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Iterates of Maps which are Non-expansive in Hilbert's Projective Metric

Jeremy Gunawardena, Cormac Walsh

The cycle time of an operator on $\mathbb{R}^n$ gives information about the long term behaviour of its iterates. We generalise this notion to operators on symmetric cones. We show that these cones, endowed with either Hilbert's projective metric or Thompson's metric, satisfy Busemann's definition of a space of non-positive curvature. We then deduce that, on a strictly convex symmetric cone, the cycle time exists for all maps which are non-expansive in both these metrics. We also review an analogue for the Hilbert metric of the Denjoy-Wolff theorem.

1 Introduction

Let (X, d) be a metric space. A map $F : X \to X$ is said to be non-expansive if $d(F(x), F(y)) \leq d(x, y)$ for all $x, y \in X$. In studying such maps, one is typically interested in the asymptotic behaviour of their iterates. In the special case when X is a normed space, a useful description of this behaviour is given by the *cycle time* of F, which is defined to be

$$\chi(F) := \lim_{i\to\infty} \frac{F^i(x_0)}{i} \tag{1}$$

when the limit exists. Since F is assumed to be non-expansive, the cycle time will of course be independent of the initial point x_0. Kohlberg and Neyman have shown [11] that if X is a Banach space, then $\chi(F)$ exists for every non-expansive map $F : X \to X$ if and only if the dual space X^* has a Fréchet differentiable norm.

Rather than imposing conditions on the normed space, one can impose conditions on the map. For example, in [9], several classes of maps are discussed, each of which is non-expansive in the l_∞ norm on $\mathbb{R}^n$. For some of these classes, in particular the so called min-max maps, it is known [8] that the cycle time always exists.

Relying as it does on the linear structure, the definition of cycle time in (1) only applies when X is a normed space. We propose to extend the definition of cycle time to another class of spaces, the *symmetric cones*. We do this by exploiting the fact that associated to each symmetric cone C of dimension m is a naturally defined map $\log : C \to \mathbb{R}^m$. The cycle time of an operator $F : C \to C$ can therefore be defined by

$$\chi(F) := \lim_{i\to\infty} \frac{\log F^i(x_0)}{i}, \tag{2}$$

where x_0 is, as before, an arbitrary initial point. The log appearing in the definition is appropriate because of the nature of the metrics we will be considering, those of Thompson and Hilbert. Iterates of maps that are non-expansive in these metrics tend to grow exponentially rather than linearly as in the case of normed spaces. An illustration of this is that multiplication by a positive scalar is non-expansive in both metrics. It is also worth noting that the symmetric cone $\mathbb{R}^m_+$ endowed with the Thompson metric is isometric to $(\mathbb{R}^m, l_\infty)$. Indeed, if F is a self map of $\mathbb{R}^m$ and $\tilde{F}$ is its translation across to $\mathbb{R}^m_+$, then the cycle time of F defined by (1) will be identical to that of $\tilde{F}$ defined by (2).

The layout of the paper is as follows. The relevant background definitions and results are recalled in Section 2. In Section 3, we establish some geometrical properties of the Thompson and Hilbert metrics on symmetric cones. These are then used in Section 4 to prove that, if $F : C \to C$ is non-expansive in one of these metrics, then the limit points of $\log F^i(x_0)/i$ all lie in a face of a ball of a certain norm on $\mathbb{R}^m$. This is an exact analogue of the main Theorem in [11]. In Section 5, we restrict our attention to a particular class of cone, the Lorentz cones. We show that the cycle time exists for maps on such cones that are non-expansive in both the Thompson and Hilbert metrics. In the final Section, we consider a different question concerning iterates: that of convergence to a ray in the boundary of the cone.

2 Preliminaries

A cone is a subset of a vector space that is convex, closed under multiplication by positive scalars, and does not contain any complete line through the origin. In this paper we consider only open cones in finite dimensional spaces. Associated with each cone $C \subset \mathbb{R}^m$ is a partial ordering on $\mathbb{R}^m$ defined as follows: $x \le y$ if and only if $y - x \in \mathrm{cl}\, C$. In addition, there are the following two metrics. For each $x \in \mathbb{R}^m$ and $y \in C$, define $M(y,x) := \inf\{\lambda \in \mathbb{R} : y \le \lambda x\}$. Then *Thompson's part metric* on the cone is defined to be

$$d_T(x,y) := \log \max \Big\{ M(x,y), M(y,x) \Big\}$$

and *Hilbert's projective metric* is defined to be

$$d_H(x,y) := \log \Big(M(x,y) M(y,x) \Big).$$

With respect to Thompson's part metric, the cone C is a complete metric space. Hilbert's projective metric, however, is only a pseudo-metric: it is possible to find two distinct points in C that are zero distance apart. Indeed it is not difficult to see that $d_H(x,y) = 0$ if and only if $x = \lambda y$ for some $\lambda > 0$. Thus d_H is a metric on the space of rays of the cone. For further details, see the monograph of Nussbaum [12].

Let S be a cross section of C, that is $S := \{x \in C : \phi(x) = 1\}$, where $\phi : \mathbb{R}^m \to \mathbb{R}^m$ is some linear functional that is positive with respect to the ordering induced by C. Suppose that x and y are a pair of distinct points in S and define a and b to be the points in the boundary of S such that a, x, y, and b are collinear and arranged in this order along the line in which they lie.

Then, it can be shown that the Hilbert distance between x and y is given by the logarithm of the cross ratio of these four points:

$$d_H(x,y) := \log \frac{|bx|\,|ay|}{|by|\,|ax|}.$$

Indeed, this was the original definition of Hilbert, who defined the metric on bounded convex open sets. If S is the open unit disk, the Hilbert metric is exactly the Klein model of the hyperbolic plane.

Operators on cones which are non-expansive in both the Hilbert's projective metric and Thompson's part metric arise quite naturally. For example, consider those self maps of the cone that are both isotone with respect to the cone ordering and homogeneous of degree one. Recall that isotone means $x \leq y$ implies $F(x) \leq F(y)$ and homogeneous of degree r means that $F(\lambda x) = \lambda^r F(x)$ for each $x \in C$ and $\lambda > 0$. Gunawardena and Keane [10] have called these maps *topical*. Topical maps are non-expansive in both the Thompson and the Hilbert metrics. In fact this is a consequence of a more general theorem [12] which states that if F is homogeneous of degree $r \neq 0$ and isotone, then $d_H(Fx, Fy) \leq |r| d_H(x,y)$ and $d_T(Fx,Fy) \leq |r| d_T(x,y)$ for all $x, y \in C$.

The Thompson and Hilbert geometries are both examples of a *Finsler* space [13]. The cone can be considered to be an m-dimensional manifold and the tangent space at each point may be identified with $\mathbb{R}^m$. If a norm

$$|v|_x^T := \inf\{\alpha > 0 : -\alpha x \leq v \leq \alpha x\},$$

is defined on the tangent space at each point $x \in C$, then the length of any C^1 curve $\phi : [a,b] \to C$ can be defined to be

$$L(\phi) := \int_a^b |\phi'(t)|_{\phi(t)}^T \, dt.$$

The Thompson distance between any two points is recovered by minimising over all paths connecting the points:

$$d_T(x,y) = \inf\{L(\phi) : \phi \in C^1[x,y]\},$$

where $C^1[x,y]$ denotes the set of all C^1 paths $\phi : [0,1] \to C$ with $\phi(0) = x$ and $\phi(1) = y$. A similar procedure yields the Hilbert metric when the norm above is replaced by the semi-norm

$$|v|_x^H := M(v,x) - m(v,x).$$

Here $M(v,x)$ is as before and $m(v,x) := \sup\{\lambda \in \mathbb{R} : v \geq \lambda x\}$. The Hilbert geometry will be Riemannian only in the case of the Lorentz cone, which will be defined shortly. The Thompson geometry will be Riemannian only in the trivial case of the 1-dimensional cone $\mathbb{R}_+$.

A particularly interesting class of cones is the class of those that are *symmetric*, in other words both self dual and homogeneous. Recall that a cone C is self dual if it equal to its dual

$$C^* := \Big\{ x \in \mathbb{R}^m : x \cdot y > 0 \text{ for all } y \in \text{cl } C \backslash \{0\} \Big\}.$$

A cone C is said to be homogeneous if its group of linear automorphisms acts transitively on it, in other words for any $x, y \in C$ there is some linear automorphism g of C such that $g(x) = y$.

An example of a symmetric cone is the Lorentz cone,

$$\Lambda_m := \{(t, \mathbf{v}) \in \mathbb{R} \times \mathbb{R}^{m-1} : |t| > |\mathbf{v}|\}, \tag{3}$$

where $|\mathbf{v}|$ denotes the Euclidean norm of $\mathbf{v}$. Another example is the set of positive definite Hermitian matrices $\mathrm{Herm}(n, E)$, where $n \in \mathbb{N}$ and the set of entries E can be either $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. The Octonians $\mathbb{O}$ are anomalous in this respect—the positive definite elements of $\mathrm{Herm}(n, \mathbb{O})$ only form a symmetric cone when $n = 3$.

Given two cones C_1 and C_2 in linear spaces V_1 and V_2, we may form the product cone $C_1 + C_2$ in the linear space $V_1 \oplus V_2$. If C_1 and C_2 are both symmetric then this cone will also be. The class of symmetric cones is actually quite small. It can be shown that all finite dimensional symmetric cones can be formed by taking products of the two classes of examples mentioned above.

Our interest in symmetric cones stems from their close connection with Jordan Algebras. A commutative algebra J over $\mathbb{R}$ with identity $\mathbb{I}$ is said to be a Jordan algebra if $x(x^2 y) = x^2(xy)$ for all $x, y \in J$. A finite dimensional Jordan algebra is called *Euclidean* if there is an inner product $(\cdot|\cdot)$ such that $(xy|z) = (y|xz)$ for all $x, y, z \in J$.

The connection with symmetric cones is that the interior of the set of square elements, $\mathrm{int}\,\{x^2 : x \in J\}$, in a Euclidean Jordan algebra is a symmetric cone. Furthermore, every symmetric cone arises in this way: for each symmetric cone $C \subset \mathbb{R}^m$ we can define a product on $\mathbb{R}^m$ such that $\mathbb{R}^m$ becomes a Euclidean Jordan algebra with $C = \mathrm{int}\,\{x^2 : x \in \mathbb{R}^m\}$.

For a reference on symmetric cones and Jordan algebras, see the book by Faraut and Korányi [7].

By way of an example, if $(\lambda, \mathbf{u})$ and $(\mu, \mathbf{v})$ are two points in a Lorentz cone, then their Jordan product is

$$(\lambda, \mathbf{u}) \circ (\mu, \mathbf{v}) = (\lambda\mu + \mathbf{u}\cdot\mathbf{v},\ \lambda\mathbf{v} + \mu\mathbf{u}).$$

For two points A and B in $\mathrm{Herm}(m, E)$, the Jordan product is

$$A \circ B = \frac{1}{2}(AB + BA).$$

In a product cone $C_1 + C_2$, two points (a, b) and (c, d) are multiplied according to the rule

$$(a, b) \circ (c, d) = (a \diamond c, b \star d),$$

where $\diamond$ and $\star$ are the Jordan products associated with cones C_1 and C_2 respectively.

Jordan algebras are power associative, that is $w^i w^j = w^{i+j}$ for every element w of the algebra and positive integers i and j. This means that the exponential function can be defined using the usual formula:

$$\exp(w) := \sum_{i=1}^{\infty} \frac{w^i}{i!}.$$

This function is a bijection between J and C. We denote its inverse by log.

Using these functions we may define the cycle time of an operator F acting on C:

$$\chi(F) := \lim_{i\to\infty} \frac{\log F^i(x_0)}{i}.$$

Note that if F is non-expansive in d_T, then $\chi(F)$ is independent of the initial point x_0. Gunawardena and Keane [10] found an example of a map on the symmetric cone $\mathbb{R}^3_+$ which is topical but for which the cycle time does not exist. Let f_i be a sequence drawn from $[0,1]$ such that $\sum_{i=1}^n f_i/n$ does not converge and define the following trajectory in $\mathbb{R}^3_+$:

$$w_n = (1, e^n, \textstyle\prod_{i=1}^n e^{f_i}).$$

Then the map

$$F(x) := \inf_{n\in\mathbb{N}} \{M(x, w_n) w_{n+1}\}$$

is clearly topical. However $F(w_n) = w_{n+1}$ for each $n \in \mathbb{N}$ and so the third component of $\log F^n(w_0)/n$ is $\sum_{i=1}^n f_i/n$, which we have assumed does not converge.

It is often useful to express an element of a Jordan algebra in terms of a *Jordan frame*. Recall that an element $c \in J$ is said to be an idempotent if $c^2 = c$ and such an element is said to be primitive if cannot be written as the sum of two non-zero idempotents. Also, a pair of idempotents c_1 and c_2 are said to be orthogonal if $c_1c_2 = 0$ and a set of primitive orthogonal idempotents $c_1, c_2, \ldots, c_n$ is said to be complete if $c_1+c_2+\cdots+c_n = \mathbb{I}$. Then, a Jordan frame is defined to be a complete set of orthogonal primitive idempotents. It is a fact that any element of a Jordan algebra can be expressed as a linear combination

$$x := \sum_{i=1}^n \lambda_i c_i,$$

of members of some Jordan frame $\{c_i : 1 \le i \le n\}$. Moreover, if x is in the interior of the cone of squares, then the coefficients $\{\lambda_i : 1 \le i \le n\}$ will be positive. In general, the frame will depend on the element in question. The exponential function will be given by

$$\exp(x) := \sum_{i=1}^n e^{\lambda_i} c_i.$$

The distance, with respect to Thompson and Hilbert metrics, between a point $x \in C$ and the unit can be expressed in terms of the coefficients of x in its Jordan frame:

$$\begin{aligned} d_T(x, \mathbb{I}) &= \max_i \{|\log \lambda_i|\}, \\ d_H(x, \mathbb{I}) &= \max_i \{\log \lambda_i\} - \min_i \{\log \lambda_i\}. \end{aligned} \tag{4}$$

We say that a cone is strictly convex if any line segment contained in its boundary is contained within a ray. We will show later that the cycle time exists for all topical operators defined on a strictly convex symmetric cone. But first we will investigate the properties of the Hilbert and Thompson geometries.

3 Geometrical Properties

Let (X,d) be a metric space. A metric line is the image of $\mathbb{R}$ under any mapping $\phi : \mathbb{R} \to X$ such that

$$d\big(\phi(t),\phi(s)\big) = |t-s|, \qquad \text{for all } t,s \in \mathbb{R}. \tag{5}$$

In addition, we call a metric line segment the image of a real interval under such a mapping. Often we wish to consider not the set of all metric lines of X but only a subset M. Suppose that for every distinct pair of points $x,y \in X$ there is a unique metric line $l \subset M$ containing both x and y. For each $t \in [0,1]$, there is a unique point $w \in l$ such that

$$\begin{array}{rrcl} & d(x,w) & = & td(x,y) \\ \text{and} & d(w,y) & = & (1-t)d(x,y). \end{array}$$

This point is denoted by $(1-t)x \oplus ty$. The metric space (X,d,M) is said to be *hyperbolic* if

$$d(\tfrac{1}{2}x \oplus \tfrac{1}{2}y,\ \tfrac{1}{2}x \oplus \tfrac{1}{2}z) \le \tfrac{1}{2}d(y,z)$$

for each $x,y,z \in X$. This nomenclature was used by Reich and Shafrir [15]. Busemann [3] has called such spaces non-positively curved.

The images of the following mappings are natural candidates to be considered special metric lines in a symmetric cone with Thompson's metric:

$$\phi(t) := \exp(t\log(x)) = x^t.$$

To have the correct parameterisation, we must have that $d_T(\mathbb{I},x) = 1$. It is obvious from expression (4), that (5) holds for these curves. Each of the above lines passes through the unit $\mathbb{I}$ and there is a unique line passing through each point (apart from $\mathbb{I}$). Using the homogeneity of the cone we may obtain a set M_T of metric lines such that there is a unique line connecting any distinct pair of points. This set of lines will be invariant under the action of the automorphisms of the cone.

Note that if y is a scalar multiple of x, then y^t is a scalar multiple of x^t. It follows that the expression above also defines curves in the projective space of the cone. We take these to be the metric lines in Hilbert's metric. Correct parameterisation now requires that $d_H(\mathbb{I},x) = 1$. As above, a set M_H of metric lines may be obtained using homogeneity. Again, there will be a unique line connecting any distinct pair of points and the set of lines will be invariant under cone automorphisms.

The following observation is of fundamental importance.

Proposition 1 *Both (C,d_T,M_T) and (D,d_H,M_H) are hyperbolic metric spaces.*

This proposition is a direct consequence of Corollary 1 below.

Lemma 1 *For r in the range $[0,1]$, the mapping $C \to C : w \mapsto w^r$ is isotone.*

Proof. The complex function $z \mapsto z^r$ is a Pick function [6] and has integral representation

$$z^r = \frac{\sin \pi r}{\pi} \int_0^\infty \Big(\frac{t}{t^2+1} - \frac{1}{t+z}\Big) t^r \, dt + \cos\frac{\pi r}{2}.$$

By writing w in terms of its Jordan frame we see that we may replace z in the formula above with any $w \in C$. Since $w \mapsto w + \mathbb{I}t$ is isotone and both $w \mapsto w^{-1}$ and $w \mapsto -w$ are antitone (order inverting), the integrand is isotone in w. The conclusion follows. $\square$

Remark 1 *This proof follows that of Bhagwat and Subramanian [2], where the result was stated for bounded positive Hermitian operators on a Hilbert space.*

Corollary 1 *For all $r \in [0,1]$, and $x, y, z \in C$,*

$$\begin{aligned} d_T((1-r)x \oplus ry,\, (1-r)x \oplus rz) &\leq r d_T(y,z) \\ d_H((1-r)x \oplus ry,\, (1-r)x \oplus rz) &\leq r d_H(y,z). \end{aligned}$$

Proof. Since the cone C is homogeneous, x may be taken to be $\mathbb{I}$ without loss of generality. Observe that $(1-r)\,\mathbb{I} \oplus ry = y^r$. The map $C \to C : w \mapsto w^r$ is isotone by the lemma above. Since it is also homogeneous of degree r, we have that $d_T(y^r, z^r) \leq r d_T(y,z)$ and $d_H(y^r, z^r) \leq r d_H(y,z)$ for all $y, z \in C$ using a result stated previously. $\square$

Remark 2 *Corach* et al. *[5] prove this result using different means for the Thompson metric on the cone of positive elements of a C^*-algebra.*

Lemma 2 *The mapping $C \to \mathbb{R}^m : w \mapsto \log w$ is isotone.*

Proof. We proceed as in the proof of the previous lemma, this time using the formula

$$\log w = \int_0^\infty \frac{dt}{(1+t)t} \Big(\mathbb{I} - (1+t)(\mathbb{I} + wt)^{-1} \Big).$$

Again the integrand is isotone in w for each $t \in (0, \infty)$ and the conclusion follows. $\square$

Corollary 2 *For each $x, y \in C$, we have that*

$$\begin{aligned} d_T(x,y) &\geq |\log x - \log y|_{\mathbb{I}}^{T} \\ \text{and} \qquad d_H(x,y) &\geq |\log x - \log y|_{\mathbb{I}}^{H}. \end{aligned}$$

Proof. Write $\delta := d_T(x,y)$. Then $e^{-\delta} x \leq y \leq e^{\delta} x$. Since the log operator is isotone,

$$-\delta\,\mathbb{I} + \log x \;\leq\; \log y \;\leq\; \delta\,\mathbb{I} + \log x.$$

Subtracting $\log x$ we find that

$$-\delta\,\mathbb{I} \leq \log y - \log x \leq \delta\,\mathbb{I}$$

which is equivalent to $|\log y - \log x|_{\mathbb{I}}^{T} \leq \delta$.

It is not hard to show that the Hilbert distance may be expressed in terms of the Thompson distance as follows:

$$d_H(x,y) = 2 \inf_{\lambda > 0} d_T(\lambda x, y), \qquad \text{for } x, y \in C.$$

The infinitesimal form of this is

$$|V|_{\mathbb{I}}^{H} = 2 \inf_{\lambda \in \mathbb{R}} |V + \lambda \mathbb{I}|_{\mathbb{I}}^{T}, \qquad \text{for } V \in \mathbb{R}^{m}.$$

The second inequality of the lemma can now be deduced by applying the first to $d_T(\lambda x, y)$ and then taking the infimum over λ. □

Remark 3 *Using a different method, Corach* et al. *[4] derive this inequality for the Thompson metric on the cone of positive elements of a C^*-algebra.*

Remark 4 *The inequalities of Corollary 2 may be seen to be a limiting case of the those of Corollary 1. Take x to be $\mathbb{I}$ and consider what happens as $r \to 0$. Using $\frac{d}{dt} w^t \big|_0 = \log w$, we see that*

$$\frac{d_T(y^r, z^r)}{r} \to |\log y - \log z|_{\mathbb{I}}^{T}.$$

Corollary 1 implies that this quantity is less than $d_T(y, z)$. The case with the Hilbert metric is similar.

4 Iterates of non-expansive maps

Let f be a linear functional on the tangent space at a point $w \in C$. Taking the norm $|\cdot|_w^T$ on this tangent space, we define the norm of f in the usual way to be

$$|f|_w^T := \sup\{f(V) : |V|_w^T = 1\}.$$

Note that if $|f|_w^T = 1$ then $f(V) \leq |V|_w^T$ for all tangent vectors V at w. If instead we take the seminorm $|\cdot|_w^H$ on the tangent space, we obtain a seminorm $|f|_w^H := \sup\{f(V) : |V|_w^H = 1\}$.

The following theorem is an analogue of a result of Kohlberg and Neyman [11]. Their result applies in the case when the metric is a norm, however the property of the Hilbert and Thompson metrics given in Corollary 2 above is actually sufficient to establish the result. Our method of proof is a modification of that of Plant and Reich [14] who also considered the normed space case.

Theorem 1 *Let C be a symmetric cone and let $F : C \to C$ be non-expansive with respect to Thompson's metric d_T. Then, for each $w \in C$, there exists a linear functional f on the tangent space of C at w such that $|f|_w^T = 1$ and*

$$\begin{aligned} \lim_{n\to\infty} f\left(\frac{\log F^n w}{n}\right) &= \lim_{n\to\infty} \left|\frac{\log F^n w}{n}\right|_w^T \\ &= \inf_{n \in \mathbb{N}} d_T(w, F^n w)/n. \end{aligned}$$

Proof. For each $k \in \mathbb{N}$, write $a_k := d_T(F^k w, w)$. For $k, i \in \mathbb{N}$,

$$\begin{aligned} d_T(F^{k+i} w, w) &\leq d_T(F^{k+i} w, F^i w) + d_T(F^i w, w) \\ &\leq d_T(F^k w, w) + d_T(F^i w, w) \end{aligned}$$

and hence the sequence a_k is sub-additive. It follows that $\lim_{k\to\infty} a_k/k$ exists and equals $L := \inf_{k\in\mathbb{N}} a_k/k$. Plant and Reich show that sub-additivity also implies that, for any $p \in \mathbb{N}$ and $\epsilon > 0$, we can find $n_p > p$ such that

$$\frac{a_{n_p} - a_{n_p-m}}{m} \geq L - \epsilon, \qquad \text{for all } m < p.$$

For each $n \in \mathbb{N}$, there exists a linear functional f_n such that $|f_n|_w^T = 1$ and $f_n(\log F^n w) = |\log F^n w|_w^T$. Now, for any $m, n \in \mathbb{N}$ with $n \geq m$,

$$\begin{aligned} f_n(\log F^n w - \log F^m w) &\leq |\log F^n w - \log F^m w|_w^T \\ &\leq d_T(F^n w, F^m w) \\ &\leq d_T(F^{n-m} w, w). \end{aligned}$$

Also $f_n(\log F^n w) = |\log F^n w|_w^T = d_T(F^n w, w)$. Thus, for any $p \in \mathbb{N}$,

$$f_{n_p}(\log F^m w) \geq (L - \epsilon)m, \qquad \text{for all } m < p.$$

Let f be the limit of some subsequence of $\{f_{n_p} : p \in \mathbb{N}\}$. Then $|f|_w^T = 1$ and $f(\log F^m w) \geq (L-\epsilon)m$ for all $m \in \mathbb{N}$. Since ϵ is arbitrary, $f(\log F^m w) \geq Lm$. We have

$$L = \lim_{m\to\infty} \frac{|\log F^m w|_w^T}{m} \geq \lim_{m\to\infty} f\left(\frac{\log F^m w}{m}\right) \geq L.$$

□

A similar result holds for the Hilbert metric.

5 Cycle times of topical operators

In this section we specialise to the case of the Lorentz cones Λ_m defined by (3). These are interesting as they are the only examples of strictly convex symmetric cones.

Theorem 2 *Let $F : \Lambda_m \to \Lambda_m$ be non-expansive in both the Thomson metric and Hilbert's projective metric. Then the cycle time $\chi(F)$ exists.*

Proof. The unit of the Jordan algebra associated with the Lorentz cone is $\mathbb{I} = (1, \mathbf{0})$. The balls of radius r about the origin of the norms $|\cdot|_{\mathbb{I}}^T$ and $|\cdot|_{\mathbb{I}}^H$ are, respectively,

$$\begin{aligned} &\{(t,\mathbf{x}) : |t| + |\mathbf{x}| \leq r\} \\ \text{and} \qquad &\{(t,\mathbf{x}) : |\mathbf{x}| \leq r/2\}. \end{aligned}$$

By Theorem 1, there exists a linear functional f such that $|f|_{\mathbb{I}}^T = 1$ and

$$\nu_T := \lim_{n\to\infty} f\left(\frac{\log F^n \mathbb{I}}{n}\right) = \lim_{n\to\infty} \left|\frac{\log F^n \mathbb{I}}{n}\right|_{\mathbb{I}}^T.$$

From the shape of the balls it is clear that the set $S_T := \{V \in \mathbb{R}^m : |V|_{\mathbb{I}}^T = f(V) = \nu_T\}$ must take one of the following three forms: either it contains a single point $\pm\nu_T \mathbb{I}$, or a single point $(0, \mathbf{x})$ where $|\mathbf{x}| = \nu_T$, or a line segment

$[\pm\nu_T \mathbb{I}, (0, \mathbf{x})]$ again with $|\mathbf{x}| = \nu_T$. Since the sequence $n^{-1} \log F^n \mathbb{I}$ is norm bounded, it must have an accumulation point.

F is also non-expansive in d_H and so there exists a linear functional g such that $|g|_{\mathbb{I}}^H = 1$ and

$$\nu_H := \lim_{n\to\infty} g\left(\frac{\log F^n \mathbb{I}}{n}\right) = \lim_{n\to\infty} \left|\frac{\log F^n \mathbb{I}}{n}\right|_{\mathbb{I}}^H .$$

The set $S_H := \{V \in \mathbb{R}^m : |V|_{\mathbb{I}}^T = g(V) = \nu_H\}$ is of the form $\{(0, \mathbf{y}) + \lambda \mathbb{I} : \lambda \in \mathbb{R}\}$ where $|\mathbf{y}| = 2\nu_H$. The set of accumulation point of $n^{-1} \log F^n \mathbb{I}$ must lie within one of these lines. Since these lines cannot be parallel to the line segment S_T , there can be at most one accumulation point and so the limit exists. □

6 The Denjoy-Wolff Result

In 1926, Denjoy and Wolff proved the following theorem. Let Δ be the open unit disk of the complex plane and suppose $F : \Delta \to \Delta$ is holomorphic and has no fixed point. Then there exists a point p in the boundary of Δ such that the iterates F^n of F converge to p uniformly on compact subsets of Δ. A connection between holomorphic and non-expansive maps had been made earlier by Pick who observed that holomorphic self maps of Δ are non-expansive in the Poincaré metric on Δ. In fact the Denjoy-Wolff result holds even if F is merely required to be non-expansive rather than holomorphic [16].

A contraction on a metric space (X, d) is a map $F : X \to X$ such that $d(F(x), F(y)) < d(x, y)$. Beardon [1] proved the following theorem.

Theorem 3 *Let (X, d) be a metric space embedded in a compact Hausdorff topological space $\overline{X}$ in such a way that X is open and dense in $\overline{X}$. Suppose that for all sequences $\{x_n\}$ and $\{y_n\}$ in X converging respectively to distinct points x and y in $\overline{X} - X$, we have that*

$$d(x_n, y_n) - \max[d(x_n, w),\, d(y_n, w)] \to \infty$$

for all $w \in X$. Let $F : X \to X$ be a contraction with respect to d. Suppose that F is the point-wise limit of a sequence of contractions with fixed points in X. Then the iterates F^n converge locally uniformly on X to some point in $\overline{X}$.

He deduced the following corollary.

Corollary 3 *Let D be a bounded strictly-convex open subset of $\mathbb{R}^k$ and let $F : D \to D$ be a contraction with respect to the Hilbert metric d_H on D. Then the iterates F^n converge locally uniformly on D to some point in the Euclidean closure of D.*

A minor modification of Beardon's proof allows a version of this corollary to be established for maps which are non-expansive rather than contractions.

Theorem 4 *Let D be a bounded strictly convex open subset of $\mathbb{R}^k$ and let $F : D \to D$ be non-expansive with respect to d_H. If F has no fixed point then its iterates converge locally uniformly on D to some point in the boundary of D.*

Proof. The key property of contractions used by Beardon is that if F is a contraction, $x \in X$, and some subsequence of $F^n(x)$ converges to $\zeta \in X$, then $F(\zeta) = \zeta$. Of course, this is not true if F is merely non-expansive. The property is used to show that if the iterates of F do not converge locally uniformly to a point in the interior, then each orbit accumulates only on the boundary. In the present case this can be established using a result of Nussbaum [12] that states that if $F : D \to D$ is non-expansive with respect to d_H and has no fixed point in D, then for any $x \in D$, the sequence $F^n(x)$ is in any compact set at most a finite number of times. $\square$

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