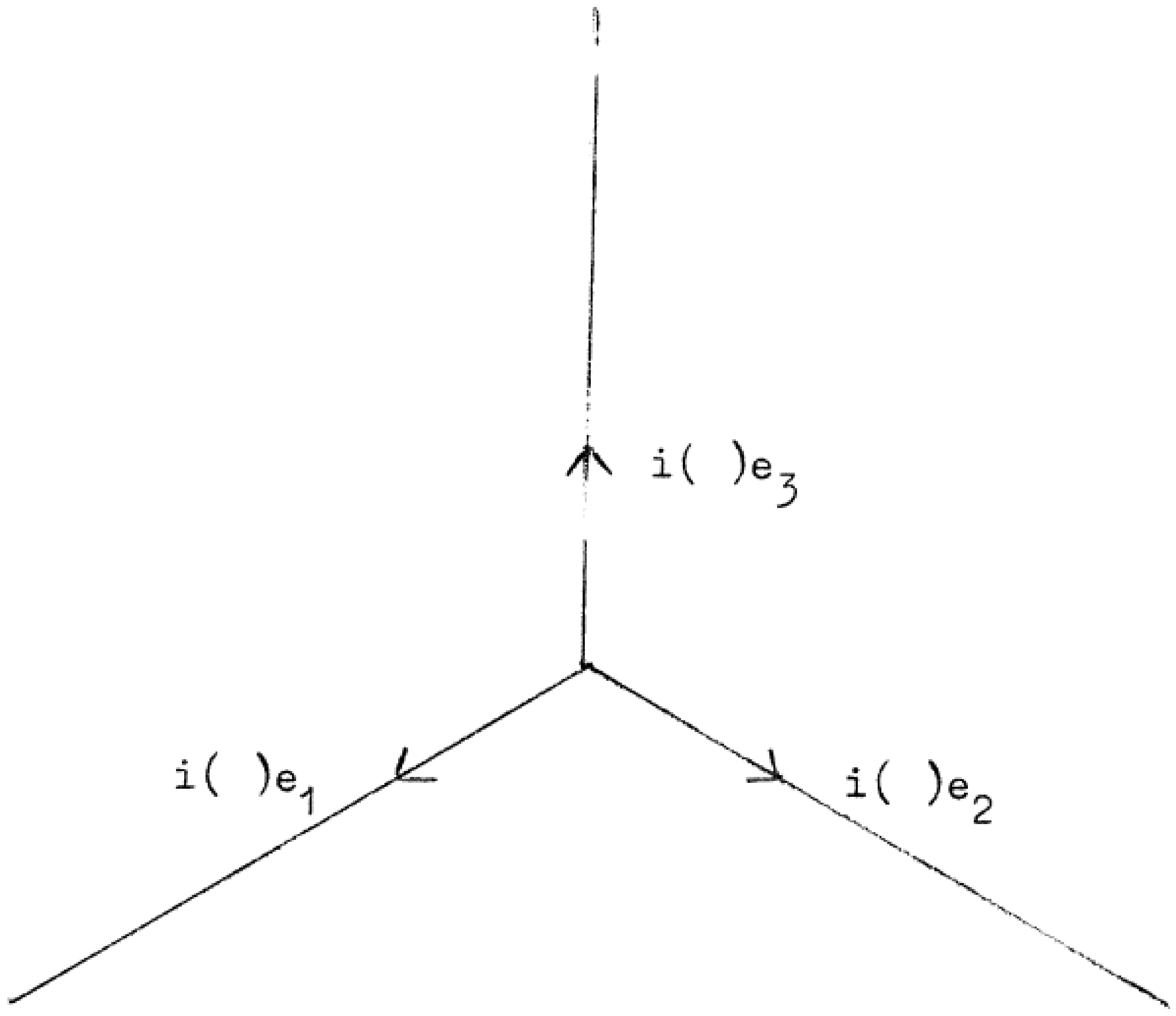

Fig. 7. Second anti-commuting triad.

First set of five anti-commuting Conway matrices:

$$c_1 = e_1(\ )e_3 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} = \gamma_5 ,$$

$$c_2 = e_2(\ )e_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = -\gamma_2 ,$$

$$c_3 = i(\ )e_1 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} = -\gamma_1 , \qquad (9.19)$$

$$c_4 = i(\ )e_2 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} = \gamma_3$$

$$c_5 = c_1 c_2 c_3 c_4 = -e_3(\ )e_3 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = -\gamma_4$$

Second set of five anti-commuting Conway matrices:

$$c'_1 = e_3(\ \ )e_1 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = i\gamma_3\gamma_4 ,$$

$$c'_2 = e_3(\ \ )e_2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} = i\gamma_1\gamma_4 ,$$

$$c'_3 = e_1(\ \ )i = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} = i\gamma_2\gamma_4 , \qquad (9.20)$$

$$c'_4 = e_2(\ \ )i = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} = i\gamma_1\gamma_2\gamma_3 = i\gamma_5\gamma_4 ,$$

$$c'_4 = c'_1\, c'_2\, c'_3\, c'_4 = -\, e_3(\ \ )e_3 = c_5 = -\gamma_4 .$$

In terms of the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad (9.21)$$

and the unit 2×2 matrix I, we have

$$c_1 = \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix}, \quad c_2 = \begin{pmatrix} 0 & i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix}, \quad c_3 = \begin{pmatrix} 0 & i\sigma_1 \\ -i\sigma_1 & 0 \end{pmatrix}, \quad c_4 = \begin{pmatrix} 0 & -i\sigma_3 \\ i\sigma_3 & 0 \end{pmatrix},$$

$$c_5 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}, \qquad (9.22)$$

and

$$c'_1 = \begin{pmatrix} 0 & -\sigma_3 \\ -\sigma_3 & 0 \end{pmatrix}, \quad c'_2 = \begin{pmatrix} 0 & -\sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}, \quad c'_3 = \begin{pmatrix} 0 & -\sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}, \quad c'_4 = \begin{pmatrix} 0 & iI \\ -iI & 0 \end{pmatrix},$$

$$c'_5 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} \qquad (9.23)$$

From the standpoint of general theory it matters little which of the six anti-commuting pentads we use, which four members of the pentad we treat as generators, and in what order we assign the numerical labels. But when it comes to detailed calculations, such freedom easily leads to confusion, since one writer may start from a different basis from another. Undoubtedly a standard set is needed, and the standard set actually used is as in (9.1). But this choice is unnatural and therefore hard to remember. On the other hand, Conway's table (9.18) has an inevitable character and, once we have decided to associate (e_3, i), considerations of symmetry direct us inevitably to two tetrads, c_1, c_2, c_3, c_4 as in (9.19) and c'_1, c'_2, c'_3, c'_4 as in (9.20). Between the two tetrads there is little to choose. Since the Dirac matrices in some form or other are an essential notation in modern physics, and are likely to remain so for the foreseeable future, one might ask whether it is too late to adopt one or other of the Conway tetrads as standard.

Much of the notational value of quaternions lies in speed of calculation, and it may be well to repeat in varied form the rule for the multiplication of the 4×4 matrices: <u>To find a continued product, the factors in front of ( ) are to be arranged in natural order and the factors behind ( ) in the reverse of natural order</u>:

$$a(\quad)a' \times b(\quad)b' \times c(\quad)c' = abc(\quad)c'b'a'. \qquad (9.24).$$

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