ISSN - 0070 - 7414

Sgríbhinní Institiúid Árd-Léinn Bhaile Átha Cliath
Sraith. A, Uimh. 27

Communications of the Dublin Institute for Advanced Studies
Series A (Theoretical Physics), No. 27

MAGNETIC POLES IN GAUGE FIELD THEORIES

By
J. BURZLAFF

DUBLIN

Institiúid Árd-Léinn Bhaile Átha Cliath
Dublin Institute for Advanced Studies
1983

TABLE OF CONTENTS

PREFACE

This communication grew out of a graduate course presented at Yale University during the Fall Semester of 1982. The purpose of the course was to explain nonperturbative methods to non-specialists, and to point out their importance in various areas of theoretical physics. Because of the prominent role of gauge theories, the potential relevance of magnetic poles, the rapid progress made during the last years, and the possible experimental observation of a monopole, we selected those methods which have been successfully applied to the study of magnetic poles in gauge field theories. The purpose of this communication is the same but the emphasis has shifted. Only if it is necessary for completeness we repeat material which can be found in other review articles. Our concern, instead, will be on recent developments in the theory of magnetic poles. The final aim is to review soliton theoretic methods as well as complex manifold techniques which have been successfully applied in the last two years to construct multi-pole solutions.

I would like to thank my colleagues at Yale University and at the Dublin Institute for Advanced Studies for many helpful discussions and especially Professor L. O'Raifeartaigh for his encouragement in producing this communication. I am indebted to Miss E.R. Wills for editing and to Mrs. M. Matthews for typing the manuscript. Financial support from the Max Kade Foundation during the time I presented the course, and from the Alexander von Humboldt-Stiftung during the preparation of the work for publication, is gratefully acknowledged.

J. Burzlaff
Dublin Institute for Advanced Studies
2 May, 1983.

CHAPTER I: TOPOLOGICAL EXCITATIONS

To familiarize the reader with the questions which will be raised in this communication, and some of the answers we will give in the case of non-Abelian gauge theories, we study analogues of magnetic poles in this introductory chapter. The models with nontrivial topology which we discuss are the ϕ^4 theory, the sine-Gordon model, the $\mathbb{C}P^n$ model, the Abelian Higgs model, and Euclidean Yang-Mills theory. In all these models the energy can be minimized in a topological sector by solving first-order differential equations. We present the corresponding topological excitations and discuss methods to find them which have counterparts in the theory of magnetic poles.

1.1. KINKS AND SINE-GORDON SOLITONS

To explain the concept of a topological quantum number we choose the ϕ^4 theory as a simple example. Its dynamics is given by the Lagrangian density

$$\mathcal{L} = \tfrac{1}{2}(\partial_t\phi)^2 - \tfrac{1}{2}(\partial_x\phi)^2 - U(\phi), \tag{1.1}$$

with

$$U = \tfrac{1}{2}\lambda(\phi^2 - a^2)^2, \ \lambda > 0. \tag{1.2}$$

The energy of the configuration $\phi(t,x)$ is therefore

$$E = \int_{-\infty}^{\infty}dx[\tfrac{1}{2}(\partial_t\phi)^2 + \tfrac{1}{2}(\partial_x\phi)^2 + U(\phi)]. \tag{1.3}$$

At any time t,

$$\lim_{x\to\pm\infty} \phi = \pm a \tag{1.4}$$

has to hold for a smooth (say C^∞) finite-energy configuration. These configurations therefore belong to four different classes according to the four different combinations of the asymptotic values $\pm a$. Within each so-called homotopy class any configuration can be continuously deformed into any other one, but configurations from two different classes cannot be continuously deformed into each other. The classes are topologically inequivalent, and can be labelled by a topological quantum number.

The solutions which minimize the energy in each topological sector can easily be found. (The method was used first by Bogomol'nyi (1976) for the ϕ^4 theory, the Abelian Higgs model, and non-Abelian Yang-Mills-Higgs theory.) For a time-independent field the inequality

$$\begin{aligned} E &= \tfrac{1}{2}\int_{-\infty}^{\infty}dx[(\partial_x\phi \mp \sqrt{2U})^2 \pm 2\sqrt{2U}\partial_x\phi] \\ &\geq \pm\int_{\phi(-\infty)}^{\phi(+\infty)}d\phi\sqrt{2U} = \pm\frac{4}{3}\sqrt{\lambda}a^3 n \end{aligned} \tag{1.5}$$

holds, where $n = 0, \pm 1$ label the trivial and nontrivial topological sectors for the ϕ^4

theory. The lower bound is attained for fields which satisfy

$$\partial_x \phi = \pm \sqrt{2U}. \tag{1.6}$$

Thus the ground states $\phi = \pm a$ with zero energy minimize the energy in the topologically trivial sectors and the kinks

$$\phi = \pm a \tanh (\sqrt{\lambda} a x) \tag{1.7}$$

with energy $E = 4\sqrt{\lambda}a^3/3$ minimize the energy in the topologically nontrivial sectors.

We have found the homotopy classes and solutions with the corresponding topological quantum numbers for the ϕ^4 theory in one space dimension. To do the same for non-Abelian gauge theories in three space dimensions is the main problem of this communication; we will use a number of different techniques to solve the problem. One of these techniques will be introduced now, in the context of the sine-Gordon theory.

The sine-Gordon theory is described by the Lagrangian density (1.1) with

$$U = 1 - \cos \phi. \tag{1.8}$$

For smooth finite-energy configurations

$$\frac{1}{2\pi} \lim_{x\to\pm\infty} \phi = k \in \mathbb{Z} \tag{1.9}$$

holds. Thus the model is topologically nontrivial and has infinitely many topologically inequivalent classes, labelled by the topological quantum number

$$n = \frac{1}{2\pi}[\phi(+\infty) - \phi(-\infty)]. \tag{1.10}$$

Finite-energy solutions with nonzero quantum number n are called sine-Gordon solitons. (For a precise definition of a soliton see for instance Scott, Chu & McLaughlin 1973.)

The ground states with topological quantum number $n = 0$ and energy $E = 0$ are $\phi = 2\pi k$, $k \in \mathbb{Z}$. To obtain the solitons with $n = \pm 1$ we saturate the lower bound for the inequality (1.5) by solving eq. (1.6). The solution

$$\phi = 4 \tan^{-1} \exp (\pm x + c) \tag{1.11}$$

has topological quantum number $n = \pm 1$ and energy $E = 8$. Before we generate solitons with $n \geqslant 2$ we rederive this 1-soliton solution.

To this end we write the equation of motion for the sine-Gordon theory in characteristic coordinates

$$\xi = \tfrac{1}{2}(t + x), \quad \eta = \tfrac{1}{2}(t - x), \tag{1.12}$$

which yields

$$\partial^2_{\xi\eta} \phi = - \sin \phi. \tag{1.13}$$

Instead of solving this second-order differential equation, we generate a family of new solutions $\phi(\gamma)$ from a known one ϕ_o by solving the first-order equations

$$\partial_\xi \tfrac{1}{2}(\phi - \phi_o) = \gamma \sin \tfrac{1}{2}(\phi + \phi_o),$$
$$\partial_\eta \tfrac{1}{2}(\phi + \phi_o) = -\gamma^{-1} \sin \tfrac{1}{2}(\phi - \phi_o). \tag{1.14}$$

These equations are only consistent if ϕ as well as ϕ_o is a solution of eq. (1.13). The transformation from ϕ_o to ϕ, $\phi = B\phi_o$, is called a Bäcklund transformation. With this Bäcklund transformation we can generate, from the vacua $\phi_o = 2\pi k$, $k \in \mathbb{Z}$, a family of 1-soliton solutions, which contains the time-independent soliton (1.11) as the special case $\gamma = 1$.

Iterating this procedure seems to be difficult because the seed solution ϕ_o becomes more and more complicated. The addition formula for sine-Gordon Bäcklund transformations however reduces the problem to an algebraic one. In fact, two Bäcklund transformations B_1, B_2 with parameters γ_1 and γ_2 commute:

$$\phi' := B_1\phi_2 := B_1B_2\phi_o = B_2B_1\phi_o =: B_2\phi_1, \tag{1.15}$$

and for ϕ'

$$\tan \tfrac{1}{4}(\phi' - \phi_o) = \frac{\gamma_1 + \gamma_2}{\gamma_1 - \gamma_2} \tan \tfrac{1}{4}(\phi_1 - \phi_2) \tag{1.16}$$

holds (see App. A). After solving the differential equation (1.14) once, which is easily done for $\phi_o = 0$, we only have to solve the algebraic equation (1.16). Using this fact Barnard (1973) has generated n-soliton solutions for the sine-Gordon theory.

1.2. $\mathbb{C}P^N$ INSTANTONS

Since the simplest $\mathbb{C}P^N$ model, the O(3) model, is related to the sine-Gordon theory and all $\mathbb{C}P^N$ models in two space dimensions have many features in common with four-dimensional Yang-Mills theories, it is natural to proceed by discussing these models. The relation between the sine-Gordon theory and the O(3) model with Lagrangian density

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu \vec{q}) \cdot (\partial^\mu \vec{q}),$$
$$\vec{q} \in S^2, \quad \mu = 0,1, \tag{1.17}$$

and metric diag (+1, −1) is the following (Pohlmeyer 1976): If $\vec{q}$ satisfies the equations of motion for the Lagrangian (1.17), then characteristic coordinates ξ,η with

$$\vec{q}_{,\xi}^{\,2} = \vec{q}_{,\eta}^{\,2} = 1 \tag{1.18}$$

($\vec{q}_{,\xi} := \partial_\xi \vec{q}$ etc.) exist. In this coordinate system

$$\phi = \cos^{-1}(\vec{q}_{,\xi}\cdot\vec{q}_{,\eta}) \tag{1.19}$$

satisfies the sine-Gordon equation (1.13).

To prove the existence of a coordinate system with the property (1.18), we use the fact that

$$(\vec{q}_{,\xi}^{\,2})_{,\eta} = (\vec{q}_{,\eta}^{\,2})_{,\xi} = 0 \tag{1.20}$$

is a consequence of the equation of motion,

$$\vec{q}_{,\xi\eta} + (\vec{q}_{,\xi}\cdot\vec{q}_{,\eta})\vec{q} = 0. \tag{1.21}$$

Hence, eq. (1.18) holds in the coordinate system (ξ',η') defined by

$$d\xi'(\xi) = \sqrt{\vec{q}_{,\xi}^{\,2}}\,d\xi, \quad d\eta'(\eta) = \sqrt{\vec{q}_{,\eta}^{\,2}}\,d\eta. \tag{1.22}$$

In this new coordinate system we express $\vec{q}_{,\xi\xi}$ and $\vec{q}_{,\eta\eta}$ in terms of $\vec{q}$, $\vec{q}_{,\xi}$ and $\vec{q}_{,\eta}$:

$$\begin{aligned}\vec{q}_{,\xi\xi} &= -\vec{q} + \phi_{,\xi}\,\vec{q}_{,\xi}\,\mathrm{ctg}\,\phi - \frac{1}{\sin\phi}\phi_{,\xi}\,\vec{q}_{,\eta},\\ \vec{q}_{,\eta\eta} &= -\vec{q} - \frac{1}{\sin\phi}\phi_{,\eta}\,\vec{q}_{,\xi} + \phi_{,\eta}\,\vec{q}_{,\eta}\,\mathrm{ctg}\,\phi,\end{aligned} \tag{1.23}$$

and derive the sine-Gordon equation for ϕ:

$$\phi_{,\xi\eta} = -(\vec{q}_{,\xi\xi}\cdot\vec{q}_{,\eta}/\sin\phi)_{,\eta} = -\sin\phi. \tag{1.24}$$

This result suggests that we look for a generalization to the O(3) model of the methods which work for the sine-Gordon model. This has already been done successfully in some cases, and we will study one of these methods and use its analogue for Yang-Mills-Higgs theory in Chapter 3. For the moment we study the topological excitations of Euclidean $\mathbb{C}P^N$ models which are in some sense similar to the topological excitations of Euclidean Yang-Mills theories (Gürsey & Tze 1980, cf. Maison 1980).

The topology of the model (1.17) is nontrivial if $\mathbb{R}^2$ can be compactified. Then the continuous maps

$$\vec{q}: S^2 \to S^2$$

belong to topologically inequivalent classes, the elements of the homotopy group $\pi_2(S^2)$, labelled by the quantum number

$$\begin{aligned} n &= \frac{1}{8\pi}\int d^2x\, \epsilon_{ijk}\,\epsilon_{\mu\nu}\, q_i\,(\partial_\mu q_j)\,(\partial_\nu q_k)\\ &= \frac{1}{4\pi}\int d^2x \sin\theta\,(\partial_1\theta\partial_2\phi - \partial_2\theta\partial_1\phi) \in \mathbb{Z},\ \vec{q}^{\,T} = (\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta).\end{aligned} \tag{1.25}$$

We now cast this result into a form which can be easily generalized.

If $\vec{p} \in \mathbb{C}^{N+1}$, $N = 1$, is the normalized eigenvector of $\vec{q}\cdot\vec{\sigma}$, with Pauli matrices σ_i, corresponding to the eigenvalue -1:

$$(\vec{q}\cdot\vec{\sigma})\vec{p} = -\vec{p}, \quad \vec{p}\in \mathbb{C}^{N+1}, \tag{1.26}$$

the spectral decomposition of $\vec{q}\cdot\vec{\sigma}$ takes the form

$$(\vec{q}\cdot\vec{\sigma})_{jk} = \delta_{jk} - 2p_j\bar{p}_k \tag{1.27}$$

($\bar{p}_i$ is the complex conjugate of p_i). In terms of $\vec{p}$ the topological quantum number n and the Lagrangian density $\mathcal{L}$ now read

$$\begin{aligned} n &= -\frac{i}{16\pi}\int d^2x\, \epsilon_{\mu\nu}\, \mathrm{tr}\,[\vec{q}\cdot\vec{\sigma}\,(\partial_\mu\vec{q}\cdot\vec{\sigma})\,(\partial_\nu\vec{q}\cdot\vec{\sigma})] \\ &= -\frac{i}{2\pi}\int d^2x\, \epsilon_{\mu\nu}(D_\mu\vec{p})^+\cdot(D_\nu\vec{p}), \end{aligned} \tag{1.28}$$

and

$$\mathcal{L} = \tfrac{1}{4}\,\mathrm{tr}\,[(\partial_\mu\vec{q}\cdot\vec{\sigma})\,(\partial_\mu\vec{q}\cdot\vec{\sigma})] = 2(D_\mu\vec{p})^+\cdot(D_\mu\vec{p}). \tag{1.29}$$

Here we have introduced the covariant derivative

$$D_\mu\vec{p} = (\partial_\mu - \bar{p}_i\partial_\mu p_i)\vec{p}. \tag{1.30}$$

The O(3) model is thus the first in a series of $\mathbb{C}P^N$ models, $N \geqslant 1$, defined by eq. (1.29) and (1.30) (D'Adda, Lüscher & DiVecchia 1978, Eichenherr 1978, Golo & Perelomov 1978).

For all N the Lagrangian (1.29) is invariant under the U(1) transformation

$$\vec{p}\rightarrow\vec{p}\,' = e^{i\theta(x)}\vec{p}. \tag{1.31}$$

If we consider, instead of $\vec{p}$, the whole equivalence class of $\vec{p}\,'$s which are equal up to a U(1) factor, we see that the topology of the $\mathbb{C}P^N$ models is interesting. Indeed, if the class goes to a constant at infinity,

$$[p] \underset{r\to\infty}{\longrightarrow} [p_\infty], \quad [p]\in\mathbb{C}P^N, \tag{1.32}$$

then

$$\vec{p} \underset{r\to\infty}{\longrightarrow} \omega(\hat{x})\vec{p}_\infty \tag{1.33}$$

has to hold with a map ω which maps the circle at infinity to U(1). These maps belong to different homotopy classes of $\pi_1(U(1))$, and are labelled by the topological quantum number given in eq. (1.28):

$$n = \frac{i}{2\pi}\int d^2x\, \epsilon_{\mu\nu}\,\partial_\mu(\bar{p}_j\partial_\nu p_j). \tag{1.34}$$

Furthermore, the action is bounded from below by the topological quantum number:

$$\begin{aligned} A = \int d^2x\, [&(D_\mu\vec{p} \pm i\epsilon_{\mu\nu}\, D_\nu\vec{p})^+\cdot(D_\mu\vec{p} \pm i\epsilon_{\mu\nu}\, D_\nu\vec{p}) \\ &\mp 2i\epsilon_{\mu\nu}(D_\mu\vec{p})^+\cdot(D_\nu\vec{p})] \geqslant 4\pi|n|, \end{aligned} \tag{1.35}$$

and the solutions to the (anti-) self-dual equations

$$D_\mu \vec{p} \; = \; \pm\, i\epsilon_{\mu\nu}\, D_\nu \vec{p} \tag{1.36}$$

minimize the action in a topological sector. These solutions are called $\mathbb{C}P^N$ (anti-) instantons.

For N = 1, Belavin & Polyakov (1975) gave solutions to eq. (1.36) in the form

$$p_i \; = \; P_n^i \,/\, \sqrt{|P_n^1|^2 + |P_n^2|^2}, \tag{1.37}$$

where the P_n^i are polynomials of degree n in $z = x_0 + ix_1$ or $\bar{z}$, respectively. These $\vec{p}$ satisfy $D_{\bar{z}}\vec{p} = 0$ or $D_z\vec{p} = 0$, respectively, and thus are $\mathbb{C}P^N$ (anti-) instantons. Garber, Ruijsenaars, Seiler & Burns (1979) have shown that the corresponding $\vec{q}$'s are the only continuous maps from $\mathbb{R}^2$ to S^2 with finite and stationary action. Thus, for N = 1 all smooth finite-action solutions are (anti-) self-dual.

That this is not true for arbitrary N can be seen as follows (Din & Zakrzewski 1980 a,b, cf. Corrigan 1980): For real $\vec{p} \in \mathbb{R}^3$ the equation of motion

$$D_\mu D_\mu \vec{p} + (D_\mu \vec{p})^+ \cdot (D_\mu \vec{p})\, \vec{p} \; = \; 0 \tag{1.38}$$

reduces to the equation of motion (1.21) for the O(3) model, because $\bar{p}_i \partial_\mu p_i = 0$ holds. Further, any finite-action O(3) solution is a finite-action $\mathbb{C}P^2$ solution. Since only constant real $\vec{p}$ can be self-dual or anti-self-dual, we conclude that for $N \geq 2$ there exist finite-action solutions which do not minimize the action in their topological sectors. We will encounter a similar situation when we study finite-energy Yang-Mills-Higgs solutions.

1.3. VORTICES AND YANG-MILLS INSTANTONS

In this section we introduce gauge field theories with which we will be concerned for the rest of this article. A simple example of a gauge field theory is the Abelian Higgs model with the Lagrangian density

$$\mathcal{L} \; = \; -\tfrac{1}{4}F_{\mu\nu}\,F^{\mu\nu} + \tfrac{1}{2}\overline{(D^\mu\phi)}\,D_\mu\phi - \tfrac{\lambda}{2}(\bar{\phi}\phi - 1)^2, \;\; \lambda > 0. \tag{1.39}$$

Here ϕ is a complex field, $\phi = \phi_1 + i\phi_2$, $F_{\mu\nu}$ is the electromagnetic field,

$$F_{\mu\nu} \; = \; \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{1.40}$$

given in terms of the gauge potential A_μ and $D_\mu\phi$ is the covariant derivative,

$$D_\mu\phi \; = \; \partial_\mu\phi + iA_\mu\phi. \tag{1.41}$$

Here and throughout we put the gauge coupling constant equal to one.

The Lagrangian (1.39) is invariant under the U(1) gauge transformation

$$\phi \rightarrow \phi' \; = \; \exp(i\theta)\phi,$$

$$A_\mu \rightarrow A'_\mu \; = \; A_\mu - \partial_\mu\theta, \tag{1.42}$$

with $\theta = \theta(x)$. Therefore we can simplify our discussion by choosing the $A_o = 0$ gauge. If we restrict our attention to time-independent fields we can in addition transform the radial component A_r to zero.

In two space dimensions ($\mu,\nu = 0,1,2$ with metric diag $(+1, -1, -1)$) this model is topologically nontrivial (see Jaffe & Taubes (1980) for a rigorous discussion). In fact, for smooth finite-energy configurations, Φ is continuous at infinity and maps the unit circle at infinity to the unit circle of complex numbers, these maps Φ_∞ therefore belong to topologically inequivalent classes, the elements of the homotopy group $\pi_1(S^1)$. The topological quantum number is the winding number

$$n = \frac{i}{2\pi} \int_o^{2\pi} d\phi \, \overline{\Phi_\infty} \frac{d\Phi_\infty}{d\phi} \in \mathbb{Z}. \tag{1.43}$$

Since

$$\frac{1}{2\pi} \lim_{r \to \infty} \int_{|x|<r} d^2x \, F_{12} = n \tag{1.44}$$

holds the corresponding topological excitations describe a magnetic flux through the x_1-x_2 plane and are called vortices.

We have studied the topology for the Lagrangian (1.39) and interpreted the topological excitations. We would like to find the corresponding solutions to the equations of motion. For the special value $\lambda = 1/4$ this task can be simplified. In this case ($\lambda = 1/4$, $A_o = \partial_t\Phi = \partial_t A_i = 0$) the energy can be cast into the form

$$\begin{aligned} E = \int d^2x \, \{ & \tfrac{1}{2}[F_{12} \pm \tfrac{1}{2}(\phi^2_1 + \phi^2_2 - 1)]^2 \\ & + \tfrac{1}{2}[(\partial_1\phi_1 - A_1\phi_2) \pm (\partial_2\phi_2 + A_2\phi_1)]^2 \\ & + \tfrac{1}{2}[(\partial_2\phi_1 - A_2\phi_2) \mp (\partial_1\phi_2 + A_1\phi_1)]^2 \pm \tfrac{1}{2}F_{12}\} \end{aligned} \tag{1.45}$$

by partial integration. Therefore, to find solutions for which the energy takes its minimal value $E = \pi|n|$ in the topological sector n, three first-order equations have to be solved. Using these equations one can prove (Jaffe & Taubes 1980) that solutions for all n exist. Furthermore all finite-energy solutions satisfy these first-order equations. However, an explicit solution is not yet known.

Let us consider in which respect non-Abelian Yang-Mills theory is similar to the Abelian Higgs model and the $\mathbb{C}P^N$ model. For SU(2) Yang-Mills theory in four Euclidean dimensions the Lagrangian density reads

$$\mathcal{L} = \tfrac{1}{2} \, \mathrm{tr} \, F_{\mu\nu}F_{\mu\nu}, \quad \mu,\nu = 1,2,3,4, \tag{1.46}$$

with metric $\delta_{\mu\nu}$. $F_{\mu\nu}$ are the gauge fields

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu] \tag{1.47}$$

given in terms of the gauge potential $A_\mu(x) \in su(2)$. $\mathcal{L}$ is invariant under the gauge transformations

$$A_\mu \to A'_\mu \;=\; \omega A_\mu \omega^{-1} + i(\partial_\mu \omega)\omega^{-1},\; \omega \in SU(2). \tag{1.48}$$

For smooth fields with finite action $F_{\mu\nu}$ must vanish at infinity, and A_μ is pure gauge

$$A_\mu \underset{r\to\infty}{\longrightarrow} i(\partial_\mu \omega)\omega^{-1},\; \omega \in SU(2). \tag{1.49}$$

Hence, ω maps the 3-sphere at infinity into SU(2), which is topologically equivalent to a 3-sphere. These maps therefore belong to topologically inequivalent classes, the elements of the homotopy group $\pi_3(S^3)$. The topological quantum number is the Pontryagin index

$$\begin{aligned} n \;&=\; \frac{1}{16\pi^2} \int d^4x \;\mathrm{tr}\; F_{\mu\nu} F^*_{\mu\nu} \\ &=\; \frac{1}{8\pi^2} \int d^4x \;\mathrm{tr}\; \epsilon_{\mu\nu\rho\sigma}\, \partial_\mu (A_\nu \partial_\rho A_\sigma + i\tfrac{2}{3} A_\nu A_\rho A_\sigma) \in \mathbb{Z}, \end{aligned} \tag{1.50}$$

where $F^*_{\mu\nu}$ are the dual fields

$$F^*_{\mu\nu} \;=\; \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}\, F_{\rho\sigma}. \tag{1.51}$$

For the action A, the equation

$$A \;=\; \tfrac{1}{4} \int d^4x \;\mathrm{tr}\; (F_{\mu\nu} \pm F^*_{\mu\nu})^2 \mp 8\pi^2 n \tag{1.52}$$

holds. The action is therefore bounded from below by $8\pi^2|n|$, and attains its minimum for (anti-) self-dual fields,

$$F_{\mu\nu} \;=\; \pm F^*_{\mu\nu}. \tag{1.53}$$

The (anti-) self-dual solutions are called (anti-) instantons. (Anti-) instantons satisfy automatically the equations of motion

$$D_\nu F_{\mu\nu} := \;\partial_\nu F_{\mu\nu} + i[A_\nu, F_{\mu\nu}] \;=\; 0. \tag{1.54}$$

This follows not only from the fact that they minimize the action in a topological sector but also from the Bianchi identities

$$D_\nu F^*_{\mu\nu} \;=\; 0, \tag{1.55}$$

which are a consequence of the definition (1.47) alone.

(Anti-) self-dual solutions for all topological quantum numbers have been found (cf. Actor 1979, Olive, Sciuto & Crewther 1979, Prasad 1980). For the ansatz

$$\begin{aligned} A_i \;&=\; -\tfrac{1}{2}\epsilon_{ijk}\, \sigma_j \partial_k \ln\psi \;\mp \tfrac{1}{2}\sigma_i \partial_4 \ln\psi, \\ A_4 \;&=\; \pm \tfrac{1}{2}\sigma_i \partial_i \ln\psi, \end{aligned} \tag{1.56}$$

the self-duality equations are satisfied if ψ satisfies the equation

$$\partial_\mu \partial_\mu \psi = 0. \tag{1.57}$$

't Hooft has shown that for the solution

$$\psi = 1 + \sum_{i=1}^{n} \frac{a_i^2}{|x-b_i|^2} \tag{1.58}$$

the singularities in the potentials (1.56) can be gauged away. These A_μ are therefore a regular 5n parameter family of solutions. This family contains, as the special case $n = 1$, the first instanton found (Belavin, Polyakov, Schwartz & Tynpkin 1975) (BPST). The most general 8n-3 parameter family of instanton solutions has been constructed by Atiyah, Drinfeld, Hitchin & Manin (1978) (ADHM). We will come back to the ADHM construction in Chapter 5.

CHAPTER II: TOPOLOGY AND MAGNETIC CHARGE

In this chapter we apply the topological concepts already discussed to the study of non-Abelian Yang-Mills-Higgs theories, and recover the Dirac monopole as a topological excitation in the form of the 't Hooft-Polyakov monopole. Coleman (1975) and Goddard & Olive (1978) have already described this in a detailed and pedagogic way and so we only repeat those facts essential for our discussion and elaborate on some less well known developments. To illustrate topological aspects in groups other than SU(2), we discuss point-singular monopoles and spherically symmetric monopole solutions which satisfy the Bogomol'nyi equations as well as those which do not.

2.1. TOPOLOGICAL EXCITATIONS IN YANG-MILLS-HIGGS THEORY

We will now study the topology of Yang-Mills-Higgs theory (see Jaffe & Taubes (1980) for the precise form of the assumptions and derivations). The Lagrangian density of a Yang-Mills-Higgs theory is

$$\mathcal{L} = -\tfrac{1}{4}F_a^{\mu\nu} F^a_{\mu\nu} + \tfrac{1}{2}(D^\mu\Phi)^+ \cdot D_\mu\Phi - U(\Phi), \qquad (2.1)$$

with $\mu,\nu = 0,1,2,3$ and metric diag $(+1, -1, -1, -1)$.

$$F^{\mu\nu} = F_a^{\mu\nu}T_a = \partial^\mu A^\nu - \partial^\nu A^\mu + i[A^\mu,A^\nu] \qquad (2.2)$$

are the gauge fields, T_a the Hermitian generators of a compact connected gauge group G which satisfy $2\mathrm{tr}(T_aT_b) = \delta_{ab}$, and $A^\mu = A_a^\mu T_a$ are the gauge potentials. Φ is an ℓ-tuple of real or complex scalar fields,

$$D^\mu\Phi = \partial^\mu\Phi + iA^\mu\Phi \qquad (2.3)$$

is the covariant derivative with an ℓ-dimensional representation A^μ, and

$$U = \tfrac{1}{4}\lambda(|\Phi|^2 - 1)^2, \quad \lambda > 0, \qquad (2.4)$$

is the Higgs potential.

This theory is invariant under the gauge transformation

$$\Phi \to \Phi' = \omega\Phi$$

$$A^\mu \to A'^\mu = \omega A^\mu\omega^{-1} + i(\partial^\mu\omega)\omega^{-1} \qquad (2.5)$$

with a group element ω. Thus, as in Section 1.3, we can study smooth time-independent finite-energy configurations in the $A_0 = A_r = 0$ gauge. Here, for G = SU(2) and three real scalar fields Φ_i, the continuous maps Φ at infinity map S^2 into the set of zeros of U which is also a 2-sphere. These maps therefore belong to the homotopy classes of $\pi_2(S^2)$ labelled by the topological quantum number

$$n = \frac{1}{8\pi} \int_{S^2} d\sigma^i \epsilon_{ijk} \epsilon_{abc} \hat{\Phi}_a (\partial_j \hat{\Phi}_b) \partial_k \hat{\Phi}_c \in \mathbb{Z}, \tag{2.6}$$

$$\hat{\Phi}_i = \Phi_i / |\Phi|, \; i = 1,2,3$$

(cf. eq. (1.25)). Where the topological current is defined its divergence is zero because $\partial_i \hat{\Phi}$ is orthogonal to $\hat{\Phi}$. Hence only the zeros of Φ contribute to the topological charge, and have to be interpreted as the locations of the topological excitations.

We have seen that the topological charge of a vortex is proportional to its flux. Analogously, the topological charge of a smooth finite-energy SU(2) field configuration is related to its magnetic charge

$$g = \lim_{R \to \infty} \int_{|x|=R} d\sigma^i \hat{\Phi}_a B^i_a \tag{2.7}$$

through the equation

$$g = 4\pi n. \tag{2.8}$$

The SU(2) magnetic field is defined in terms of $F^{\mu\nu}$ as

$$B^i = -\tfrac{1}{2} \epsilon_{ijk} F^{jk}, \tag{2.9}$$

and the B^i in direction of $\hat{\Phi}$, corresponding to the unbroken U(1) symmetry, is interpreted as the U(1) magnetic field. Because the eigenvalues $e_0 = \pm\tfrac{1}{2}$ of $\tfrac{1}{2}\hat{\Phi}_a \sigma_a$ are the charge quanta in this theory we have recovered Dirac's quantisation condition (Dirac 1931, cf. Goddard & Olive 1978, Coleman 1982).

According to Dirac's quantisation condition the minimal magnetic charge a monopole can carry is $2\pi/e_0$. We will see in Section 2.3 that this is still true in a more realistic model than that just discussed. If such a monopole passes through a superconducting loop it has two flux quanta, as defined in eq. (1.44), added because the unit of charge in this equation is the charge of a Cooper pair and therefore $2e_0$. Hence the current which flows in the superconducting loop to sustain the magnetic flux, changes by the corresponding amount. This change has been observed in an experiment by Cabrera (1982) which constitutes the only candidate event for a magnetic monopole.

Let us go back to the problem of finding a corresponding classical solution. In the spirit of Chapter I we again minimize the energy in a topological sector. From

$$\begin{aligned} E &= \int d^3x [\tfrac{1}{2} B^i_a B^i_a + \tfrac{1}{2}(D^i\Phi)_a (D^i\Phi)_a + U] \\ &= \int d^3x \, \{ \tfrac{1}{2}[B^i_a \pm (D^i\Phi)_a]^2 \mp B^i_a (D^i\Phi)_a + U\} \end{aligned} \tag{2.10}$$

it follows that the energy is bounded from below by

$$\begin{aligned} | \int d^3x B^i_a (D^i\Phi)_a | &= | \int d^3x D^i (B^i_a \Phi_a)| \\ &= | \int d^3x \partial^i (B^i_a \Phi_a)| = 4\pi |n| \end{aligned} \tag{2.11}$$

for smooth finite-energy fields. In the Bogomol'nyi-Prasad-Sommerfield (BPS) limit of vanishing Higgs potential, $\lambda \to 0$, the energy attains its lower bound for fields which satisfy the Bogomol'nyi equations

$$B^i = \pm D^i\phi = \pm (\partial^i\phi + i[A^i,\phi]), \qquad (2.12)$$
$$\phi = \phi_i\sigma_i/2.$$

If we identify ϕ with a new A_4, our Yang-Mills-Higgs theory in the BPS limit is merely a Euclidean Yang-Mills theory for time-independent fields, and the Bogomol'nyi equations become the self-duality conditions.

We have reduced the problem of finding magnetic pole solutions in the BPS limit to one of solving the Bogomol'nyi equations. The asymptotic behaviour of $|\phi|^2$,

$$|\phi|^2 \xrightarrow[r\to\infty]{} 1 - \frac{2|n|}{r}, \qquad (2.13)$$

then gives us the topological quantum number of the solution, because (2.13) yields

$$E = \int d^3x (D^i\phi)_a (D^i\phi)_a = \int d^3x \tfrac{1}{2}\partial^i\partial^i|\phi|^2$$
$$= 4\pi|n|. \qquad (2.14)$$

Hence the solution with asymptotic behaviour (2.13) is an n-pole with magnetic charge $g = 4\pi n$ and energy $E = 4\pi|n|$. To find these n-poles is the final aim of this communication.

2.2 POINT-SINGULAR MONOPOLES

Because smooth finite-energy solutions are hard to find we relax our conditions for a moment and look for point-singular solutions first. This turns out to be a much easier problem, and a relevant one if we are only interested in n-poles far away from their cores. In this case we can also easily carry our analysis beyond SU(2).

Far away from the cores of the magnetic poles

$$U(\phi) = 0, \; D^i\phi = 0 \qquad (2.15)$$

is a good approximation. These equations are solved by

$$\vec{\phi} = e^{-in\phi T_3} e^{-i\theta T_2} \begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}\cos n\phi \sin\theta\\ \sin n\phi \sin\theta\\ \cos\theta\end{pmatrix},$$

$$A^i = -i[\phi,\partial^i\phi], \qquad (2.16)$$

where T_i are the SU(2) generators in the adjoint representation. Obviously the topological quantum number is n for $\vec{\phi}$ given by eq. (2.16).

The potentials A^i lead to fields

$$F^{ij} = -i[\partial^i\phi, \partial^j\phi] = -\frac{n}{r^2}\epsilon_{ijk}\hat{x}^k\phi, \qquad (2.17)$$

which satisfy $D_iF^{ij} = 0$. The fields (2.16) and (2.17) are therefore a point-singular solution to the equations of motion

$$D_iD^i\phi = -\frac{\partial U}{\partial \phi},$$
$$D_jF^{ij} = i[\phi, D^i\phi] . \quad (2.18)$$

Because F^{ij} diverges like r^{-2} at the origin, the energy of this solution is infinite.

We have found point-singular solutions for arbitrary magnetic charge in SU(2) Yang-Mills-Higgs theory. These solutions can be generalized to higher SU(N) groups (for SU(3) solutions with arbitrary quantum numbers see Cho (1980); for generalizations to SU(4) and SU(5) see Kim, Koh & Park (1982) and Koh, Kim, Park, Kim & Kim (1981)). In general, the topology of a gauge theory with gauge group G broken down to the little group

$$H = \{h \in G: h\vec{\Phi}_o = \vec{\Phi}_o\}, \quad (2.19)$$

with $\vec{\Phi}_o$ a zero of the Higgs potential, is given by $\pi_2(G/H)$. For SU(2) broken down to U(1), for each pair of zeros of the Higgs potential $\vec{\Phi}_1$, $\vec{\Phi}_2$, a group element ω_{12} exists with $\vec{\Phi}_1 = \omega_{12}\vec{\Phi}_2$. SU(2) acts transitively on the zeros of U. In this case ϕ alone determines the topology, which is therefore given by $\pi_2(S^2)$.

This is no longer true for G = SU(3) with a Higgs field in the adjoint representation, $\Phi = \Phi_a\lambda_a/2$, a = 1, ..., 8, and the potential (2.4). (λ_a are the Gell-Mann matrices.) If a λ_3-like Φ breaks down the symmetry to U(1) x U(1), the formula

$$\pi_2(G/H) = \ker \pi_1(H) \rightarrow \pi_1(G)$$
$$= \pi_1(H) \text{ for } \pi_1(G) = 0 \quad (2.20)$$

tells us that the magnetic poles are classified by the two topological quantum numbers of $\pi_1(U(1)) \times \pi_1(U(1))$. Furthermore, the generalized quantisation condition reads in this case

$$\exp\left[i \left(\tfrac{1}{2}g_3\lambda_3 + \tfrac{1}{2}g_8\lambda_8\right)\right] = 1, \quad (2.21)$$

which implies

$$g_3 = 4\pi(n-\tfrac{1}{2}n'), \; g_8 = 2\pi\sqrt{3}n', \quad (2.22)$$

with integers n, n'. The magnetic charges are defined as

$$g_3 = \int d\sigma^i \epsilon_{ijk} 2 \,\mathrm{tr}\,(\hat{\Phi}F^{jk}),$$
$$g_8 = \int d\sigma^i \epsilon_{ijk} 2 \,\mathrm{tr}\,(\hat{\Phi}'F^{jk}), \quad (2.23)$$

with a λ_8-like $\hat{\Phi}'$ orthonormal to the normalized $\hat{\Phi}$.

A Φ which can be gauge-transformed to $\lambda_3/2$, and exhibits the full homotopy group, is

$$\Phi = \omega\frac{\lambda_3}{2}\omega^{-1},$$
$$\omega = e^{-in'\phi(-\frac{1}{4}\lambda_3 + \frac{1}{4}\sqrt{3}\lambda_8)} e^{-i\theta\lambda_7/2} e^{-i(n-\frac{1}{2}n')\phi\lambda_3/2} e^{-i\phi\lambda_2/2}. \quad (2.24)$$

The orthonormal λ_8-like $\hat{\Phi}'$ is

$$\hat{\Phi}' = 2/\sqrt{3}\hat{\Phi}^2 - \frac{1}{\sqrt{3}}I = \omega\frac{\lambda_8}{2}\omega^{-1}. \tag{2.25}$$

To find the gauge potentials we solve $D^i\phi = 0$ for A^i, using (2.24) and (2.25). That the solution is

$$A^i = a^i\hat{\Phi} + a'^i\hat{\Phi}' - i[\hat{\Phi},\partial^i\hat{\Phi}] - i[\hat{\Phi}',\partial^i\hat{\Phi}'], \tag{2.26}$$

with arbitrary functions a^i and a'^i, can be proved by writing $D^i\phi$ in terms of λ_3, λ_8, and ω. In the singular $\hat{\Phi} = \lambda_3/2$ gauge these potentials read

$$A'^i = a^i\lambda_3/2 + a'^i\lambda_8/2 - 2i[\mathrm{tr}(\omega^{-1}\partial^i\omega\lambda_3/2) + \mathrm{tr}(\omega^{-1}\partial^i\omega\lambda_8/2)]. \tag{2.27}$$

It is easy to calculate the traces in eq. (2.27) for ω given in (2.24), and to see that for

$$a^i = -\frac{n'}{2e}\cos^2\theta\partial^i\phi, \quad a'^i = 0, \tag{2.28}$$

the gauge potentials take the form

$$A'^i = -[(n-\tfrac{1}{2}n')\frac{\lambda_3}{2} + \frac{1}{2}\sqrt{3}n'\frac{\lambda_8}{2}]\cos\theta\partial^i\phi. \tag{2.29}$$

This equation implies that the gauge fields are

$$F^{ij} = \frac{1}{r^2}\epsilon_{ijk}\hat{x}^k[(n-\tfrac{1}{2}n')\hat{\Phi} + \tfrac{1}{2}\sqrt{3}n'\hat{\Phi}'] \tag{2.30}$$

in the nonsingular gauge. These F^{ij} satisfy $D_jF^{ij} = 0$. Thus we have found point-singular SU(3) solutions for arbitrary magnetic quantum numbers.

2.3. SPHERICALLY SYMMETRIC MONOPOLES

To find a solution which minimizes the energy in a topological sector, i.e., satisfies the Bogomol'nyi equations in the BPS limit, we smooth out the point-singular SU(2) solution for $n = \pm 1$. The ansatz

$$\begin{aligned} \phi &= \frac{H(r)}{r}\hat{x}_iT_i, \\ A^i &= -i[1 - K(r)][\hat{x}_jT_j,\partial^i\hat{x}_kT_k], \end{aligned} \tag{2.31}$$

is spherically symmetric because it satisfies the equations

$$\begin{aligned} [-i\epsilon_{ijk}x_j\partial^k + T_i,\phi] &= 0, \\ [-i\epsilon_{ijk}x_j\partial^k + T_i,A_l] &= i\epsilon_{ilm}A_m. \end{aligned} \tag{2.32}$$

This ansatz leads to

$$D^i\phi = \left(\frac{d}{dr}\frac{H}{r}\right)\hat{x}_i\hat{x}_jT_j + \frac{HK}{r}\partial^i\hat{x}_jT_j,$$

$$B^i = (1-K^2)\frac{\hat{x}_i}{r^2}\hat{x}_jT_j - \frac{dK}{dr}\partial^i\hat{x}_jT_j, \tag{2.33}$$

and to the following solution to the Bogomol'nyi equations:

$$H = \pm(r\coth r - 1),\quad K = \frac{r}{\sinh r}. \tag{2.34}$$

This solution to the equations of motion in the BPS limit was first found by Prasad & Sommerfield (1975) (PS) without the use of the Bogomol'nyi equations.

$$H \to \pm(r-1) \text{ for } r \to \infty \tag{2.35}$$

shows that (2.31) is an $n = \pm 1$ monopole solution. It is in fact the most general 1-pole solution for SU(2) up to translation. For arbitrary semisimple gauge groups other spherically symmetric solutions to the Bogomol'nyi equations are known (cf. Bais 1979, Olive 1980).

For nonvanishing Higgs potential, $\lambda \neq 0$, we cannot use the Bogomol'nyi equations. Nevertheless we can prove that a solution of the form (2.31) exists. To this end we calculate the energy and obtain

$$E = 4\pi\int_0^\infty dr\Big[K'^2 + \frac{1}{2r^2}(rH'-H)^2 + \frac{1}{2r^2}(1-K^2)^2 + \alpha\frac{K^2H^2}{r^2} + \frac{\lambda r^2}{2}\Big(\frac{H^2}{r^2}-1\Big)^2\Big]. \tag{2.36}$$

For later reference we have introduced a constant α, which is equal to one for the ansatz (2.31). The corresponding Euler-Lagrange equations are the equations of motion (2.18) for our ansatz (2.31). Therefore, the configuration which minimizes the energy (2.36) is a solution to the equations of motion. That the energy attains its minimum was proved by Tyupkin, Fateev & Shvarts (1975) for $\alpha = 1$, and the corresponding solution is the 't Hooft-Polyakov monopole ('t Hooft 1974, Polyakov 1974).

The 't Hooft-Polyakov solution has analogues in SU(2) theories with Higgs fields which do not lie in the adjoint representation (Michel, O'Raifeartaigh & Wali 1977, O'Raifeartaigh & Rawnsley 1978, see App. B for a detailed discussion). A spherically symmetric ansatz is

$$\phi_M = \sqrt{\frac{4\pi}{2I+1}}\; Y^I_M(\phi,\theta)\,\frac{H(r)}{r},\quad I = 1,2,\ldots,\quad M = -I,\ldots,I, \tag{2.37}$$

where the Y^I_M are spherical harmonics of order I, together with the potentials of (2.31). For this ansatz the equations of motion reduce to the Euler-Lagrange equations obtained from the energy (2.36), where $\alpha = \frac{1}{2}I(I+1)$.

Since only α is different for different I, the technique used by Tyupkin et al. applies for arbitrary I. In fact, it can be shown in all cases that for a minimizing sequence

($h_n = \frac{1}{r}H_n - 1,\ k_n = K_n$),

$$||(h_n,k_n)|| \;=\; [\int_0^\infty dr(r^2h_n'^2 + k_n'^2) + h_n^2(1) + k_n^2(1)]^{\frac{1}{2}} < \infty \qquad (2.38)$$

holds. From this it follows that a limit (h_o,k_o) with

$$E(h_o,k_o) \leq \lim_{n\to\infty} E(h_n,k_n) \;=\; \inf^* E(h,k) \qquad (2.39)$$

exists. The energy attains its minimum for (h_o,k_o), which hence is a solution. These solutions for arbitrary I are the only known monopole solutions in the realistic case of nonvanishing Higgs potential.

For vanishing Higgs potential we can still prove the existence of a solution for the ansatz (2.37) by restricting our attention to minimizing sequences with $h_n(\infty) = 0$. Since there is no Higgs potential to guarantee that $h_o(\infty) = 0$ holds, we need the inequality

$$|h_n(r)| \leq (\int_r^\infty dr r^2 h_n'^2 \int_r^\infty dr \frac{1}{r^2})^{\frac{1}{2}} \leq \frac{C}{\sqrt{r}} \qquad (2.40)$$

to ensure the correct asymptotic behaviour for h_o. Thus even in the BPS limit solutions for arbitrary I exist.

These solutions can be embedded into SU(I + 1) gauge theory with a Higgs field in the adjoint representation. In these theories the Bogomol'nyi equations make sense, and we may ask whether they are satisfied by the embedded solutions. The answer is negative (see Burzlaff 1981 for I = 2 and App. B). We have a situation similar to the one we encountered in our discussion of the $\mathbb{C}P^N$ model.

Except for the PS monopole none of the solutions (2.37) is known in closed form. The asymptotic form of the 't Hooft-Polyakov monopole is

$$K \sim O(e^{-r}), \quad H - r \sim O(e^{-2\sqrt{\lambda}r}). \qquad (2.41)$$

Since the mass of the massive gauge particle is m = 1, and the mass of the Higgs particle is $\mu = 2\sqrt{\lambda}$ in our units, according to eq. (2.36) the mass of the monopole is

$$M \;=\; 4\pi m f(\lambda), \qquad (2.42)$$

with f(0) = 1 in the BPS limit. Monopoles in Grand Unified Theories with $m \sim 10^{15}$ Gev, which might be realistic models, are therefore very heavy. (On Grand Unified Theories see e.g. Ellis (1980), Tye (1982).)

Since in all realistic models the symmetry group G is broken down to a symmetry which contains the electromagnetic U(1) group, and G is semisimple for a unified theory, monopoles should exist according to formula (2.20). In physical terms, monopoles are created as topological defects during the symmetry breakdown (Kibble 1981). A corresponding classical monopole solution can be easily given for SU(5) Grand Unified Theory by embedding the 't Hooft-Polyakov solution.

SU(5) contains as subgroups

$$SU(3) = \begin{pmatrix} SU(3) & \\ & I_2 \end{pmatrix}, \quad SU(2) = \begin{pmatrix} I_3 & \\ & SU(2) \end{pmatrix},$$

and the two U(1) subgroups generated by the weak hypercharge Y and the charge matrix Q respectively:

$$Y = \frac{1}{\sqrt{15}}\,\mathrm{diag}(1,1,1,-\frac{3}{2},-\frac{3}{2}), \quad Q = \frac{1}{2}\sigma_3 - \sqrt{\frac{5}{3}}Y. \qquad (2.43)$$

A Higgs field in the adjoint representation with asymptotic form

$$\Phi \sim \frac{1}{\sqrt{15}}\,\mathrm{diag}(1,1,1,-\frac{3}{2},-\frac{3}{2}), \qquad (2.44)$$

and a Higgs field in the fundamental representation

$$H \sim H_o = c\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \qquad (2.45)$$

break the symmetry down to $SU(3) \times U(1)/Z_3$.

The three possible embeddings of the 't Hooft-Polyakov monopole with $H = H_o$ and asymptotic field Φ, which at each point at infinity can be gauge transformed to (2.44), are

$$H = H_o, \quad \Phi = \frac{H(\sqrt{5}\,r/\sqrt{12})}{r}\hat{x}_i T_i + T,$$

$$A^i = -i[1-K(\sqrt{5}\,r/\sqrt{12})]\,[\hat{x}_j T_j, \partial^i \hat{x}_k T_k], \qquad (2.46)$$

with

$$T = \frac{1}{\sqrt{15}}\,\mathrm{diag}(1,1,-\frac{1}{4},-\frac{1}{4},-\frac{3}{2}), \quad T_i = \frac{1}{2}\begin{pmatrix} 0 & & \\ & 0 & \\ & & \sigma_i \\ & & & 0 \end{pmatrix}, \qquad (2.47)$$

or

$$T = \frac{1}{\sqrt{15}}\,\mathrm{diag}(1,-\frac{1}{4},1,-\frac{1}{4},-\frac{3}{2}), \quad T = \frac{1}{\sqrt{15}}\,\mathrm{diag}(-\frac{1}{4},1,1,-\frac{1}{4},-\frac{3}{2}),$$

and the corresponding SU(2) generators T_i.

For the solution (2.46) and (2.47) the asymptotic behaviour of the gauge fields is

$$F^{ij} \sim \mp \frac{1}{r^2}\varepsilon_{ijk}\hat{x}_k \hat{x}_l T_l, \qquad (2.48)$$

which at each point is gauge equivalent to

$$F^{ij} \sim \pm \frac{1}{r^2}\epsilon_{ijk}\hat{x}_k\left(\frac{1}{2}Q + \frac{1}{\sqrt{3}}\frac{1}{2\sqrt{3}}\lambda_8\right). \tag{2.49}$$

Hence the charges $q = -1/3, -1/3, -1/3, 1, 0$, the magnetic charge $g = 2\pi$, the color charges $q_c = 1/2\sqrt{3}, 1/2\sqrt{3}, -1/\sqrt{3}, 0, 0$, and the color magnetic charge $g_c = 4\pi/\sqrt{3}$, satisfy the generalized quantisation condition

$$gq + g_c q_c = 2\pi n. \tag{2.50}$$

According to (2.50) the minimal magnetic charge is 2π, which corresponds to the flux change Cabrera has detected.

CHAPTER III: SOLITON THEORETIC METHODS

Since spherically symmetric ansätze do not yield solutions with magnetic charge greater than one, we start with axisymmetric ones. These reduce the Bogomol'nyi equations to the Ernst equation. Using solution generating techniques known from General Relativity for the Ernst equation, axisymmetric n-pole solutions have been constructed by Forgács et al. Following these authors, we set up a Riemann-Hilbert problem to generate a family of n-pole solutions with the maximal number of degrees of freedom. Here our aim is to make the basic ideas transparent (cf. Forgács, Horvath & Palla 1981c), and a comparison with other techniques possible. In the following chapter we will concentrate also on the explicit form of all SU(2) n-pole solutions.

3.1. AXISYMMETRIC CONFIGURATIONS

In Section 1.3 we saw that SU(2) n-instanton solutions are not hard to obtain. To find their time-independent analogues, which are the magnetic n-pole solutions to the SU(2) Bogomol'nyi equations, proves to be a harder task. So far we have solved this problem for $n = \pm 1$ only using a spherically symmetric ansatz. If we want to go to higher magnetic charge a spherically symmetric ansatz will not be sufficient (Guth & Weinberg 1976). (On axisymmetric configurations see also O'Raifeartaigh & Rouhani 1981a.)

In fact, one can put an upper bound of 1 on the topological quantum number $|n|$ for a spherically symmetric ansatz. Here the topological quantum number is the generalization of formula (2.6) with the long-range magnetic field $\hat{b} = b/|b|$,

$$r^2 \hat{x}_i B_i \xrightarrow[r\to\infty]{} b(\phi,\theta) \neq 0, \tag{3.1}$$

instead of $\hat{\Phi}$, which need not lie in the adjoint representation. Spherical symmetry means that

$$[-i\epsilon_{ijk}x_j\partial_k + t_i(\phi,\theta),\ \hat{b}] = 0$$

and

$$[-i\epsilon_{ijk}x_j\partial_k + T_i, A_m] = i\epsilon_{imn}A_n - i\partial_m T_i \tag{3.2}$$

hold, with $T_i \in SU(2)$ and rA_i approaching $t_i(\phi,\theta)$ or $a_i(\phi,\theta)$ respectively at infinity. Equations (2.32) are the special case with constant T_i of the general definition of spherically symmetry.

Using eqs. (3.2) the formula for the magnetic charge can be cast into the form (see O'Raifeartaigh (1977) for details)

$$|n| = \left|\frac{1}{2\pi}\int d\phi d\theta \sin\theta \operatorname{tr}(\hat{b}\hat{x}_i t_i)\right| \leq \frac{1}{4\pi}\int d\phi d\theta \sin\theta \sqrt{2\operatorname{tr}(\hat{x}_i t_i)^2}. \tag{3.3}$$

The spherical symmetry of B^i on the other hand implies

$$[2\mathrm{tr}(\hat{x}_i T_i)^2 - 1]\epsilon_{jkl}\hat{x}_k B_l = 0. \tag{3.4}$$

Since $\hat{x} \wedge \vec{B}$ cannot be identically zero for a $\vec{B}$ which is neither identically zero nor singular at the origin, $2\mathrm{tr}(\hat{x}_i T_i)^2 = 1$ holds for all r, and therefore at infinity. This establishes the upper bound of 1 on $|n|$ in eq. (3.3).

The proof just sketched shows that for n-poles with $|n| \geq 2$ a spherically symmetric ansatz will not be sufficient. We therefore try an axisymmetric ansatz, with Φ and A_i satisfying

$$[-i\partial_\phi + T_0, \Phi] = [-i\partial_\phi + T_0, A_\rho] = [-i\partial_\phi + T_0, A_\phi] = [-i\partial_\phi + T_0, A_z] = 0 \tag{3.5}$$

in cylindrical coordinates (ρ, ϕ, z). After a gauge transformation with $\omega = \exp(i\phi T_0)$, $T_0 \in su(2)$, the new fields satisfy eqs. (3.5) with $T_0 = 0$. In this gauge

$$\Phi = \Phi_1 T_2 + \Phi_2 T_3, \quad A_\rho = -a_1 T_1,$$
$$A_\phi = -b_1 T_2 - b_2 T_3, \quad A_z = -a_2 T_1, \quad T_i = \sigma_i/2, \tag{3.6}$$

with functions Φ_α, a_α, and b_α depending on ρ and z only, is obviously a subset of the axisymmetric ansätze. It is the most general axisymmetric ansatz in the $T_0 = 0$ gauge which is also mirrorsymmetric, i.e., symmetric with respect to the transformation $x \to -x$, $T_1 \to -T_1$, $A_\rho \to -A_\rho$, $A_z \to -A_z$. In the $T_0 = T_3$ gauge, the ansatz is still of the form (3.6), with T_ρ instead of T_1 and T_ϕ/ρ instead of T_2 ($T_\rho d\rho + T_\phi d\phi = T_1 dx + T_2 dy$).

For this ansatz, which was first given by Jang, Park & Wali (1978) and Manton (1978), the Bogomol'nyi equations,

$$\rho D_z\Phi = -F_{\rho\phi}, \quad \rho D_\rho\Phi = F_{z\phi}, \quad D_\phi\Phi = -\rho F_{z\rho}, \tag{3.7}$$

reduce to five equations for Φ_α, a_α, b_α ($\alpha = 1,2$). Three of these equations are solved if we express Φ_α, a_α, and b_α in terms of two unknown functions $f(\rho,z)$ and $\psi(\rho,z)$ as follows:

$$\Phi_1 = \frac{1}{f}\partial_z\psi, \quad \Phi_2 = -\frac{1}{f}\partial_z f,$$
$$a_1 = -\frac{1}{f}\partial_\rho\psi, \quad a_2 = -\frac{1}{f}\partial_z\psi,$$
$$b_1 = -\frac{\rho}{f}\partial_\rho\psi, \quad b_2 = \frac{\rho}{f}\partial_\rho f. \tag{3.8}$$

The two remaining equations can then be combined into one complex equation for $\epsilon = f + i\psi$ (Witten 1979b, Forgács, Horvath & Palla 1980):

$$(\mathrm{Re}\ \varepsilon)\ \vec{\nabla}(\rho\ \vec{\nabla}\ \varepsilon) - \rho(\vec{\nabla}\ \varepsilon)\cdot(\vec{\nabla}\ \varepsilon) = 0,\ \vec{\nabla} = (\partial_\rho, \partial_z). \qquad (3.9)$$

We have reduced our problem to one of solving eq. (3.9). A generalization of this result to arbitrary groups has been given by Bais and Sasaki (see Bais 1981).

Equation (3.9) is the Ernst equation (Ernst 1968), which has been carefully studied in General Relativity (see Cosgrove 1980). This is why many properties of the solutions and solution generating techniques for this equation are known. The Ernst equation is furthermore related to the O(3) model. This can be seen when eq. (3.9) is expressed in terms of

$$\tau = \begin{pmatrix} f^{-1}(f^2 + \psi^2) & -f^{-1}\psi \\ -f^{-1}\psi & f^{-1} \end{pmatrix}, \qquad (3.10)$$

and reads

$$\vec{\nabla}(\rho\tau^{-1}\ \vec{\nabla}\ \tau) = 0. \qquad (3.11)$$

In this form the similarity to the equation of motion of the model (1.17):

$$\partial^\mu\ (\tau^{-1}\partial_\mu\tau) = 0,\ \det\ \tau = -1,\ \tau = \vec{q}\cdot\vec{\sigma}, \qquad (3.12)$$

is obvious. It therefore comes as no surprise that some of the methods used for the sine-Gordon model can be generalized beyond the O(3) model.

Forgács, Horvath & Palla (1981a) used two of the symmetry transformations already known for the Ernst equation to construct solutions for axially symmetric magnetic n-poles. Houston & O'Raifeartaigh (1980) have shown that axisymmetric and mirrorsymmetric monopoles with charges of equal sign cannot be separated. The generating technique we are going to discuss in the next section can therefore yield only solutions corresponding to monopoles located at one point.

3.2. THE HARRISON TRANSFORMATION AND THE NEUGEBAUER-KRAMER MAPPING

To apply Harrison's Bäcklund transformation (Harrison 1978), we first cast the Ernst equation into a new form. After the coordinate transformation

$$\zeta_1 = \rho + iz,\ \zeta_2 = \rho - iz,\ \partial_\alpha = \partial/\partial\zeta_\alpha,\ \alpha = 1,2, \qquad (3.13)$$

eq. (3.9) reads

$$\partial_2\left(\frac{1}{2f}\partial_1\varepsilon\right) = \frac{i}{2f^2}\partial_1\varepsilon\partial_2\psi - \frac{1}{4f(\zeta_1+\zeta_2)}(\partial_1\varepsilon + \partial_2\varepsilon). \qquad (3.14)$$

In terms of

$$M_1 = \frac{1}{2f}\partial_1\varepsilon,\quad M_2 = \frac{1}{2f}\partial_1\bar{\varepsilon} \qquad (3.15)$$

and

$$N_\alpha = \bar{M}_\alpha, \; \alpha = 1,2, \tag{3.16}$$

eq. (3.14) and its complex conjugate equation can both be written in two different ways:

$$\partial_2 M_1 = M_1(N_2 - N_1) - \frac{1}{2(\zeta_1 + \zeta_2)}(M_1 + N_2),$$

$$\partial_2 M_2 = M_2(N_1 - N_2) - \frac{1}{2(\zeta_1 + \zeta_2)}(N_1 + M_2),$$

$$\partial_1 N_1 = N_1(M_2 - M_1) - \frac{1}{2(\zeta_1 + \zeta_2)}(N_1 + M_2),$$

$$\partial_1 N_2 = N_2(M_1 - M_2) - \frac{1}{2(\zeta_1 + \zeta_2)}(M_1 + N_2). \tag{3.17}$$

These equations are a system of first-order equations for M_α and N_α. Any solution (M_α, N_α) to these equations which satisfies the reality condition (3.16) yields a solution to the Ernst equation.

The next step is based on Harrison's proposal of a pseudopotential. Harrison was guided by the following considerations: To linearize the problem of solving nonlinear equations we need an associated linear system,

$$d\psi_1 = \tau_\alpha^\beta \psi_\alpha d\zeta_\beta, \; d\psi_2 = \mu_\alpha^\beta \psi_\alpha d\zeta_\beta, \tag{3.18}$$

for ψ_α, whose consistency condition is the nonlinear equation. Given the so-called Lax-pair (3.18), $q = \psi_2/\psi_1$ satisfies the Riccati type equation

$$dq = [\mu_1^\alpha + (\mu_2^\alpha - \tau_1^\alpha)q - \tau_2^\alpha q^2] d\zeta_\alpha. \tag{3.19}$$

Being aware of the usefulness of a Lax-pair, we suspect that a pseudopotential q might also prove useful, and in fact our suspicions will prove correct.

To find a pseudopotential for the system (3.17), Harrison used, not a Lax-pair, but the ansatz (3.19) directly with τ_α^β and μ_α^β depending on M_α and N_α linearly only. This assumption simplified the problem considerably. Using the equations (3.17) it is not difficult to show that one solution to the consistency condition is

$$\partial_1 q = (M_2 - M_1)q + p(w)\,(M_2 - M_1 q^2),$$

$$\partial_2 q = (N_1 - N_2)q + p^{-1}(w)\,(N_1 - N_2 q^2), \tag{3.20}$$

with

$$p(w) = \sqrt{\frac{w - i\zeta_2}{w + i\zeta_1}} \tag{3.21}$$

and an arbitrary constant $w \in \mathbb{C}$. (The calculations involved in the derivation of eqs. (3.20), (3.23) and (3.25) though long are elementary.)

The system (3.20) for q, M_α, N_α is consistent if M_α, N_α satisfy eqs. (3.17). This implies that $\tilde{q}$ is another pseudopotential if it satisfies (3.20) with

$$IM_1 = -M_2 + \frac{1}{4\rho}, \quad IM_2 = -M_1 + \frac{1}{4\rho},$$

$$IN_1 = -N_1 + \frac{1}{4\rho}, \quad IN_2 = -N_2 + \frac{1}{4\rho}, \tag{3.22}$$

instead of M_α, N_α, because (3.22) is a symmetry transformation for eqs. (3.17), the so-called Neugebauer-Kramer mapping (Neugebauer & Kramer 1962). In terms of q and p, $\tilde{q}$ reads

$$\tilde{q} = -\frac{p+q}{1+pq}, \tag{3.23}$$

because the right-hand side satisfies eqs. (3.20) for the Neugebauer-Kramer transforms IM_α and IN_α if q satisfies (3.20) for M_α and N_α.

To generate a new solution, we substitute

$$HM_\alpha = HM_\alpha(\zeta_\beta, M_\beta, N_\beta, q), \quad HN_\alpha = HN_\alpha(\zeta_\beta, M_\beta, N_\beta, q) \tag{3.24}$$

for M_α and N_α into eqs. (3.17), and expand. After using the equations for M_α, N_α and q, we are left with equations for their coefficients in (3.24). Harrison gave one solution to these equations, the Harrison transformation

$$HM_1 = \frac{q}{\tilde{q}}M_1 - \frac{q(p^2-1)}{4\rho(p+q)},$$

$$HM_2 = \frac{\tilde{q}}{q}M_2 - \frac{p^2-1}{4\rho(1+pq)},$$

$$HN_1 = \frac{1}{q\tilde{q}}N_1 - \frac{1-p^2}{4\rho p(p+q)},$$

$$HN_2 = q\tilde{q}N_2 - \frac{q(1-p^2)}{4\rho p(1+pq)}. \tag{3.25}$$

Another solution, in fact the one we shall use, is the Neugebauer-Kramer mapping (3.22) of the Harrison transformation (3.25), which reads

$$M_1' = IHM_1 = -\tilde{q}\left(\frac{1}{q}M_2 + \frac{p}{4\rho}\right),$$

$$M_2' = IHM_2 = -\frac{1}{\tilde{q}}\left(qM_1 + \frac{p}{4\rho}\right),$$

$$N_1' = IHN_1 = -\frac{1}{\tilde{q}}\left(\frac{1}{q}N_1 + \frac{1}{4p\rho}\right),$$

$$N_2' = IHN_2 = -\tilde{q}\left(qN_2 + \frac{1}{4p\rho}\right). \tag{3.26}$$

In order to iterate this Backlund transformation in the next section we need to know the following commutation property:

$$IH(p,q) = H(p,\tilde{q})I. \tag{3.27}$$

3.3. THE MONOPOLE AND THE AXISYMMETRIC 2-POLE

We are now left with the concrete task of picking a seed solution (M_α, N_α), integrating eqs. (3.20), and substituting the result into eqs. (3.26). In our formulation of the problem the simplest seed solution is given by a constant, pure imaginary M_1, equal to M_2, with a Higgs field which satisfies

$$\phi^2 = -4(M_1 - N_2)(M_2 - N_1) = 1 \tag{3.28}$$

and therefore describes the vacuum without magnetic poles. Using the reality condition we find that that seed solution is

$$M_1 = M_2 = -\frac{i}{4},\ N_1 = N_2 = \frac{i}{4},\ \epsilon = c_1 e^z + ic_2,\ \phi = -\tfrac{1}{2}\sigma_3,\ A_i = 0, \tag{3.29}$$

up to a sign. (Cf. Chakrabarti (1982), who works with spherical coordinates (r,θ) instead of (ρ,z), and starts from the spherical symmetric vacuum e^r instead of e^z in a slightly different framework.)

For (3.29) the pseudopotential is

$$q = -\tanh\left(\tfrac{1}{2}\sqrt{(w-z)^2+\rho^2} - \beta\right). \tag{3.30}$$

Since $q = \bar{q}$ holds for real constants w and β, (M'_α, N'_α) satisfy the reality condition (3.16). The solution has furthermore magnetic charge one, as

$$\begin{aligned}\phi^2 &= 4|M'_1 - N'_2|^2 = \left| -\frac{i}{2}\left(\frac{1}{q}+q\right) + \frac{1}{2\rho}\left(p - \frac{1}{p}\right)\right|^2 \\ &= \left| \coth\left(\sqrt{(w-z)^2+\rho^2} - 2\beta\right) - \frac{1}{\sqrt{(w-z)^2+\rho^2}}\right|^2 \xrightarrow[r\to\infty]{} 1 - \frac{2}{r}\end{aligned} \tag{3.31}$$

shows for $\beta = 0$. To avoid a singularity β has to be chosen equal to zero. Since the zero of ϕ is at $(\rho = 0,\ z = w)$ the monopole is located there. We have recovered the PS monopole, whose ϕ^2 is given by eq. (3.31).

To iterate the Bäcklund transformation (3.25) (Forgács, Horvath & Palla 1981b) we do not have to integrate eqs. (3.20) again for a new seed solution. As for the sine-Gordon theory, a composition theorem reduces the problem to an algebraic one (see App. A for details). In fact, if (q_1, p_1) and (q_2, p_2) with different constants (w_α, β_α) are solutions to the eqs. (3.20) with (M_α, N_α), then

$$\left(q = \frac{\hat{q}_1 p_2 - \hat{q}_2 p_1}{q_1(\hat{q}_1 p_1 - \hat{q}_2 p_2)},\ p_2\right) \tag{3.32}$$

is a solution to (3.20) with (HM_α, HN_α) generated by (q_1, p_1). Thus iterating (3.26) once results in

$$IH(p_2, \hat{q})\, IH(p_1, q_1) = H(p_2, q)\, H(p_1, q_1) \tag{3.33}$$

with q given in (3.32).

After this iteration $M_1^{(2)}$ and $N_2^{(2)}$ read explicitly

$$M_1^{(2)} = qq_1 \left(\frac{p_1q_1 - p_2q_2}{p_2q_1 - p_1q_2} M_1 + \frac{p_1^2 - p_2^2}{4\rho(p_2q_1 - p_1q_2)} \right),$$

$$N_2^{(2)} = qq_1 \left(\frac{p_2q_1 - p_1q_2}{p_1q_1 - p_2q_2} N_2 + \frac{p_2^2 - p_1^2}{4\rho p_1 p_2(p_1q_1 - p_2q_2)} \right). \tag{3.34}$$

The analogous formulas for $M_2^{(2)}$ and $N_1^{(2)}$ show that the reality condition (3.16) can be satisfied for the seed solution (3.29) and real (w_α, β_α). However for this choice of parameters ϕ^2 is singular. The 2-pole therefore cannot be generated from the 1-pole for which w_1 is real.

The reality condition for the seed solution (3.29) can also be satisfied by choosing

$$w_1 = \overline{w}_2 = i\pi/2, \ \beta_1 = 0, \ \beta_2 = -i\pi/2. \tag{3.35}$$

For this choice of parameters

$$\overline{p}_1 = p_2^{-1}, \ \overline{q}_1 = q_2^{-1}, \ |qq_1| = 1, \tag{3.36}$$

and the generated ϕ^2 is regular. On the z-axis ϕ^2 reads

$$\phi^2 = \left| \tanh z - \frac{2z}{z^2 + \pi^2/4} \right|^2 \xrightarrow[z\to\infty]{} 1 - \frac{4}{z}, \tag{3.37}$$

and therefore belongs to the 2-pole solution.

We have succeeded in going beyond the PS monopole solution. Iterating the algebraic technique used to construct the 2-pole Forgács et al. (1981a) generated n-pole solutions for arbitrary n (see App. C, cf. also Lee 1981). It has not been proved so far that these solutions are regular. We do know however that the monopoles are all located at one point, because these solutions are axi- and mirrorsymmetric. To generate solutions for separated monopoles, whose existence has been proved (Jaffe & Taubes 1980), we have to use a different technique.

3.4. THE RIEMANN-HILBERT TRANSFORMATION

We have seen that from a Lax-pair of a special form a pseudopotential can be constructed and that a pseudopotential can be useful for generating solutions. To demonstrate that an associated linear system itself can lead to a generating technique, we consider again, as a simple example, the O(3) model with its nonlinear equation of motion (3.12). Since in characteristic coordinates (1.12) the consistency condition for

$$\partial_\xi Y = \frac{\zeta}{1-\zeta} \tau^{-1} \partial_\xi \tau Y = \frac{\zeta}{1-\zeta} \Omega_\xi Y,$$

$$\partial_\eta Y = \frac{-\zeta}{1+\zeta} \tau^{-1} \partial_\eta \tau Y = \frac{-\zeta}{1+\zeta} \Omega_\eta Y, \tag{3.38}$$

$$\det Y = 1, \; Y(0) = I, \; \zeta \in \mathbb{C}, \tag{3.39}$$

is the equation of motion (3.12), the linear system (3.38) constitutes the Lax-pair we are seeking.

From a solution Y to (3.38) we can generate a new solution Y' by means of the Riemann-Hilbert (RH) transformation (see Ueno & Nakamura 1981): We assume that Y is holomorphic in $C \cup C_+$, where C is an annulus in the ζ-plane, and $C_+(C_-)$ is the inside (outside, respectively) of C, and that a pair $\chi_\pm$, holomorphic in $C \cup C_\pm$, with

$$\chi_-(\zeta) = \chi_+(\zeta)\, Y(\zeta)\, g(\zeta)\, Y^{-1}(\zeta), \; \zeta \in C, \tag{3.40}$$

$$\chi_+(0) = I,$$

exists. Here g is a function of ζ alone, which is analytic on C and satisfies the conditions

$$g^+(\bar{\zeta})\, g(\zeta) = I, \; \det g = 1. \tag{3.41}$$

Since $\chi_\pm$ is analytic in $C \cup C_\pm$,

$$\det \chi_+(\zeta) = \det \chi_-(\zeta) = \det \chi_+(0) = 1, \; \zeta \in C, \tag{3.42}$$

holds and therefore

$$Y'(\zeta) := \chi_+(\zeta)\, Y(\zeta) = \chi_-(\zeta)\, Y(\zeta)\, g^{-1}(\zeta), \; \zeta \in C, \tag{3.43}$$

satisfies the conditions (3.39). Furthermore

$$\begin{aligned} &(\pm 1 - \zeta)\, (\partial_\alpha \chi_+)\, \chi_+^{-1} + \zeta \chi_+\, \tau^{-1} \partial_\alpha\, \tau \chi_+^{-1} \\ &= (\pm 1 - \zeta)\, (\partial_\alpha \chi_-)\, \chi_-^{-1} + \zeta \chi_-\, \tau^{-1} \partial_\alpha\, \tau \chi_-^{-1} \\ &= \pm \zeta \partial_\alpha\, \dot{\chi}_+(0) + \zeta \tau^{-1} \partial_\alpha \tau, \\ &\dot{\chi}_+ = \frac{\partial}{\partial \zeta} \chi_+, \; \zeta \in C, \end{aligned} \tag{3.44}$$

holds because of the analyticity properties of $\chi_\pm$. This implies that Y' is a solution to (3.38) with

$$\Omega'_\xi = \partial_\xi \dot{\chi}_+(0) + \tau^{-1} \partial_\xi \tau, \; \Omega'_\eta = -\partial_\eta \dot{\chi}_+(0) + \tau^{-1} \partial_\eta \tau. \tag{3.45}$$

To set up the analogous RH problem for the self-duality equations (1.53) we need an associated linear system. In the next section we will see how Ward's geometrical interpretation leads to the Lax-pair

$$(A_{\alpha'1} - \zeta A_{\alpha'2})\, k = i(\partial_{\alpha'1} - \zeta \partial_{\alpha'2})\, k =: iD_{\alpha'} k, \tag{3.46}$$

$$\alpha' = 1,2,\ x^{\alpha'\alpha} = (x_\mu e_\mu)^{\alpha'\alpha},\ e = (i\vec{\sigma},\ I_2),$$

$$A_{\alpha'\beta} dx^{\alpha'\beta} = A_\mu dx^\mu,\ \mu = 1,2,3,4,\ \zeta \in C.$$

The Lax-pair (3.46) was found independently by Belavin & Zakharov (1978). Whichever method we use to reach eqs. (3.46), given these equations it is easy to check that the compatibility conditions are the self-duality equations.

Equations (3.46) imply that for any $k(\zeta,x)$ with $(D_{\alpha'}k)\,k^{-1}$ linear in ζ and $\det k = 1$, the $A_{\alpha'\beta}(x)$ defined in (3.46) are automatically self-dual and traceless. The problem is to find a k with these properties. This can be reduced to the RH problem of finding a $G(x,\zeta)$ and $k_\pm$ holomorphic in $C \cup C_\pm$ with $\det G = 1$ and

$$D_{\alpha'}G = 0,\ k_+G = k_-,\ \zeta \in C. \tag{3.47}$$

Since for $k_\pm$

$$(D_{\alpha'}k_+)\,k_+^{-1} = (D_{\alpha'}k_-)\,k_-^{-1},\ \zeta \in C, \tag{3.48}$$

holds, $(D_{\alpha'}\ k_+)\,k_+^{-1}$ is indeed linear in ζ because of the analyticity properties of $k_\pm$, and since

$$\det k_+(\zeta) = \det k_-(\zeta) = \det k_\pm(0) \tag{3.49}$$

holds, we can always normalize $k_\pm$ to guarantee $\det k_\pm = 1$. Given a solution k holomorphic in C,

$$G = kgk^{-1},\ D_{\alpha'}g = 0,\ \det g = 1, \tag{3.50}$$

is a possible starting point for the solution of the RH problem (3.47), which will yield a new solution, the RH transform $k' = k_+k$.

The splitting of G in eq. (3.47) into $k_\pm$ is determined only up to a ζ-independent matrix Ω with determinant one, because $k_\pm' = \Omega k_\pm$ solves the RH problem if $k_\pm$ does. The transformation $k_\pm \to k_\pm'$ is a gauge transformation

$$A_{\alpha'\alpha} \to A'_{\alpha'\alpha} = \Omega A_{\alpha'\alpha}\Omega^{-1} + i(\partial_{\alpha'\alpha}\Omega)\Omega^{-1} \tag{3.51}$$

for the potentials. We can use the gauge freedom to impose the condition $k_-^0 := k_-(\zeta = \infty) = I$, which implies $A_{\alpha'2} = 0$, since

$$A_{\alpha'1} = i(\partial_{\alpha'1}k_+^0)\ (k_+^0)^{-1},$$

$$A_{\alpha'2} = i(\partial_{\alpha'2}k_-^0)\ (k_-^0)^{-1} \tag{3.52}$$

hold with $k_+^0 := k_+^0(\zeta = 0)$. Furthermore, for time-independent configurations we can assume $\partial_{11} = -\tfrac{1}{2}\partial_3$, and identify the Higgs field Φ with A_4.

In the $A_{\alpha'2} = 0$ gauge the Lax-pair (3.46) was given by Pohlmeyer (1980). This is the gauge Forgács, Horvath & Palla (1982) work in to construct solutions for separated monopoles.

They start from the ground state (3.29) in the $A_{\bar{a}'2} = 0$ gauge, which in terms of k_+^0 reads $k_+^0 = \mathrm{diag}\,(e^{-z}, e^{z})$, and calculate $k(\zeta)$ from eqs. (3.46) and (3.52). For the ansatz

$$k_+(\zeta) = I + \sum_{r=1}^{n} \frac{R_r}{\zeta - \mu_r}, \qquad (3.53)$$

where R_r and μ_r are ζ-independent and the μ_r satisfy certain differential equations, they write down the RH transform $k_+'^0 = k_+(0)k_+^0$ and identify the axially symmetric solutions from Section 3.3. Imposing constraints, which are not yet solved explicitly, they are also able to identify a 4n-1 parameter family of n-pole solutions. Since 4n-1 is the maximal number of degrees of freedom (Weinberg 1979) this family might be the most general n-pole solution.

We will not discuss this approach further here because we will study a different solution to the RH problem in detail in the next chapter. What we aimed to do in this section was to set up the problem and supply the information needed to show the relation between the two methods of solving it. The reader who would like further details should consult the original articles by Forgács, Horvath & Palla.

CHAPTER IV: WARD'S METHOD

In this chapter we discuss Ward's geometrical interpretation of self-dual fields and the method based on it (see Ward 1982b, O'Raifeartaigh & Rouhani 1981a). This interpretation will lead to the linear system and the Riemann-Hilbert problem discussed in the previous section. The solutions to the Riemann-Hilbert problem for magnetic poles given by Ward, by Prasad & Rossi and by Corrigan & Goddard are reviewed. We also study the relation of this complex manifold technique to Yang's approach, which is interesting in its own right, and to the method based on it which was applied by Prasad & Rossi to the monopole problem.

4.1. GEOMETRY OF SELF-DUAL FIELDS

Ward's method is based on the observation that self-dual fields vanish on anti-self-dual planes (Ward 1977, Atiyah & Ward 1977). An anti-self-dual plane is defined by the two equations

$$\chi = x\eta, \quad \eta \neq 0, \tag{4.1}$$

with two 2-spinors $\chi^{\alpha'}, \eta_\alpha$ ($\alpha, \alpha' = 1,2$), and the quaternion

$$x = x_4 + i\vec{x}\cdot\vec{\sigma} = \begin{pmatrix} x_4 + ix_3 & x_2 + ix_1 \\ -x_2 + ix_1 & x_4 - ix_3 \end{pmatrix} = \sqrt{2}\begin{pmatrix} y & -\bar{z} \\ z & \bar{y} \end{pmatrix} \tag{4.2}$$

for the complex coordinates $x_\mu \in \mathbb{C}$ of 4-dimensional Euclidean space-time. It is called anti-self-dual because

$$\Omega_{\mu\nu} := dx_\mu\, dy_\nu - dx_\nu\, dy_\mu = -\Omega^*_{\mu\nu}, \quad dx\,\eta = dy\,\eta = 0, \tag{4.3}$$

holds for any displacements in the plane; this can be proved easily for the three different combinations of indices.

Equation (4.3) implies, for self-dual field $F_{\mu\nu}$, the equation

$$F_{\mu\nu}dx_\mu dy_\nu = \tfrac{1}{2}F_{\mu\nu}\Omega_{\mu\nu} = \tfrac{1}{4}(F_{\mu\nu} - F^*_{\mu\nu})\Omega_{\mu\nu} = 0, \tag{4.4}$$

and therefore yields the result we were trying to prove: A self-dual field $F_{\mu\nu}$ vanishes on any anti-self-dual plane. This holds of course in particular for the anti-self-dual planes given by $\eta^T = (1,0), (0,1), (1,1)$. In these special cases the self-duality equations (4.4) read

$$F_{\bar{z}\bar{y}} = F_{yz} = F_{y\bar{y}} + F_{z\bar{z}} = 0 \tag{4.5}$$

($F_{\mu\nu}dx_\mu dy_\nu = F_{\alpha'\alpha,\beta'\beta}dx^{\alpha'\alpha}dy^{\beta'\beta}$). Equation (4.5) is the starting point for Yang's method, to which we will return in the next section.

In this section we elaborate on eq. (4.4) (see Corrigan, Fairlie, Yates & Goddard 1978a). That the gauge fields vanish on an anti-self-dual plane means that the potentials are pure gauge on anti-self-dual planes:

$$A_\mu dx_\mu \;=\; i(\partial_\mu \omega)\omega^{-1} dx_\mu,\; dx\,\eta \;=\; 0. \tag{4.6}$$

Here ω is an element of the complexification of the gauge group, i.e. an element of $SL(N,\mathbb{C})$ for the gauge group $SU(N)$. Equation (4.6) allows us to integrate between any two points x and y on the plane to obtain a group element as the path-ordered exponential

$$\omega_{[\theta]}(x,y) \;=\; P\exp\left(-i\int_x^y dx_\mu A_\mu\right), \tag{4.7}$$

where

$$[\theta] \;=\; \{\lambda\theta : \lambda \in \mathbb{C}\},\; \theta = \begin{pmatrix}\eta\\ \chi\end{pmatrix} \in \mathbb{C}^4,\; \eta \neq 0, \tag{4.8}$$

labels the planes.

On the one hand eqs. (4.8) tell us that the set of anti-self-dual planes is isomorphic to $\mathbb{C}P^3 \setminus \mathbb{C}P^1$ because we have to omit those θ for which $\eta = 0$ holds. Equation (4.7) on the other hand permits us to define the parallel transport of 2-spinors ψ over the plane as

$$\psi_{[\theta]}(x) \;=\; \omega_{[\theta]}(x,y)\,\psi_{[\theta]}(y),\; \psi_{[\theta]} \in V_{[\theta]}. \tag{4.9}$$

If parallel transported ψ's at different points are identified, $V_{[\theta]}$ is an N-dimensional vector space in the case of the gauge group SU(N). We have constructed an N-dimensional vector bundle over $\mathbb{C}P^3 \setminus \mathbb{C}P^1$: This is the geometrical interpretation of self-dual gauge fields we were looking for.

We can set up coordinates for this bundle in the following way: $\mathbb{C}P^3 \setminus \mathbb{C}P^1$ can be covered by the two coordinate patches

$$x^1_{[\theta]} \;=\; \begin{pmatrix}\chi^1/\eta_1 & 0\\ \chi^2/\eta_1 & 0\end{pmatrix},\quad \eta_1 \neq 0,$$

$$x^2_{[\theta]} \;=\; \begin{pmatrix}0 & \chi^1/\eta_2\\ 0 & \chi^2/\eta_2\end{pmatrix},\quad \eta_2 \neq 0, \tag{4.10}$$

and $\psi_{[\theta]}(x^1_{[\theta]})$ and $\psi_{[\theta]}(x^2_{[\theta]})$ are the corresponding coordinates in the fibre. $\psi_{[\theta]}(x^1_{[\theta]})$ and $\psi_{[\theta]}(x^2_{[\theta]})$ are related by

$$\psi_{[\theta]}(x^1_{[\theta]}) \;=\; \omega_{[\theta]}(x^1_{[\theta]},x)\,\omega_{[\theta]}(x,x^2_{[\theta]})\,\psi_{[\theta]}(x^2_{[\theta]}). \tag{4.11}$$

$$G(\chi,\eta) \;=\; \omega_{[\theta]}(x^2_{[\theta]},\,x^1_{[\theta]}) \tag{4.12}$$

is the so-called transition function into which we have coded the information about the self-dual field we began with. Since G is homogeneous:

$$G(\chi,\eta) = G(\lambda\chi,\lambda\eta), \; \lambda \in \mathbb{C}; \tag{4.13}$$

it in fact depends only on

$$\zeta = \eta_1/\eta_2, \; u = i\chi^2/\eta_2, \; v = i\chi^1/\eta_1. \tag{4.14}$$

This means that $D_{\alpha'}G$ with $D_{\alpha'}$ defined in eq. (3.46) vanishes, and eq. (4.11) turns out to be the solution to the RH problem (3.47) with

$$k^{-1} := k_+ = \omega_{[\theta]}(x, x^2_{[\theta]}), \; h^{-1} := k_- = \omega_{[\theta]}(x, x^1_{[\theta]}), \tag{4.15}$$

where $k_\pm$ have the correct analyticity properties. We have seen that to each self-dual field there corresponds a Laurent-decomposable transition function $G(u,v,\zeta)$.

Given a Laurent-decomposable G, we find the corresponding self-dual potentials with the help of eqs. (4.11) and (4.6). By solving the RH problem (4.11), we obtain $k_\pm$ with the correct analyticity properties. For $k_\pm$ eq. (4.6) reads

$$A_{\alpha'1} - \zeta A_{\alpha'2} = i(D_{\alpha'}k_\pm)\, k_\pm^{-1}, \tag{4.16}$$

because $dx_{\alpha'2} = -\zeta dx_{\alpha'1}$ holds on an anti-self-dual plane. So we have recovered the linear eigenvalue problem (3.46). It remains to choose transition functions which can be split into $k_\pm$ and correspond to magnetic poles.

4.2. CONSTRUCTION OF MAGNETIC POLES AND RELATION TO YANG'S FORMULATION

If, for SU(2), $g = G^{-1}$ is of the special triangular form

$$g = \begin{pmatrix} \zeta^n & \rho(\zeta,u,v) \\ 0 & \zeta^{-n} \end{pmatrix}, \; \zeta = \sum_{r=-\infty}^{\infty} \rho_r(x)\zeta^r, \tag{4.17}$$

the RH problem is easily solved. (That $D_\alpha G$ vanishes is reflected in the equations $\partial_\mu\partial_\mu\rho_r = 0$.) The analyticity properties of ($k_+^{-1} = k$, $k_-^{-1} = h$) and the equation $gk = h$ imply

$$k_{2\alpha} = \sum_{r=0}^{n} k^r_{2\alpha}(x)\zeta^r, \; h_{2\alpha} = \sum_{r=-n}^{0} k^{r+n}_{2\alpha}\,\zeta^r,$$

$$k_{1\alpha} = -\sum_{r=0}^{n} k^r_{2\alpha} \sum_{s=-r+n}^{\infty} \rho_s\, \zeta^{r+s-n},$$

$$h_{1\alpha} = \sum_{r=0}^{n} k^r_{2\alpha} \sum_{s=-\infty}^{-r} \rho_s\, \zeta^{r+s}, \quad \alpha = 1,2, \tag{4.18}$$

together with the 2(n-1) constraints

$$\sum_{r=0}^{n} k^r_{2\alpha}\, \rho_{s-r} = 0, \; 1 \leqslant s \leqslant n-1, \tag{4.19}$$

on the 2(n+1) functions $k^r_{2\alpha}(x)$. (For the generalization to SU(N) see Burzlaff 1982). After the condition det $k = 1$ is imposed, 3 arbitrary functions remain. This reflects the gauge freedom (3.51).

We have taken the first step in our construction of magnetic poles by constructing self-dual fields. Next we impose the conditions of time-independence and reality. For this we avail ourselves of the following equivalence relation:

g and $\tilde{g}$ are equivalent, $g \sim \tilde{g}$,

if $\Lambda_\pm \in SL(2,\mathbb{C})$ holomorphic away from $\zeta = \infty$ or $\zeta = 0$, respectively, exist with det $\Lambda_\pm = 1$ and

$$\tilde{g}(\zeta,u,v) = \Lambda_-(\zeta,u,v)\, g(\zeta,u,v)\, \Lambda_+(\zeta,u,v). \tag{4.20}$$

g and $\tilde{g}$ are called equivalent because $\tilde{g}$ can be split using $\tilde{k}_- = k_-\Lambda_-^{-1}$, $\tilde{k}_+ = k_+\Lambda_+$, and leads to the same gauge potentials (3.52).

The time-independence of the solution is now guaranteed if $\tilde{g}$ satisfies

$$\tilde{g}(\zeta,u,v) = \tilde{g}(\zeta,\gamma), \quad \gamma = u - v. \tag{4.21}$$

Since

$$\gamma = 2x_3 + i(x^{21}\zeta - x^{12}\zeta^{-1}) \tag{4.22}$$

is independent of x_4, so also is $\tilde{g}$, and finally also the potentials (3.52).

To guarantee that the potentials are real in some gauge for real x_μ, we impose the condition

$$\tilde{g}(\zeta,u,v) = \tilde{g}^+(-\bar{\zeta}^{-1},-\bar{v},-\bar{u}) \tag{4.23}$$

for $x_\mu = \bar{x}_\mu \in \mathbb{R}$. That this condition is sufficient can be proved as follows (Ward 1980): In terms of $\tilde{k}$ and $\tilde{h}$ we can rewrite eq. (4.23) for $x_\mu \in \mathbb{R}$ as

$$\tilde{k}_-(\zeta,x)\, \tilde{k}_+^+(-\bar{\zeta}^{-1},x) = \tilde{k}_+(\zeta,x)\, \tilde{k}_-^+(-\bar{\zeta}^{-1},x) = \lambda(x). \tag{4.24}$$

The analyticity properties of $\tilde{k}_\pm$ imply that the left and right-hand side of (4.24) are independent of ζ.

Under the gauge transformation

$$\tilde{k}_\pm \to \Omega\tilde{k}_\pm, \tag{4.25}$$

λ changes to $\Omega\, \lambda\, \Omega^+$. We can therefore always choose an Ω such that

$$\lambda = \pm I \tag{4.26}$$

holds in eq. (4.24). This leads to the equation

$$[i(D_1\tilde{k}_+)\, \tilde{k}_+^{-1}]^+ = i\bar{\zeta}[(\partial_{21} + \bar{\zeta}^{-1}\partial_{22})\, \tilde{k}_-(-\bar{\zeta}^{-1})]\, \tilde{k}_-^{-1}(-\bar{\zeta}^{-1}), \tag{4.27}$$

and therefore to

$$[A_{11} - \zeta A_{12}]^+ = A_{22} + \bar{\zeta}A_{21}, \tag{4.28}$$

that is, to the reality condition $A_\mu^+ = A_\mu$. (The condition (4.23) is also necessary, but since this will not arise in our work we will not prove it here.)

Given a $\tilde{g}$ which satisfies the condition (4.23), it therefore follows that the potentials constructed from $\tilde{g}$ are real in some gauge. From the proof given above we learn also how to construct the potentials in the real gauge. Equations (4.24) and (4.26) tell us that in the real gauge

$$\tilde{k}_-^+(-\bar{\zeta}^{-1}) = \pm \tilde{k}_+^{-1}(\zeta) \tag{4.29}$$

holds. Using for $k_\pm$ the expressions

$$\tilde{k}_+ = \Omega k_+ \Lambda_+, \quad \tilde{k}_- = \Omega k_- \Lambda_-^{-1}, \tag{4.30}$$

we calculate

$$\Omega^+\Omega = \pm (k_-^{o+})^{-1} \Lambda_-^{o+} (\Lambda_+^o)^{-1} (k_+^o)^{-1}, \tag{4.31}$$

where the superscript zero labels the ζ-independent term of the corresponding Taylor expansion in ζ^{-1} or ζ respectively. From this equation we determine Ω up to a unitary transformation

$$\Omega = \omega\Omega_o, \quad \omega\omega^+ = 1, \tag{4.32}$$

which reflects the gauge freedom we still have in specifying the gauge potentials calculated from (4.30).

To the conditions guaranteeing time-independence and reality we must now add the condition of regularity and (2.13) for the right asymptotic behaviour of the n-pole solutions. Since for Laurent coefficients of the form $\rho_r = e^{ix^4}\hat{\rho}_r(\vec{x})$,

$$\Phi^2 = 1 - \partial_i\partial_i \ln \det D^{(n)} \tag{4.33}$$

holds (Prasad 1981, cf. O'Raifeartaigh & Rouhani 1981a), where $D^{(n)}$ is an n x n matrix with elements

$$D_{rs}^{(n)} = \rho_{n+1-r-s}, \quad 1 \leq r,\ s \leq n, \tag{4.34}$$

the second condition is a condition on the Laurent coefficients ρ_r. The regularity condition will be discussed in Section 4.3. Thus at this point one must find a transition matrix which will yield the correct asymptotic behaviour for det $D^{(n)}$, in addition to satisfying all the other conditions discussed above. This can be done for arbitrary n by using the transition matrix itself or by using the $k_\pm^o$ in the framework of Section 3.4, or in Yang's R gauge (Yang 1977, cf. Prasad 1980).

Yang's formulation of self-dual fields starts from eqs. (4.5). Two of these equations can be integrated, and then yield pure gauge potentials

$$A_y = i(\partial_y D)\, D^{-1}, \quad A_z = i(\partial_z D)\, D^{-1},$$

or

$$A_{\bar{y}} = i(\partial_{\bar{y}}\bar{D})\bar{D}^{-1}, \quad A_{\bar{z}} = i(\partial_{\bar{z}}\bar{D})\bar{D}^{-1}, \tag{4.35}$$

for fixed $\bar{y},\bar{z}$ or fixed y,z respectively. Since $\mathrm{tr}\, A_\mu = 0$ holds we can normalize:

$$\det D = \det \bar{D} = 1. \tag{4.36}$$

We see that eqs. (4.35) can be obtained from the Lax-pair (3.46) and the solution to the RH problem (3.47) by identifying D with k_+^0 and $\bar{D}$ with k_-^0 (see eqs. (3.52)).

If we now define

$$J = D^{-1}\bar{D}, \tag{4.37}$$

the third of the equations (4.5) can be written

$$\partial_y(J\partial_{\bar{y}}J^{-1}) + \partial_z(J\partial_{\bar{z}}J^{-1}) = 0 \tag{4.38}$$

(Notice the similarity to the equation of motion (3.12) of the O(3) model.) Since J can be parametrized as

$$J = \begin{pmatrix} \frac{1}{\phi_0} & \frac{\rho^2}{\phi_0} \\ \frac{\rho^1}{\phi_0} & \frac{\phi_0^2 + \rho^1\rho^2}{\phi_0} \end{pmatrix}, \tag{4.39}$$

eq. (4.38) is equivalent to the three equations

$$(\partial_y\partial_{\bar{y}} + \partial_z\partial_{\bar{z}}) \ln \phi_0 + \frac{1}{\phi_0^2}(\partial_y\rho^1\partial_{\bar{y}}\rho^2 + \partial_z\rho^1\partial_{\bar{z}}\rho^2) = 0,$$

$$\partial_{\bar{y}}(\frac{1}{\phi_0^2}\partial_y\rho^1) + \partial_{\bar{z}}(\frac{1}{\phi_0^2}\partial_z\rho^1) = 0,$$

$$\partial_y(\frac{1}{\phi_0^2}\partial_{\bar{y}}\rho^2) + \partial_z(\frac{1}{\phi_0^2}\partial_{\bar{z}}\rho^2) = 0, \tag{4.40}$$

for the three functions ϕ_0, ρ^1, and ρ^2.

If one can solve these equations for ϕ_0, ρ^1, and ρ^2, one will obtain J. Then J can for example be split into D and $\bar{D}$ of the special triangular form:

$$R = D = \begin{pmatrix} \sqrt{\phi_0} & 0 \\ -\frac{\rho^1}{\sqrt{\phi_0}} & \frac{1}{\sqrt{\phi_0}} \end{pmatrix}, \bar{R} = \bar{D} = \begin{pmatrix} \frac{1}{\sqrt{\phi_0}} & \frac{\rho^2}{\sqrt{\phi_0}} \\ 0 & \sqrt{\phi_0} \end{pmatrix}. \tag{4.41}$$

From (4.41) we obtain the potentials (4.35) in the R gauge. This formulation therefore corresponds to the solution of the RH problem with $k_\pm^0$ of the form (4.41) into which they can always be cast by using the gauge freedom (4.25). Another form of the gauge potentials which leads to eqs. (4.40) is

$$A_\mu = -\frac{1}{2\Phi_o}\begin{pmatrix} \eta^3_{\mu\nu}\partial_\nu\Phi_o & -\eta^-_{\mu\nu}\partial_\nu\rho^1 \\ \eta^+_{\mu\nu}\partial_\nu\rho^2 & -\eta^3_{\mu\nu}\partial_\nu\Phi_o \end{pmatrix}, \tag{4.42}$$

$$\eta^i_{\mu\nu} = \epsilon_{4i\mu\nu} + \partial_{i\mu}\partial_{\nu 4} - \partial_{i\nu}\partial_{\mu 4}, \quad \eta^\pm = \eta^1 \pm i\eta^2.$$

Starting from the transition matrix (4.17) the potentials (4.16) can be cast into the form (4.42). In this case the solutions Φ_o, ρ^1, and ρ^2 of eqs. (4.40) read

$$\Phi_o = (D^{(n)-1})_{1n}, \quad \rho^1 = -(D^{(n)-1})_{11}, \quad \rho^2 = (D^{(n)-1})_{nn} \tag{4.43}$$

in terms of the Laurent coefficients of the transition matrix (Corrigan et al. 1978a). (Yang's parametrization of self-dual fields has been generalized to SU(3) by Brihaye, Fairlie, Nuyts & Yates (1978), Prasad (1978) and Singh & Tchrakian (1981).)

4.3. SU(2) MAGNETIC n-POLE SOLUTIONS

The remaining problem is to find the right transition matrices (4.17). We use the transition matrix for the PS monopole for this task. This matrix reads (Ward 1981a, Manton 1978)

$$g = \begin{pmatrix} \zeta & \rho^{(1)} \\ 0 & \zeta^{-1} \end{pmatrix}, \quad \rho^{(1)} = e^u + v\,\frac{e^\gamma - e^{-\gamma}}{\gamma}. \tag{4.44}$$

With

$$\Lambda_- = \begin{pmatrix} e^{-v} & 0 \\ 0 & e^v \end{pmatrix}, \quad \Lambda_+ = \begin{pmatrix} 0 & -e^u \\ e^{-u} & \zeta\gamma e^{-u} \end{pmatrix}, \tag{4.45}$$

the equivalent $\tilde{g}$ is

$$\tilde{g} = \Lambda_- g \Lambda_+ = \begin{pmatrix} \frac{e^\gamma - e^{-\gamma}}{\gamma} & -\zeta e^{-\gamma} \\ \zeta^{-1} e^{-\gamma} & \gamma e^{-\gamma} \end{pmatrix}. \tag{4.46}$$

$\tilde{g}$ is time-independent and satisfies the reality condition (4.23).

Since no other monopole solutions exist within the ansatz (4.17) with n = 1 (Manton 1978), Ward (1981a) chose the following ansatz with n = 2:

$$g = \begin{pmatrix} \zeta^2 & \rho^{(2)} \\ 0 & \zeta^{-2} \end{pmatrix}, \quad \rho^{(2)} = e^u + v\,\frac{e^\gamma + e^{-\gamma}}{\gamma^2 + \pi^2/4} \tag{4.47}$$

which generalizes to arbitrary n as

$$\rho^{(n)} = e^{u+v}\,\frac{e^{\gamma} + (-1)^n e^{-\gamma}}{H^{(n)}},\; H^{(n)} = \prod_{s=1}^{n} (\gamma - \gamma_s),\; \gamma_s = i\frac{\pi}{2}(n + 1 - 2s). \quad (4.48)$$

The n-pole solutions corresponding to (4.48) for $n \geqslant 3$ have been found by Prasad & Rossi (Prasad 1981, Prasad & Rossi 1981a,c, cf. Rossi 1982, and see also Narain 1981) much of whose work is in Yang's R gauge.

Using the following generalization of (4.45):

$$\Lambda_- = \begin{pmatrix} e^{-v} & 0 \\ 0 & e^{v} \end{pmatrix}, \quad \Lambda_+ = \begin{pmatrix} 0 & -e^{u} \\ e^{-u} & \zeta^n H_n e^{-u} \end{pmatrix}, \quad (4.49)$$

we obtain

$$\tilde{g} = \begin{pmatrix} \frac{e^{\gamma} + (-1)^n e^{-\gamma}}{H_n} & (-\zeta)^n e^{-\gamma} \\ \zeta^{-n} e^{-\gamma} & H_n e^{-\gamma} \end{pmatrix}. \quad (4.50)$$

This establishes the proof of time-independence and reality.

To find the magnetic charge of these solutions is a more difficult problem. One must use the the asymptotic form of the Laurent coefficients $\rho_s^{(n)}(x)$, which read in terms of Cauchy integrals

$$\rho_s = \frac{e^{i(2x_4 + s\phi)}}{\pi} \int_0^{2\pi} d\psi\, e^{-is\psi - 2\rho\cos\psi}\, \frac{f_n(2x_3 - 2i\rho\sin\psi)}{H_n(2x_3 - 2i\rho\sin\psi)}, \quad (4.51)$$

$$x_1 + ix_2 = \rho e^{i\phi},$$

$$f_n(x) = \begin{cases} \sinh x & \text{for odd } n \\ \cosh x & \text{for even } n. \end{cases}$$

Prasad & Rossi (1981a,b) extracted the asymptotic behaviour of ρ_s from (4.51) by neglecting exponentially damped terms, with the result

$$\phi^2 \underset{r\to\infty}{\longrightarrow} 1 - 2\sum_{p=1}^{n}\frac{1}{r_p},\; r_p^2 = x_1^2 + x_2^2 + (x_3 - \gamma_p)^2. \quad (4.52)$$

This is the asymptotic behaviour of an n-pole. We still have to prove the regularity of the solutions, which means we must prove that the determinant of the matrix (4.34) does not vanish. For $n = 2,3$ this has been shown explicitly, but a complete proof for $n \geqslant 4$ is still being sought (see O'Raifeartaigh & Rouhani (1981a) for the problem of establishing regularity).

We do know, however, that these solutions possess axial and mirrorsymmetry (Prasad & Rossi 1981b), and cannot describe separated monopoles. (4.51) and (4.42) already show that the potentials are independent of ϕ. Ward (1981b) therefore generalized the ansatz (4.47) to include solutions for two separated monopoles, and Corrigan & Goddard (1981) extended his ansatz to magnetic n-poles. The Corrigan-Goddard ansatz, which satsifies the time-independence and reality condition, reads

$$\hat{g} = \begin{pmatrix} \hat{\rho} & (-\zeta)^n e^{-K_{n-1}} \\ \zeta^{-n} e^{-K_{n-1}} & H_n e^{-K_{n-1}} \end{pmatrix}, \tag{4.53}$$

with

$$\hat{\rho} = \frac{e^{K_{n-1}} + (-1)^n e^{-K_{n-1}}}{H_n},$$

$$H_n = \gamma^n + a_{n-1}\gamma^{n-1} + \dots + a_1\gamma + a_o = \prod_{r=1}^{n} (\gamma - \gamma_r),$$

$$K_{n-1} = i\frac{\pi}{2} \sum_{r=1}^{n} n_r \prod_{s\neq r}^{n} \frac{\gamma - \gamma_s}{\gamma_r - \gamma_s} = b + b_o\gamma + \dots + b_{n-2}\gamma^{n-1}. \tag{4.54}$$

The integers n_r,

$$n_r = \begin{cases} (0, \pm 2, \dots, \pm n \mp 1) & \text{for odd } n \\ \pm(1,3,\dots, n-1) & \text{for even } n, \end{cases} \tag{4.55}$$

are the smallest with the property that the zeros of H_n will be cancelled by the zeros in the numerator. The $a_r(\zeta, \zeta^{-1})$ are polynomials of degree $(n - r)$ in ζ and ζ^{-1} which satisfy

$$\overline{a_r(\zeta, \zeta^{-1})} = a_r(-\bar{\zeta}^{-1}, -\bar{\zeta}). \tag{4.56}$$

Because of this condition, $\overline{H_n(\zeta)} = H_n(-\bar{\zeta}^{-1})$ holds and the number of real parameters in the ansatz is $n(n + 2)$.

In order to solve the RH problem for the ansatz, further constraints must be imposed. The transition matrix (4.53) can be cast into the triangular form (4.17) by using the matrices

$$\Lambda_-^{-1} = \begin{pmatrix} e^{f_2} & 0 \\ 0 & e^{-f_2} \end{pmatrix}, \quad \Lambda_+^{-1} = \begin{pmatrix} H_n\zeta^n e^{-f_1} & e^{f_1} \\ -e^{-f_1} & 0 \end{pmatrix}, \tag{4.57}$$

if K_{n-1} allows for the decomposition $K_{n-1} = f_1 - f_2$ with Taylor series $f_1(u,v,\zeta)$ and $f_2(u,v,\zeta)$ in ζ and ζ^{-1} respectively. This decomposition is possible if the $n(n-2)$ conditions

$$\frac{1}{2\pi i}\oint \frac{d\zeta}{\zeta} b_r \zeta^s = 0, \quad r = 1,\dots,n-2, -r \leq s \leq r, \tag{4.58}$$

are satisfied. Together with the normalization condition for ϕ^2 at infinity this reduces the number of free parameters to 4n-1, which is the degree of freedom of magnetic n-poles (Weinberg 1979). That the ansatz (4.53) indeed describes an n-pole can be deduced from the asymptotic form of the ansatz and the equation

$$\partial_i \partial_i \rho_r = \rho_r \tag{4.59}$$

whose spherically symmetric solutions are of the form e^r/r. Formula (4.33) then leads to the correct asymptotic behaviour.

The solutions to the constraint equations (4.58) in general are not known explicitly. One special class of solutions describing a one-parameter family of separated n-poles (see Ward (1981b) for n = 2 and Brown, Prasad & Rossi (1981) for n ≥ 3), and the solution for small values of the parameters (O'Raifeartaigh, Rouhani & Singh 1982a), have been found. However, even for given solutions to the constraints, the non-axisymmetric potentials could not yet be constructed explicitly. For n = 2 the problem has been reduced to the solution of a quartic polynomial (O'Raifeartaigh & Rouhani 1981b, O'Raifeartaigh, Rouhani & Singh 1982b, Brown 1983). For n ≥ 3 all the details of the solutions have yet to be worked out.

However, the structure of the SU(2) n-pole solutions is known up to the proof of regularity, due to the ansatz (4.53), because this ansatz is guaranteed to contain all magnetic n-pole solutions (Hitchin 1982). As far as the twistor construction of all SU(N) magnetic poles is concerned only the first steps have been taken (Ward (1981c, 1982a) and Athorne (1983, 1982)). To get more control of the SU(2) solutions, and to take a different approach to SU(N) solutions, we will now study yet another technique.

CHAPTER V: THE ATIYAH-HITCHIN-DRINFELD-MANIN CONSTRUCTION

In this chapter we study the formulation of the self-duality conditions given by Atiyah, Hitchin, Drinfeld and Manin (ADHM) (see Atiyah 1979, Rawnsley 1978), and the relation of this method to the twistor approach. We apply this method to the instanton problem, which will be reduced to purely algebraic conditions. We then develop the technique of Nahm, who adapted the ADHM instanton construction for monopoles.

5.1. THE RELATION TO THE TWISTOR METHOD

In Section 4.1 we have seen that to any self-dual field correspond fields (4.9) which are covariantly constant on anti-self-dual planes defined by eq. (4.1). Indeed the Lax-pair (3.46) means exactly that the covariant derivative of $k(x) = \omega[\theta](x,y)$ along the anti-self-dual plane $S[\theta]$, i.e., in the direction of the tangent vectors,

$$n_1^T = (i\zeta, -\zeta, -i, 1), \quad n_2^T = (i, 1, i\zeta, \zeta), \tag{5.1}$$

vanishes. Looking at the twistor construction in this way, we can say that the covariantly constant fields on anti-self-dual planes contain all the information about self-dual fields.

In the ADHM technique (Atiyah, Hitchin, Drinfeld & Manin 1978, Drinfeld & Manin 1978) on the other hand, all the information about self-dual fields is carried by three linear spaces A, B and C. Following Witten (1979a) and Osborn (1982), we want to show how to construct these linear spaces, and how to relate them to the covariantly constant fields discussed above. To recover the covariantly constant fields, we first solve the linear equation

$$D_{\alpha\alpha'}\,\psi^{\alpha'} := (\partial_\mu + iA_\mu)\,(\bar{e}_\mu)_{\alpha\alpha'}\,\psi^{\alpha'} = 0, \tag{5.2}$$

$$\bar{e} = (-i\sigma, I_2), \quad e = (i\vec{\sigma}, I_2).$$

For an SU(N) self-dual field with topological quantum number n, there are n linearly independent solutions to (5.2) with asymptotic behaviour

$$\psi^{\alpha'}(x) \sim \frac{x^{\alpha'\alpha}}{|x|^4}\, c_{\alpha\kappa}(\hat{x}). \tag{5.3}$$

(See Osborn 1982 and references therein for a discussion of the asymptotic behaviour and of the dimensions of the linear spaces.) These n spinors $\psi^{\alpha'}$ span the space C.

The space B is spanned by the 2n + N linearly independent solutions $(\Omega, \Omega^{\alpha'\alpha}, \Omega^{\alpha'}_{\beta'})$ of the linear equations

$$D_{\alpha\alpha'}\,\Omega^{\alpha'\beta} = D_{\alpha\alpha'}\,\Omega^{\alpha'}_{\beta'} = 0, \tag{5.4}$$

$$D^{\alpha'\alpha}\,\Omega = \Omega^{\alpha'\alpha} + \Omega^{\alpha'}_{\beta'}\,x^{\beta'\alpha}, \tag{5.5}$$

with asymptotic behaviour

$$\Omega \sim b\,\kappa(\hat{x}). \tag{5.6}$$

In terms of the solutions $\psi^{\alpha'}$, which we assume to be assembled as a row vector, eq. (5.5) for the row vector of solutions Ω reads

$$D^{\alpha'\alpha}\Omega \;=\; \psi^{\alpha'}\Delta'^{\alpha}\,, \tag{5.7}$$

where Δ'^{α} is the n x (2n+N) matrix

$$\Delta'^{\alpha} \;=\; a'^{\alpha} + b'_{\alpha'}\,x^{\alpha'\alpha}. \tag{5.8}$$

To find the consistency conditions for eqs. (5.2) and (5.7) we use the following identities:

$$D_{\alpha\alpha'}D^{\alpha'\beta} \;=\; D_{\mu}D^{\mu}\delta_{\alpha}^{\beta} + \frac{i}{4}(F^{\mu\nu} - F^{*\mu\nu})\,(\bar{\eta}_{\mu\nu})_{\alpha}^{\beta}\,, \tag{5.9}$$

with the anti-self-dual tensor

$$\bar{\eta}_{\mu\nu} :\;=\; \bar{e}_{\mu}e_{\nu} - \delta_{\mu\nu}I_2\,, \tag{5.10}$$

and

$$(e_{\mu})^{\alpha'\alpha}\,(\bar{e}_{\mu})_{\beta\beta'} \;=\; 2\delta_{\beta}^{\alpha}\delta_{\beta'}^{\alpha'}. \tag{5.11}$$

They show that we can define the two linear spaces spanned by the solutions to eqs. (5.2) and (5.7) for self-dual fields only.

Given ($\psi^{\alpha'}$, Δ'^{α}, Ω), we solve the equation

$$\psi^{\alpha'}(a'^{\alpha}\eta_{\alpha} + b'_{\beta'}\chi^{\beta'})\,v' \;=\; 0 \tag{5.12}$$

and define $\omega := \Omega v'$ which satisfies

$$D^{\alpha'\alpha}\eta_{\alpha}\omega \;=\; \psi^{\alpha'}b'_{\beta'}\,(x^{\beta'\alpha}\eta_{\alpha} - \chi^{\beta'})\,v'. \tag{5.13}$$

Using the relations (5.11) and

$$e_{\mu} \;=\; -\,\epsilon\,\bar{e}_{\mu}^{T}\,\epsilon \tag{5.14}$$

we can show that the $(e_{\mu})^{\alpha'\alpha}\eta_{\alpha}$ are two tangent vectors to the anti-self-dual plane $S_{[\theta]}$. The left-hand side of eq. (5.13) is therefore the covariant derivative along the anti-self-dual plane. Since the right-hand side vanishes on $S_{[\theta]}$, ω is covariantly constant on $S_{[\theta]}$. Conversely, it can be shown that all covariantly constant fields on $S_{[\theta]}$ can be constructed from eq. (5.12), i.e., are in the kernel of the map $g : B \to C$,

$$(\Omega v',\ \Omega^{\alpha'\alpha}v',\ \Omega_{\beta'}^{\alpha'}v') \xrightarrow{g(\eta,\chi)}$$

$$(\Omega^{\alpha'\alpha}\eta_{\alpha} + \Omega_{\beta'}^{\alpha'}\chi^{\beta'})\,v' \;=\; \psi^{\alpha'}w'. \tag{5.15}$$

However, in general ω is not a function of $[\theta]$ alone. Only when we have identified all covariantly constant fields on $S_{[\theta]}$ which differ by fields vanishing on $S_{[\theta]}$ will the resulting equivalence class depend on $[\theta]$ only. To characterize the fields which vanish on $S_{[\theta]}$ we consider the linear space A spanned by the n linearly independent solutions to

$$D_{\alpha\alpha'}D^2\lambda_{\beta'}\,\epsilon^{\beta'\alpha'} \;=\; 0, \tag{5.16}$$

with asymptotic behaviour

$$\lambda_{\alpha'} \sim \epsilon_{\alpha'\beta'}\,\frac{x^{\beta'\alpha}}{|x|^4}\,a_\alpha\kappa(\hat{x}). \tag{5.17}$$

In terms of $\lambda_{\alpha'}$ we can define

$$\lambda^\alpha := -\lambda_{\alpha'}\,x^{\alpha'\alpha}, \tag{5.18}$$

$$P^{\alpha'} := \tfrac{1}{2}D^2\lambda_{\delta'}\,\epsilon^{\delta'\alpha'}\epsilon_{\beta'\gamma'},$$

$$(P^{\alpha'\beta\alpha} + P^{\alpha'\beta}{}_{\beta'}\,x^{\beta'\alpha},\; P^{\alpha'}{}_{\beta'}{}^{\alpha} + P^{\alpha'}_{\beta'\gamma'}\,x^{\gamma'\alpha}) := (-D^{\alpha'\alpha}\lambda_{\beta'}\,x^{\beta'\beta},\; D^{\alpha'\alpha}\lambda_{\beta'}),$$

$$P^{\alpha'\alpha}{}_{\beta'} \;=\; -P^{\alpha'}{}_{\beta'}{}^{\alpha}, \tag{5.19}$$

and the map $f : A \to B$,

$$(\lambda_{\alpha'}w,\; P^{\alpha'\alpha\beta}w,\; P^{\alpha'\alpha}{}_{\beta'}w,\; P^{\alpha'}{}_{\beta'}{}^{\alpha}w,\; P^{\alpha'}_{\beta'\gamma'}w) \xrightarrow{f(\eta,\chi)}$$

$$(-\lambda_{\beta'}\,x^{\beta'\alpha}\eta_\alpha w + \lambda_{\beta'}\,\chi^{\beta'}w,\; P^{\alpha'\beta\alpha}\eta_\beta w + P^{\alpha'}{}_{\beta'}{}^{\alpha}\chi^{\beta'}w,$$

$$P^{\alpha'\alpha}{}_{\beta'}\eta_\alpha w + P^{\alpha'}_{\gamma'\beta'}\,\chi^{\gamma'}w) \;=\; (\phi v',\; \Omega^{\alpha'\alpha}v',\; \Omega^{\alpha'}_{\beta'}v'). \tag{5.20}$$

The definition of $P^{\alpha'}_{\beta'\gamma'}$ and eq. (5.16) imply

$$D_{\alpha\alpha'}\,P^{\alpha'}_{\beta'\gamma'} \;=\; 0. \tag{5.21}$$

That this equation also holds for the other components of $P^{\alpha'}$ can be deduced from their definitions. If we compare these equations and eq. (5.19) to eqs. (5.4) and (5.7), respectively, we see that Ω can be written in the following way:

$$(-\lambda_{\alpha'}\,x^{\alpha'\alpha},\; \lambda_{\alpha'}) \;=\; (\Omega a^\alpha,\; \Omega b_{\alpha'}). \tag{5.22}$$

Because of this identity, the equation

$$\Omega\Delta^\alpha \;=\; 0 \tag{5.23}$$

holds for Ω, and

$$\Delta^\alpha \;=\; a^\alpha + b_{\alpha'}\,x^{\alpha'\alpha}. \tag{5.24}$$

It will be shown that we need know only Δ^{α} for the construction of instanton solutions. But now we will finish our discussion of the relation between the ADHM and the twistor method.

It is obvious from the definition of f that all fields $\Omega v'$ which are elements of im $f(\eta, \chi)$, i.e., for which

$$\Omega v' = \lambda_{\beta'}(\chi^{\beta'} - x^{\beta'\alpha}\eta_{\alpha})w \tag{5.25}$$

holds, vanish on $S_{[\theta]}$. What is not obvious, but nevertheless true, is that all fields which vanish on $S_{[\theta]}$ can be constructed this way. Hence,

$$E_{[\theta]} = \ker g(\eta, \chi) / \operatorname{im} f(\eta, \chi) \tag{5.26}$$

depends only on $[\theta]$ and includes all covariantly constant scalars on $S_{[\theta]}$. This completes the construction of the N-dimensional vector bundle over $\mathbb{C}P^3 \setminus \mathbb{C}P^1$ from Section 4.1 in terms of the linear spaces and maps A,B,C,f and g. In the next section we will see how we can reconstruct the instanton fields from Δ^{α} alone.

5.2. ALGEBRAIC CONSTRUCTION OF INSTANTON SOLUTIONS

Without relying on our previous analysis, or the mathematical background for the ADHM construction, we will now discuss the conditions the matrices A and B in

$$\Delta(x) = A + Bx \tag{5.27}$$

must satisfy to make it possible to construct SU(2) instanton solutions from Δ alone (see Corrigan, Fairlie, Goddard & Templeton 1978b, or Christ, Weinberg & Stanton 1978; cf. Corrigan 1979, and see Osborn (1982) for a discussion of the corresponding conditions on the linear spaces A, B and C). In eq. (5.27), A and B are constant $(n + 1) \times n$ matrices whose entries A_{ij} and B_{ij} are quaternions:

$$A_{ij} = a^{\mu}_{ij}e_{\mu}, \quad B_{ij} = b^{\mu}_{ij}e_{\mu}. \tag{5.28}$$

Bx is the matrix whose entries are $B_{ij}x$. The conditions on A and B are such that the entries of the $n \times n$ matrix $\Delta^{+}\Delta$ commute with quaternions,

$$e_{\mu}(\Delta^{+}\Delta) = (\Delta^{+}\Delta)e_{\mu}, \tag{5.29}$$

i.e., are real numbers times the unit matrix, and that

$$\det \Delta^{+}\Delta \neq 0 \tag{5.30}$$

holds.

Given Δ, one has to solve the linear equations

$$\Omega^{+}(x)\,\Delta(x) = 0 \tag{5.31}$$

for a vector of quaternions $\Omega^{+} = (\Omega_0^{+}, \Omega_1^{+}, \ldots, \Omega_n^{+})$ which satisfies the normalization condition

$$\Omega^+\Omega = \Omega_0^+\Omega_0 + \dots + \Omega_n^+\Omega_n = I_2. \tag{5.32}$$

In terms of Ω, the potentials read

$$A_\mu = -i\Omega^+\partial_\mu\Omega. \tag{5.33}$$

Since these potentials define self-dual fields with Pontryagin number n, as we shall see in the following, we can construct the spaces A, B and C from them. Hence A carries all the information about the instanton solutions.

Because of the normalization condition (5.32), A_μ is Hermitian and traceless, i.e., the gauge potentials $A_\mu^a(x)$ are real. The corresponding gauge fields are

$$iF_{\mu\nu} = \partial_\mu\Omega^+(1 - \Omega\Omega^+)\,\partial_\nu\Omega - \partial_\nu\Omega^+(1 - \Omega\Omega^+)\,\partial_\mu\Omega. \tag{5.34}$$

To show that $F_{\mu\nu}$ is self-dual we use the identity

$$I_{n+1} - \Omega\Omega^+ = \Delta(\Delta^+\Delta)^{-1}\,\Delta^+. \tag{5.35}$$

Since the condition (5.30) holds, the right-hand side in eq. (5.35) is defined, and is equal to the left-hand side because both are projection operators onto the n-dimensional subspace orthogonal to Ω, and their product in either order equals $\Delta(\Delta^+\Delta)^{-1}\,\Delta^+$. Using the condition (5.29), and again (5.31), we obtain

$$e_\mu(\Delta^+\Delta)^{-1} = (\Delta^+\Delta)^{-1}e_\mu, \tag{5.36}$$

$$(\partial_\mu\Omega^+)\,\Delta = -\,\Omega^+Be_\mu, \tag{5.37}$$

and finally

$$iF_{\mu\nu} = \Omega^+B(\Delta^+\Delta)^{-1}\,[e_\mu\bar{e}_\nu - e_\nu\bar{e}_\mu]\,B^+\Omega. \tag{5.38}$$

This $F_{\mu\nu}$ is self-dual because $e_\mu\bar{e}_\nu - e_\nu\bar{e}_\mu$ is manifestly self-dual.

We will not prove here that this construction leads to potentials whose singularities can be removed by a gauge transformation, and that it yields all SU(2) instanton solutions (see Atiyah et al. 1978, Drinfeld & Manin 1978). We will however show that it yields a family of solutions with the maximal number of degrees of freedom, that is, 8n-3 (Atiyah, Hitchin & Singer 1977, Brown, Carlitz & Lee 1977, Jackiw & Rebbi 1977, Schwartz 1977): To begin with, A and B have 8n(n+1) parameters. However, for constant matrices a and b,

$$aAb,\ aBb,\ a \in Sp(n+1),\ b \in GL(n, \mathbb{R}), \tag{5.39}$$

lead to $a\Omega$ for the solution of eq. (5.31), and therefore to the same potential (5.33) as A and B itself. This removes $3n^2 + 5n + 3$ parameters; since the condition (5.29), which must hold for arbitrary x, adds 5n(n-1) constraints, only 8n-3 free parameters are left.

It remains to be shown that n really is the instanton number. To show this, we use the identity (Osborn 1979)

$$\mathrm{tr}\,(F_{\mu\nu}\,F^{*}_{\mu\nu}) \;=\; -\,\partial^2\partial^2\,\ln\det\,(\Delta^+\Delta), \tag{5.40}$$

and write the instanton number (1.50) as a surface integral:

$$n \;=\; \frac{-1}{16\pi^2}\int d\sigma\;|x|^2\;x_\mu\partial_\mu\partial^2\,\ln\det\Delta^+\Delta. \tag{5.41}$$

Since at infinity

$$\Delta^+\Delta \xrightarrow[|x|\to\infty]{} B^+B|x|^2,$$

$$\ln\det\Delta^+\Delta \xrightarrow[|x|\to\infty]{} \ln\det B^+B + 2n\ln|x| \tag{5.42}$$

holds, the right-hand side of (5.41) is equal to n.

We see that the instanton problem has been reduced to one of finding all constant matrices A and B which satisfy the conditions (5.29) and (5.30). So far there is no explicit solution to this problem. However, the ability to reduce the self-duality equations to purely algebraic conditions represents enormous progress. This is especially true because we know that all stable, finite-action solutions are self-dual (Uhlenbeck 1978, Bourguignon, Lawson & Simons 1979). Therefore we would like to develop a formalism for monopoles, on similar lines to that developed in Sections 5.1 and 5.2 for instantons.

5.3. NAHM'S CONSTRUCTION FOR MONOPOLES

To show with a simple example how the formalism developed in Section 5.1 can be adapted to the monopole problem we will construct explicitly the transition matrix for the PS monopole solution (Osborn 1982, Burzlaff & Hornos 1982): the PS monopole solution can be written in the form (Manton 1978)

$$A_\mu \;=\; \frac{i}{2}(\partial_{\mu\nu} - \bar{e}_\mu e_\nu)\,\partial_\nu\;\ln a \tag{5.43}$$

with

$$a \;=\; \frac{1}{r}\sinh r\quad e^{it}. \tag{5.44}$$

For these potentials, eq. (5.2) is satisfied by

$$\epsilon_{\alpha'\beta'}\psi^{\beta'}_{\alpha} \;=\; \sqrt{a}\,\partial_{\alpha\alpha'}(b/a), \tag{5.45}$$

where

$$b \;=\; -\frac{i}{r}\sinh sr\quad e^{ist}. \tag{5.46}$$

That the instanton number is infinite is reflected in the parameter s in (5.46), and in the form of Δ'^{α},

$$\Delta'^{\beta}{}_{\alpha'}\,\epsilon_{\beta\alpha} \;=\; -i\,\overset{+}{\partial}_s\,\partial_{\alpha\alpha'} \;-\; \bar{x}_{\alpha\alpha'}, \tag{5.47}$$

and of Ω,

$$\Omega_{\alpha\alpha'} = a^{-1/2} (e^{isx})_{\alpha\alpha'}. \qquad (5.48)$$

Here we have assembled the two solutions $\Omega_{\alpha'}$ of eq. (5.7) as a 2 x 2 matrix.

Given $\psi^{\alpha'}_{\alpha}$ and $\Delta'^{\alpha}_{\alpha'}$, we write eq. (5.12) in the form

$$\int_0^1 ds\ \psi^{\alpha'}_{\alpha} \ (a'^{\beta}_{\beta'} \eta_\beta + b'_{\beta'\gamma'} x^{\gamma'})\, v^{\beta'} = 0, \qquad (5.49)$$

which by partial integration can be reduced to

$$i\, \epsilon^{\alpha\alpha'} \eta_\alpha \partial_s\ v'^{\alpha'} - 2\epsilon_{\alpha'\beta'} x^{\alpha'}\ v'^{\beta'} = 0. \qquad (5.50)$$

This equation admits two solutions

$$v'_1 = \begin{pmatrix} 0 \\ e^{-isx^1/\eta_1} \end{pmatrix}, \qquad v'_2 = \begin{pmatrix} e^{-isx^2/\eta_2} \\ 0 \end{pmatrix}, \qquad (5.51)$$

which we assemble as a 2 x 2 matrix v'. Since $\omega = \int_0^1 \Omega\, v'$ depends on $[\theta]$ only, this is the covariantly constant field we are looking for. Now we can calculate the transition matrix

$$\tilde{g} = \omega(x^1_{[\theta]})\ \omega^{-1}(x^2_{[\theta]}) \qquad (5.52)$$

for the two coordinate patches (4.10). After we apply the equivalence transformation $\Lambda_-\tilde{g}\Lambda_+ = g$ with

$$\Lambda_- = \begin{pmatrix} \sqrt{\dfrac{2(1-e^v)}{v}} & 0 \\ -\sqrt{\dfrac{v}{2(1-e^v)}}\,\zeta^{-1} & \sqrt{\dfrac{v}{2(1-e^v)}} \end{pmatrix},$$

$$\Lambda_+ = \begin{pmatrix} -\sqrt{\dfrac{u}{2(1-e^u)}}\,\zeta & \sqrt{\dfrac{2(1-e^u)}{u}} \\ -\sqrt{\dfrac{u}{2(1-e^u)}} & 0 \end{pmatrix}, \qquad (5.53)$$

g has the required form (4.44).

As could have been expected, for the PS monopole an infinite dimensional space tensored with the quaternion space replaces the (n+1)-dimensional quaternionic vector space for n-instanton fields. Nahm (1980) chose the complex Hilbert space $\mathcal{L}^2(-\tfrac{1}{2}, \tfrac{1}{2})$ as the infinite dimensional space, and

$$(i\partial_s + \bar{x})\ v(x,s) = 0,$$

$$\int_{-1/2}^{1/2} ds\ v^+(x,s)\ v(x,s) = 1,$$

$$A_\mu = -i\int_{-1/2}^{1/2} ds\ v^+(x,s)\ \partial_\mu v(x,s). \qquad (5.54)$$

This is the ADHM formulation of the PS monopole (for a complete description in terms of the spaces A,B and C see Osborn (1982)).

Nahm (1981a,b) also generalized the formulas (5.54) to include magnetic poles of arbitrary magnetic charge n and all gauge groups SU(N), Sp(N), and O(N). For SU(N) the matrix differential operator Δ reads

$$\Delta = i\partial_s \otimes I_{2n} + x \otimes I_n + \sigma_i \otimes T_i(s). \tag{5.55}$$

In terms of the normalized solutions v_i,

$$\int_{s_-}^{s_+} ds\; v_i^\dagger v_j = \delta_{ij}, \tag{5.56}$$

to the equation

$$\Delta^+ v_i = 0, \tag{5.57}$$

the gauge potentials are defined as

$$(A_\mu)_{jk} = -i\int_{s_-}^{s_+} ds\; v_j^+ \,\partial_\mu\, v_k. \tag{5.58}$$

The proof of self-duality given in Section 5.2 can now be transcribed to the monopole case by replacing the scalar product in a finite-dimensional vector space by the scalar product in a Hilbert space. The proof still stands if $(\Delta^+\Delta)^{-1}$ is defined and commutes with quaternions. The first condition is satisfied for anti-Hermitian T_i, if

$$\Delta^+\Delta = (i\partial_s + x_4)^2 + (x_i + iT_i)^+ \cdot (x_i + iT_i) \tag{5.59}$$

is bounded below by a positive constant. The second condition is satisfied if the anti-Hermitian T_i obey the non-linear equations

$$\partial_s T_i = \epsilon_{ijk}\, T_j T_k,\; T_i^\dagger = -T_i,\; i = 1,2,3. \tag{5.60}$$

$s_\pm$ has to be chosen in such a way that N normalizable solutions over the interval $s_- \leq s \leq s_+$ exist. Under these conditions, A_μ is an SU(N) potential and Nahm's construction yields a family of n-poles with the maximal number of degrees of freedom (Nahm 1981a).

The equations (5.60), which are analogues of the algebraic constraints for instantons, are integrable (cf. Atiyah 1981). For axially symmetric configurations they reduce to the integrable Toda lattice equations. For n = 2, they have been solved explicitly (Nahm 1981a, Brown, Panagopoulos & Prasad 1982), and the problem of solving eq. (5.57) has been reduced to one of finding the zeros of the same quartic polynomial which occurs in the twistor construction (Brown et al. 1982, Panagopoulos 1983). Nahm's ADHM technique therefore seems to be as powerful a method as those we discussed previously. Probably the question of regularity can be decided even more easily within this framework.

APPENDIX A: COMPOSITION FORMULAS FOR THE SINE-GORDON THEORY AND THE ERNST EQUATION

To derive eq. (1.16) we assume that the commutation relation (1.15) holds. This yields two algebraic equations for Φ':

$$\begin{aligned}\tfrac{1}{2}\partial_\xi(\Phi' - \Phi_o) &= \gamma_1 \sin \tfrac{1}{2}(\Phi_1 + \Phi_o) + \gamma_2 [\sin \tfrac{1}{2}(\Phi' - \Phi_o) \cos \tfrac{1}{2}(\Phi_1 + \Phi_o) \\ &+ \cos \tfrac{1}{2}(\Phi' - \Phi_o) \sin \tfrac{1}{2}(\Phi_1 + \Phi_o)] = \gamma_2 \sin \tfrac{1}{2}(\Phi_2 + \Phi_o) \\ &+ \gamma_1 [\sin \tfrac{1}{2}(\Phi' - \Phi_o) \cos \tfrac{1}{2}(\Phi_2 + \Phi_o) + \cos \tfrac{1}{2}(\Phi' - \Phi_o) \sin \tfrac{1}{2}(\Phi_2 + \Phi_o)],\end{aligned}$$

$$\begin{aligned}\tfrac{1}{2}\partial_\eta(\Phi' - \Phi_o) &= \gamma_1^{-1} \sin \tfrac{1}{2}(\Phi_1 - \Phi_o) - \gamma_2^{-1} [\sin \tfrac{1}{2}(\Phi' - \Phi_o) \cos \tfrac{1}{2}(\Phi_1 - \Phi_o) \\ &- \cos \tfrac{1}{2}(\Phi' - \Phi_o) \sin \tfrac{1}{2}(\Phi_1 - \Phi_o)] = \gamma_2^{-1} \sin \tfrac{1}{2}(\Phi_2 - \Phi_o) \\ &- \gamma_1^{-1}[\sin \tfrac{1}{2}(\Phi' - \Phi_o) \cos \tfrac{1}{2}(\Phi_2 - \Phi_o) - \cos \tfrac{1}{2}(\Phi' - \Phi_o) \sin \tfrac{1}{2}(\Phi_2 - \Phi_o)]. \quad \text{(A1)}\end{aligned}$$

The roots of the two quadratic equations are

$$\tan \tfrac{1}{4}(\Phi' - \Phi_o) = \frac{(\gamma_1 \mp \gamma_2)\ (\pm \cos \tfrac{1}{2}(\Phi_1 + \Phi_o) + \cos \tfrac{1}{2}(\Phi_2 + \Phi_o))}{(\gamma_1 - \gamma_2)\ (\sin \tfrac{1}{2}(\Phi_1 + \Phi_o) + \sin \tfrac{1}{2}(\Phi_2 + \Phi_o))},$$

and

$$\tan \tfrac{1}{4}(\Phi' - \Phi_o) = \frac{(\gamma_1 \pm \gamma_2)\ (- \cos \tfrac{1}{2}(\Phi_1 - \Phi_o) \pm \cos \tfrac{1}{2}(\Phi_2 - \Phi_o))}{(\gamma_1 - \gamma_2)\ (\sin \tfrac{1}{2}(\Phi_1 - \Phi_o) + \sin \tfrac{1}{2}(\Phi_2 - \Phi_o))}, \quad \text{(A2)}$$

respectively. Their common root is

$$\tan \tfrac{1}{4}(\Phi' - \Phi_o) = \frac{\gamma_1 + \gamma_2}{\gamma_1 - \gamma_2} \tan \tfrac{1}{4}(\Phi_1 - \Phi_2). \quad \text{(A3)}$$

This is the necessary and sufficient condition for the commutation relation to hold. That Φ' is indeed the Bäcklund transform $\Phi' = B_2B_1\Phi_o$ can now be shown by using the explicit form of Φ' in eq. (A3).

To prove the composition formula (3.32) for the Ernst equation we proceed analogously (Forgács, Horvath & Palla 1981a). We have to solve the Riccati equation (3.20) with the new seed solution, which means we have to solve the equation

$$\begin{aligned}dq = {} & [- M_1 \frac{q_1}{\tilde{q}_1} q\ (1 + p_2 q) + M_2 \frac{\tilde{q}_1}{q_1} (q + p_2) + \frac{p_1^2 - 1}{4\rho} (\frac{q_1 q (1 + p_2 q)}{p_1 + q_1} - \frac{q + p_2}{1 + p_1 q_1})]\, d\xi_1 \\ & + [N_1 \frac{1}{q_1 \tilde{q}_1} (q + \frac{1}{p_2}) - N_2 q_1 \tilde{q}_1 q\ (1 + \frac{q}{p_2}) + \frac{1 - p_1^2}{4\rho p_1} (\frac{q_1 q\ (p_2 + q)}{(1 + p_1 q_1) p_2} - \frac{1 + p_2 q}{(p_1 + q_1) p_2})]\ d\xi_2.\end{aligned}$$

(A4)

From the assumption $q = q(p_1, p_2, q_1, q_2)$ we get

$$dq = \frac{\partial q}{\partial q_1} dq_1 + \frac{\partial q}{\partial q_2} dq_2 + \frac{\partial q}{\partial p_1} dp_1 + \frac{\partial q}{\partial p_2} dp_2, \tag{A5}$$

with dq_1 and dq_2 given by eq. (3.20) and

$$dp_\alpha = \frac{p_\alpha^2 - 1}{4\rho} [p_\alpha d\xi_1 + \frac{1}{p_\alpha} d\xi_2]. \tag{A6}$$

To find out whether our assumption about the form of q is correct we equate the coefficient functions of $M_\alpha d\xi_1$, $N_\alpha d\xi_2$, and $d\xi_\alpha$, and see whether the resulting equations have a solution $q = q(p_1, p_2, q_1, q_2)$.

The resulting equations are

$$\frac{\partial q}{\partial q_1} q_1(1 + p_1 q_1) + \frac{\partial q}{\partial q_2} q_2(1 + p_2 q_2) = \frac{q_1}{\tilde{q}_1} q(1 + p_2 q),$$

$$\frac{\partial q}{\partial q_1} (q_1 + p_1) + \frac{\partial q}{\partial q_2} (q_2 + p_2) = \frac{\tilde{q}_1}{q_1} (q + p_2),$$

$$\frac{\partial q}{\partial q_1} (q_1 + \frac{1}{p_1}) + \frac{\partial q}{\partial q_2} (q_2 + \frac{1}{p_2}) = \frac{1}{q_1 \tilde{q}_1} (q + \frac{1}{p_2}),$$

$$\frac{\partial q}{\partial q_1} q_1(1 + \frac{q_1}{p_1}) + \frac{\partial q}{\partial q_2} q_2(1 + \frac{q_2}{p_2}) = q_1 \tilde{q}_1 q(1 + \frac{q}{p_2}),$$

$$\frac{\partial q}{\partial p_1} p_1(p_1^2 - 1) + \frac{\partial q}{\partial p_2} p_2(p_2^2 - 1) = (p_1^2 - 1) (\frac{q_1 q(1 + p_2 q)}{p_1 + q_1} - \frac{q + p_2}{1 + p_1 q_1}),$$

$$\frac{\partial q}{\partial p_1} \frac{p_1^2 - 1}{p_1} + \frac{\partial q}{\partial p_2} \frac{p_2^2 - 1}{p_2} = \frac{1 - p_1^2}{p_1} (\frac{q_1 q(p_2 + q)}{(1 + p_1 q_1)p_2} - \frac{1 + p_2 q}{(p_1 + q_1)p_2}). \tag{A7}$$

The first two equations as well as the third and fourth equation yield expressions for $\partial q/\partial q_\alpha$. The consistency conditions are

$$\frac{p_1(\tilde{q}_1 q_2 - \tilde{q}_2 q_1)}{p_2(q_1 \tilde{q}_1 - q_2 \tilde{q}_2)} = \frac{q_1^2 \tilde{q}_2 q - \tilde{q}_1^2 q_2 \tilde{q}}{q_2 \tilde{q}_2 - q_1^2 \tilde{q}_1^2 q \tilde{q}},$$

$$\frac{\tilde{q}_1 q_2 - \tilde{q}_2 q_1}{q_1 \tilde{q}_1 - q_2 \tilde{q}_2} = \frac{q_1 q - \tilde{q}_1 \tilde{q}}{1 - q_1 \tilde{q}_1 q \tilde{q}}. \tag{A8}$$

Their common root is given by eq. (3.32). That (3.32) indeed solves the last four equations of (A7), and therefore (A4), can be shown by an explicit calculation.

APPENDIX B: NON-MINIMAL SPHERICALLY SYMMETRIC FINITE-ENERGY SOLUTIONS

The spherical harmonics in eq. (2.37) are defined recursively in terms of Clebsch-Gordan coefficients as

$$Y^I_M = \sqrt{\frac{2I+1}{4\pi}}\, y^I_M, \quad M = -I, -I+1, \ldots, I,$$

$$y^I_M = \sqrt{\frac{2I-1}{I}} \sum_{m,m'} y^1_m\, y^{I-1}_{m'} \langle 1\ I{-}1\ m\ m' | IM \rangle$$

$$y^0_0 = 1,\ y^1_0 = \cos\theta,\ y^1_{\pm 1} = \mp\frac{1}{\sqrt{2}} \sin\theta\, e^{\pm i\phi} \tag{B1}$$

(see e.g. Brink & Satchler 1968). They satisfy the following commutation relations with the angular momentum operator $\vec{L} = -i\hat{x} \wedge \vec{\nabla}$:

$$[L_1 \pm iL_2, Y^I_M] = \sqrt{I(I+1) - M(M \pm 1)}\; Y^I_{M\pm 1},$$

$$[L_3, Y^I_M] = M\, Y^I_M, \tag{B2}$$

and have the properties

$$\overline{y^I_M} = (-)^M y^I_{-M}, \quad \sum_M |y^I_M|^2 = 1. \tag{B3}$$

We are now looking for a $(2I+1)$-plet of operators T^I_M which satisfies the commutation relations (B2) with, instead of $\vec{L}$, the generators $\vec{T}$ of isospin rotations for isospin $I/2$. Using the maximal SU(2) subalgebra of SU(I + 1) the adjoint representation of SU(I + 1) can be SU(2) decomposed as

$$I(I+2) \rightarrow 3 \oplus 5 \oplus \ldots \oplus (2I+1). \tag{B4}$$

Therefore, the Lie algebra of SU(I + 1) contains a $(2I+1)$-plet T^I_M with the properties

$$(T^I_M)^+ = (-)^M T^I_{-M}, \quad \mathrm{tr}\, T^I_M = 0,$$

$$\mathrm{tr}\,(T^I_M\, T_i) \sim \partial_{I1}, \quad \mathrm{tr}\,(T^I_M\, T^I_{-M'}) \sim \partial_{MM'}. \tag{B5}$$

A $(2I+1)$-tuple of real Higgs fields which transform according to the adjoint representation of SU(I + 1) can be defined in the following way (cf. eq. (2.37)):

$$\Phi = c\frac{H(r)}{r} \sum_M y^I_M\, (T^I_M)^+ = \frac{H}{r}\hat{\Phi} \tag{B6}$$

(c is a normalization constant for $\hat{\Phi}$). Since $\mathrm{tr}\,\hat{\Phi} = 0$ and $\Phi^+ = \Phi$ hold, Φ lies in the algebra of SU(I + 1). The real Higgs fields are the coefficient functions of the normalized Hermitian generators T^I_0, $T^I_M + (T^I_M)^+$, and $i(T^I_M - (T^I_M)^+)$.

The Higgs field (B6) satisfies the equation

$$[\vec{L} + \vec{T}, \Phi] = 0, \tag{B7}$$

i.e., Φ is spherically symmetric. From this equation and the definition of the gauge potentials (2.31),

$$A^i = \frac{1}{r}(1 - K)\hat{x}_j \epsilon_{jik} T_k, \tag{B8}$$

we obtain

$$D^i\Phi = \hat{x}^i \left(\frac{H}{r}\right)'\hat{\Phi} + i\frac{KH}{r}\epsilon_{ijk}\hat{x}_j [T_k, \hat{\Phi}]. \tag{B9}$$

Since the covariant derivative is a linear combination of the operators of the (2I + 1) -plet, and the magnetic field

$$B^i = -K'\partial_i \hat{x}\cdot\hat{T} + \frac{K^2 - 1}{r^2}\hat{x}_i \hat{x}\cdot\hat{T} \tag{B10}$$

is a linear combination of the three SU(2) generators, the Bogomol'nyi equations (2.12) are satisfied only by the trivial solution $D^i\Phi = B^i = 0$ for $I \neq 1$.

On the other hand, a nontrivial solution of the equations of motion (2.18) exists for arbitrary I. This can be proved as follows (Tyupkin et al. 1975): Equations (B9) and (B10) lead to the energy functional (2.36), which can be cast into the form

$$E = 4\pi \int_0^\infty dr\, (\sigma'^2 + \tfrac{1}{2}r^2\tau'^2 + \alpha^2 + 2\beta^2 + \gamma^2) \tag{B11}$$

with

$$\sigma := K, \quad \tau := \frac{1}{r}H - 1,$$

$$\alpha := \sqrt{\frac{I(I+1)}{2}}\sigma(\tau + 1), \quad \beta := \frac{1 - \sigma^2}{2r}, \quad \gamma := \sqrt{\frac{\lambda}{2}}\, r\tau(\tau + 2).$$

Now any solution to the corresponding Euler-Lagrange equations is a solution to the equations of motion (2.18) for our ansatz (B6) and (B8). This can be proved by an explicit calculation, or by using the Coleman-Fadeev principle (see Goddard & Olive 1978). Therefore, if the energy (B11) attains its minimum, a finite-energy solution not only to the submodel (B11) but also to the equations of motion (2.18) exists.

To prove that the energy attains its minimum we consider a minimizing sequence (σ_n, τ_n), which by definition has the property

$$\lim_{n\to\infty} E(\sigma_n, \tau_n) = \inf E(\sigma, \tau), \tag{B12}$$

and restrict our attention to those elements of the sequence which satisfy

$$E(\sigma_n, \tau_n) < \infty. \tag{B13}$$

From the upper bound (B13) we conclude that the sequence is bounded in the Sobolev space with norm

$$\|(\sigma, \tau)\| = [\int_0^\infty dr\, (\sigma'^2 + r^2\tau'^2) + \sigma^2(1) + \tau^2(1)]^{\frac{1}{2}}. \tag{B14}$$

Bounded sequences in this space have a subsequence which converges weakly in the Sobolev space and strongly in the space of continuous functions on the open half-line $(0,\infty)$. The inequality

$$E(\sigma_o, \tau_o) \leq \lim_{n\to\infty} E(\sigma_n, \tau_n), \tag{B15}$$

which holds for the limit (σ_o, τ_o), now completes the proof of existence. Following Maison's line of reasoning (Maison 1981), which applies to the case of arbitrary I, the regularity of Φ and A_i can be established.

This proof can be extended to the PS limit $\lambda \to 0$. In this case we restrict our attention to minimizing sequences (σ_n, τ_n) with

$$\lim_{r\to\infty} \tau_n(r) = 0, \tag{B16}$$

and with the same technique we show that a weak limit (σ_o, τ_o) exists. Since furthermore

$$|\tau_n(r)| \leq \left(\int_r^\infty ds\, s^2 \tau_n'^2 \int_r^\infty \frac{ds}{s^2}\right)^{1/2} \leq \frac{c}{\sqrt{r}}, \tag{B17}$$

with c independent of n, holds because $E(\sigma_n, \tau_n)$ is bounded we obtain

$$\lim_{r\to\infty} \tau_o(r) = 0. \tag{B18}$$

We have found smooth nontrivial finite-energy solutions to SU(I + 1) Yang-Mills-Higgs theory (cf. Michel et al. 1977a,b, O'Raifeartaigh & Rawnsley 1978) which do not satisfy the Bogomol'nyi equations for $I \neq 1$. (For a non-minimal SU(2) solution see Taubes (1982).)

APPENDIX C: GENERATION OF n-POLE SOLUTIONS

We list the formulas Forgács et al. (1981a,c) obtained for n-pole solutions. In terms of determinants the 2-pole solution (3.34) can be written

$$M_1^{(2)} = \frac{\begin{vmatrix} \tilde{q}_1 & \tilde{q}_2 \\ p_1 & p_2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ p_1\tilde{q}_1 & p_2\tilde{q}_2 \end{vmatrix}} \left(\frac{\begin{vmatrix} 1 & 1 \\ p_1 q_1 & p_2 q_2 \end{vmatrix}}{\begin{vmatrix} q_1 & q_2 \\ p_1 & p_2 \end{vmatrix}} M_1^{(0)} + \frac{1}{4\rho} \frac{\begin{vmatrix} 1 & 1 \\ p_1^2 & p_2^2 \end{vmatrix}}{\begin{vmatrix} q_1 & q_2 \\ p_1 & p_2 \end{vmatrix}} \right),$$

$$N_2^{(2)} = \frac{\begin{vmatrix} \tilde{q}_1 & \tilde{q}_2 \\ p_1 & p_2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ p_1\tilde{q}_1 & p_2\tilde{q}_2 \end{vmatrix}} \left(\frac{\begin{vmatrix} q_1 & q_2 \\ p_1 & p_2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ p_1 q_1 & p_2 q_2 \end{vmatrix}} N_2^{(0)} + \frac{1}{4\rho} \frac{\begin{vmatrix} p_1^{-1} & p_2^{-1} \\ p_1 & p_2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ p_1 q_1 & p_2 q_2 \end{vmatrix}} \right). \quad \text{(C1)}$$

These formulas generalize to an arbitrary number n of Bäcklund transformations (3.26) as follows ($n = 2k + \varepsilon$; $\varepsilon = 0, 1$):

$$M^{(n)} = (-)^n \frac{D_1^{(n)}\ (\tilde{q}_i)}{D_2^{(n)}\ (\tilde{q}_i)} \left(\frac{D_2^{(n)}\ (q_i)}{D_1^{(n)}\ (q_i)} M_{1+\varepsilon}^{(0)} + \frac{1}{4\rho} \frac{D_3^{(n)}\ (q_i)}{D_1^{(n)}\ (q_i)} \right),$$

$$N_2^{(n)} = (-)^n \frac{D_1^{(n)}\ (\tilde{q}_i)}{D_2^{(n)}\ (\tilde{q}_i)} \left(\frac{D_1^{(n)}\ (q_i)}{D_2^{(n)}\ (q_i)} N_2^{(0)} + \frac{1}{4\rho} \frac{D_4^{(n)}\ (q_i)}{D_2^{(n)}\ (q_i)} \right), \quad \text{(C2)}$$

where the $D^{(n)}$ are determinants whose i-th row is given as

$$D_1^{(2k+1)}\ (q_i) = |\ q_i,\ p_i,\ p_i^2 q_i,\ p_i^3,\ p_i^4 q_i, \ldots,\ p_i^{2k} q_i\ |,$$

$$D_2^{(2k+1)}\ (q_i) = |\ 1,\ p_i q_i,\ p_i^2,\ p_i^3 q_i,\ p_i^4, \ldots,\ p_i^{2k}\ |,$$

$$D_3^{(2k+1)}\ (q_i) = |\ 1,\ p_i q_i,\ p_i^2,\ p_i^3 q_i,\ p_i^4, \ldots,\ p_i^{2k-1}\ q_i,\ p_i^{2k+1}\ q_i\ |,$$

$$D_4^{(2k+1)}\ (q_i) = |\ p_i^{-1},\ p_i,\ p_i^2 q_i,\ p_i^3,\ p_i^4 q_i, \ldots,\ p_i^{2k}\ q_i\ | \quad \text{(C3)}$$

for odd $n = 2k + 1$, or

$$D_1^{(2k)}(q_i) = | q_i, p_i, p_i^2 q_i, p_i^3, p_i^4 q_i, \ldots, p_i^{2k-1} | ,$$

$$D_2^{(2k)}(q_i) = | 1, p_i q_i, p_i^2, p_i^3 q_i, p_i^4, \ldots, p_i^{2k-1} q_i | ,$$

$$D_3^{(2k)}(q_i) = | 1, p_i q_i, p_i^2, p_i^3 q_i, p_i^4, \ldots, p_i^{2k-2}, p_i^{2k} | ,$$

$$D_4^{(2k)}(q_i) = | p_i^{-1}, p_i, p_i^2 q_i, p_i^3, p_i^4 q_i, \ldots, p_i^{2k-1} | \qquad \text{(C4)}$$

for even $n = 2k$.

Using the reality condition we can write for the norm of the Higgs field

$$|\Phi| = 2 \Big| \frac{D_2^{(n)}(q_i)}{D_1^{(n)}(q_i)} M_{1+\epsilon}^{(0)} - \frac{D_1^{(n)}(q_i)}{D_2^{(n)}(q_i)} N_2^{(0)}$$

$$+ \frac{1}{4\rho} \Big(\frac{D_3^{(n)}(q_i)}{D_1^{(n)}(q_i)} - \frac{D_4^{(n)}(q_i)}{D_2^{(n)}(q_i)} \Big) \Big| . \qquad \text{(C5)}$$

The regularity condition of $|\Phi|$ leads to the following values of the parameters w_i and β_i:

$$w_1 = 0, \ \bar{w}_{2r} = w_{2r+1} = ir\pi, \ q_1 = - \tanh [\tfrac{1}{2} R(0)],$$

$$q_{4r-2} = - \coth [\tfrac{1}{2} R(w_{4r-1})] = \bar{q}_{4r-1},$$

$$q_{4r} = - \tanh [\tfrac{1}{2} R(w_{4r+1})] = \bar{q}_{4r+1}, \qquad \text{(C6)}$$

$$R(w) = \sqrt{(w-z)^2 + \rho^2},$$

for odd $n = 2k + 1$, or

$$w_{2r-1} = \bar{w}_{2r} = i(2r-1)\pi/2 ,$$

$$q_{4r-3} = - \tanh [\tfrac{1}{2} R(w_{4r-3})] = (\bar{q}_{4r-2})^{-1}$$

$$q_{4r-1} = - \coth [\tfrac{1}{2} R(w_{4r-1})] = (\bar{q}_{4r})^{-1}, \qquad \text{(C7)}$$

for even $n = 2k$.

On the z-axis $|\Phi|$ reads

$$|\Phi| = \Big| \coth z - \frac{1}{z} - \sum_{r=1}^{k} \frac{2z}{z^2 + (r\pi)^2} \Big| \underset{z\to\infty}{\longrightarrow} 1 - \frac{2k+1}{z} \qquad \text{(C8)}$$

for odd $n = 2k + 1$, or

$$|\Phi| = \left| \tanh z - \sum_{r=1}^{k} \frac{2z}{z^2 + [\frac{1}{2}(2r-1)\pi]^2} \right| \underset{z\to\infty}{\longrightarrow} 1 - \frac{2k}{z} \qquad (C9)$$

for even $n = 2k$. We have generated axi- and mirrorsymmetric n-pole solutions.

REFERENCES

Actor, A. 1979 Rev. Mod. Phys. 51, 461.

Athorne, C. 1982 Factorization of an SU(3) Monopole Patching Function, Univ. Durham Preprint.

——— 1983 Comm. Math. Phys. 88, 43.

Atiyah, M.F., Hitchin, N.J. & Singer, I.M. 1977 Proc. Nat. Acad. Sci. 74, 2662.

Atiyah, M.F. & Ward, R.S. 1977 Comm. Math. Phys. 55, 117.

Atiyah, M.F., Hitchin, N.J., Drinfeld, V.G. & Manin, Yu. I. 1978 Phys. Lett. 65A, 185.

Atiyah, M.F. 1979 Geometry of Yang-Mills Fields, Lez. Fermi, Pisa, Accad. Naz. d. Lincei.

——— 1981 Proc. Monopole Meeting, Trieste 1981, Monopoles in Quantum Field Theory, eds. N.S. Craigie, P. Goddard & W. Nahm (Singapore: World Scient. Publ. 1982) p. 3.

Bais, F.A. 1979 O(3) Symmetric Monopoles and Instantons, Univ. Leuven Preprint KUL-TF-79/021.

——— 1981 Proc. Monopole Meeting, Trieste 1981, Monopoles in Quantum Field Theory, eds. N.S. Craigie, P. Goddard & W. Nahm (Singapore: World Scient. Publ. 1982) p. 111.

Barnard, T. 1973 Phys. Rev. A7, 373.

Belavin, A.A. & Polyakov, A.M. 1975 JETP Lett. 22, 245.

Belavin, A.A., Polyakov, A.M., Schwartz, A.S. & Tyupkin, Yu. S. 1975 Phys. Lett. 59B, 85.

Belavin, A.A. & Zakharov, V.E. 1978 Phys. Lett. 73B, 53.

Bogomol'nyi, E.B. 1976 Sov. J. Nucl. Phys. 24, 449.

Bourguignon, J.P., Lawson, H.B. & Simons, J. 1979 Proc. Nat. Acad. Sci. 76, 1550.

Brihaye, Y., Fairlie, D.B., Nuyts, J. & Yates, R.G. 1978 J. Math. Phys. 19, 2528.

Brink, D.M. & Satchler, G.R. 1968 Angular Momentum (Oxford: Clarendon).

Brown, L., Carlitz, R. & Lee, C. 1977 Phys. Rev. D16, 417.

Brown, S.A., Prasad, M.K. & Rossi, P. 1981 Phys. Rev. D24, 2217.

Brown, S.A., Panagopoulos, H. & Prasad, M.K. 1982 Phys. Rev. D26, 854.

Brown, S.A. 1983 The Atiyah-Ward 2 Monopole Solution, MIT Preprint CTP # 1055.

Burzlaff, J. 1981 Acta Phys. Austr. Suppl. XXIII, 599.

——— 1982 Lett. Math. Phys. 6, 141.

Burzlaff, J. & Hornos, J.E.M. 1982 On the Relation between Complex Manifold and Soliton Theoretic Constructions for Self-Dual Fields, Yale Univ. Preprint YTP 82-28.

Cabrera, B. 1982 Phys. Rev. Lett. 48, 1378.

Chakrabarti, A. 1982 Phys. Rev. D25, 3282.

Cho, Y.M. 1980 Phys. Rev. Lett. 44, 1115.

Christ, N.H., Weinberg, E. & Stanton, N.K. 1978 Phys. Rev. D18, 2013.

Coleman, S. 1975 Proc. Int. School of Subnuclear Physics, Erice 1975, New Phenomena in Subnuclear Physics, ed. A. Zichichi (New York: Plenum 1977) p. 297.

Coleman, S. 1982 The Magnetic Monopole Fifty Years Later, Harvard Univ. Preprint HUTP-82/A 032.

Corrigan, E., Fairlie, D.B., Yates, R.G. & Goddard, P. 1978a Comm. Math. Phys. 58, 223.

Corrigan, E., Fairlie, D.B., Goddard, P. & Templeton, S. 1978b Nucl. Phys. B140, 31.

Corrigan, E. 1979 Phys. Rep. 49, 95.

__________ 1980 Proc. Int. Summer Inst. on Theor. Physics, Bad Honnef 1980, Current Topics in Elementary Particle Physics, eds. K.H. Mütter & K. Schilling (New York: Plenum 1981) p. 219.

Corrigan, E. & Goddard, P. 1981 Comm. Math. Phys. 80, 575.

Cosgrove, C.M. 1980 J. Math. Phys. 21, 2417.

D'Adda, A., Lüscher, M. & DiVecchia, P. 1978 Nucl. Phys. B146, 63.

Din, A.M. & Zakrzewski, W.J. 1980a Nuovo Cim. Lett. 28, 121.

__________ 1980b Nucl. Phys. B174, 397.

Dirac, P.A.M. 1931 Proc. Roy. Soc. A133, 60.

Drinfeld, V.G. & Manin, Yu. I. 1978 Comm. Math. Phys. 63, 177.

Eichenherr, H. 1978 Nucl. Phys. B146, 215.

Ellis, J. 1980 Proc. 21st Scottish Univ. Summer School, St. Andrews 1980, Gauge Theories and Experiments at High Energies, eds. K.C. Bowler & D.G. Sutherland (Edinburgh: SUSSP 1981) p. 201.

Ernst, F.J. 1968 Phys. Rev. 167, 1175.

Forgács, P., Horvath, Z. & Palla, L. 1980 Phys. Rev. Lett. 45, 505.

__________ 1981a Ann. Phys. 136, 371.

__________ 1981b Phys. Lett. 99B, 232.

__________ 1981c Proc. Monopole Meeting, Trieste 1981, Monopoles in Quantum Field Theory, eds. N.S. Craigie, P. Goddard & W. Nahm (Singapore: World Scient. Publ. 1982) p. 21.

__________ 1982 Phys. Lett. 109B, 200.

Garber, W.D., Ruijsenaars, S., Seiler, E. & Burns, D. 1979 Ann. Phys. 119, 305.

Goddard, P. & Olive, D. 1978 Reps. Prog. Phys. 41, 1357.

Golo, V. & Perelomov, A. 1978 Phys. Lett. 79B, 112.

Gürsey, F. & Tze, H.C. 1980 Ann. Phys. 128, 29.

Guth, A. & Weinberg, E. 1976 Phys. Rev. D14, 1660.

Harrison, B.K. 1978 Phys. Rev. Lett. 41, 1197.

Hitchin, N.J. 1982 Comm. Math. Phys. 83, 579.

't Hooft, G. 1974 Nucl. Phys. B79, 276.

Houston, P. & O'Raifeartaigh, L. 1980 Phys. Lett. 93B, 151; 94B, 153.

Jackiw, R. & Rebbi, C. 1977 Phys. Lett. 67B, 189.

Jaffe, A. & Taubes, C. 1980 Vortices and Monopoles, Prog. in Phys. V. 2 (Boston: Birkhäuser).

Jang, P., Park, S. & Wali, K.C. 1978 Phys. Rev. D17, 1641.

Kibble, T.W.B. 1981 Proc. Monopole Meeting, Trieste 1981, Monopoles in Quantum Field Theory, eds. N.S. Craigie, P. Goddard & W. Nahm (Singapore: World Scient. Publ. 1982) p. 341.

Kim, Y.D., Koh, I.D. & Park, Y.J. 1982 Phys. Rev. D25, 587.

Koh, I.G., Kim, Y.D., Park, Y.J., Kim, C.H. & Kim, Y.S. 1981 Complete Set of SU(5) Monopole Solutions, Fermilab Preprint 81/71-THY.

Lee, S.-C. 1981 Phys. Rev. D24, 2200, 2209.

Maison, D. 1980 Proc. 17th Winter School of Theor. Physics, Karpacz 1980, Developments in the Theory of Fundamental Interactions, eds. L. Turko & A. Pokalski (New York: Harwood; in press).

———— 1981 Nucl. Phys. B182, 144.

Manton, N.S. 1978 Nucl. Phys. B135, 319.

Michel L., O'Raifeartaigh, L. & Wali, K.C. 1977a Phys. Lett. 67B, 198.

———— 1977b Phys. Rev. D15, 3641.

Nahm, W. 1980 Phys. Lett. 90B, 413.

1981a All Self-Dual Multimonopoles for Arbitrary Gauge Group, CERN Preprint Ref. TH. 3172.

———— 1981b Proc. Monopole Meeting, Trieste 1981, Monopoles in Quantum Field Theory, eds. N.S. Craigie, P. Goddard & W. Nahm (Singapore: World Scient. Publ. 1982) p. 87.

Narain, K.S. 1981 Phys. Rev. D24, 2211.

Neugebauer, G. & Kramer, D. 1962 Ann. Phys. 24, 62.

Olive, D., Sciuto, S. & Crewther, R.J. 1979 Nuovo Cim. Rev. Vol. 2 N. 8.

Olive, D. 1980 Proc. Int. Summer Inst. on Theor. Physics, Bad Honnef 1980, Current Topics in Elementary Particle Physics, eds. K.H. Mütter & K. Schilling (New York: Plenum 1981) p. 199.

O'Raifeartaigh, L. 1977 Nuovo Cim. Lett. 18, 205.

O'Raifeartaigh, L. & Rawnsley, J. 1978 Phys. Lett. 72B, 465.

O'Raifeartaigh, L. & Rouhani, S. 1981a Acta Phys. Austr. Suppl. XXIII, 525.

———— 1981b Phys. Lett. 105B 177, 489.

O'Raifeartaigh, L., Rouhani, S. & Singh, L.P. 1982a Phys. Lett. 112B, 369.

———— 1982b Nucl. Phys. B206, 137.

Osborn, H. 1979 Nucl. Phys. B159, 497.

———— 1982 Comm. Math. Phys. 86, 195.

Panagopoulos, H. 1983 Multimonopoles in Arbitrary Gauge Groups and the Complete SU(2) 2-Monopole System, MIT Preprint CTP # 1068.

Pohlmeyer, K. 1976 Comm. Math. Phys. 46, 207.

———— 1980 Comm. Math. Phys. 72, 37.

Polyakov, A.M. 1974 JETP Lett. 20, 194.

Prasad, M.K. & Sommerfield, C.M. 1975 Phys. Rev. 35, 760.

Prasad, M.K. 1978 Phys. Rev. D17, 3243.

_____________ 1980 Physica 1D, 167.

_____________ 1981 Comm. Math. Phys. 80, 137.

Prasad, M.K. & Rossi, P. 1981a Phys. Rev. Lett. 46, 806.

_____________ 1981b Phys. Rev. D23, 1795.

_____________ 1981c Phys. Rev. D24, 2182.

Rawnsley, J.H. 1978 Comm. DIAS, A, No. 25.

Rossi, P. 1982 Phys. Rep. 86, 317.

Schwartz, A.S. 1977 Phys. Lett. 67B, 172.

Scott, A.C., Chu, F.Y.F. & McLaughlin, D.W. 1973 Proc. IEEE 61, 1443.

Singh, L.P. & Tchrakian, D.H. 1981 Phys. Lett. 104B, 463.

Taubes, C.H. 1982 Comm. Math. Phys. 86, 257, 299.

Tye, S.-H.H. 1982 Introduction to the SU(5) Grand Unified Theory and Related Topics, Cornell Univ. Preprint CLNS-82/527.

Tyupkin, Yu. S., Fateev, V.A. & Shvarts, A.S. 1975 JETP Lett. 21, 41.

Ueno, K. & Nakamura, Y. 1981 Proc. RIMS Symposium, Kyoto 1981, Non-linear Integrable Systems-Classical Theory and Quantum Theory, eds. M. Jimbo & T. Miwa (Singapore: World Scient. Publ. 1983) p. 241.

Uhlenbeck, K. 1978 Bull. Amer. Math. Soc. 1, 579.

Ward, R.S. 1977 Phys. Lett. A61, 81.

_____________ 1981a Comm. Math. Phys. 79, 317.

_____________ 1981b Phys. Lett. 102B, 136.

_____________ 1981c Phys. Lett. 107B, 281.

_____________ 1982a Comm. Math. Phys. 86, 437.

_____________ 1982b Twistor Theory and Magnetic Poles, State Univ. New York, Stony Brook, Preprint ITP-82-SB-24.

Weinberg, E. 1979 Phys. Rev. D20, 936.

Witten, E. 1979a Complex Manifold Techniques in Theor. Physics, eds. D.W. Lerner & P.D. Sommers (San Francisco: Pitman) p. 207.

Witten, L. 1979b Phys. Rev. D19, 718.

Yang, C.N. 1977 Phys. Rev. Lett. 38, 1377.